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SOME RESULTS ON BOUNDEDNESS IN LOCALLY CONVEX CONES

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Abstract. We characterize bounded sets and bounded operators in locally convex cones. Also, we study some relations between barreledness and boundedness in locally convex cones.

1. **Introduction**

The general theory of locally convex cones as developed in [1] deals with preordered cones. We review some of the main concepts and refer to [1] for details.

A cone is defined to be a commutative monoid $\mathcal{C}$ together with a scalar multiplication by nonnegative real numbers satisfying the same axioms as for vector spaces; that is, $\mathcal{C}$ is endowed with an addition $(x, y) \mapsto x+y : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ which is associative, commutative and admits a neutral element $0 \in \mathcal{C}$, and with a scalar multiplication $(r, x) \mapsto r.x : \mathbb{R}_+ \times \mathcal{C} \to \mathcal{C}$ satisfying the usual associative and distributive properties, where $\mathbb{R}_+$ is the set of nonnegative real numbers. We have $1x = x$ and $0x = 0$, for all $x \in \mathcal{C}$. A preordered cone, ordered cone for short, is a cone with a preorder, that is, a reflexive transitive relation $\leq$ which is compatible with the algebraic operations.

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Cones may occur as a subset of real vector spaces; such a subset $\mathcal{C}$ is a cone if it satisfies $0 \in \mathcal{C}$, $a, b \in \mathcal{C} \Rightarrow a+b \in \mathcal{C}$ and $a \in \mathcal{C}, r \in \mathbb{R}_+ \Rightarrow ra \in \mathcal{C}$. Every product of (ordered) cones with pointwise addition and scalar multiplication (and order) is again a(n ordered) cone. But unlike the case for vector spaces, addition in cones need not satisfy the cancelation property, in general, and cones need not be emeddable in vector spaces. For example, $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is an ordered cone that is not embeddable in any vector space. Thus, our notion of a cone is more general than that used in classical functional analysis.

For cones $\mathcal{C}$ and $\mathcal{D}$, a mapping $T : \mathcal{C} \to \mathcal{D}$ is called a linear operator if $T(a + b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for $a, b \in \mathcal{C}$ and $\alpha \geq 0$. A linear functional on $\mathcal{C}$ is a linear operator $\mu : \mathcal{C} \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

A subset $\mathcal{V}$ of a preordered cone $\mathcal{C}$ is called an abstract 0-neighborhood system, if the following properties hold:

(v_1) $0 < v$ for all $v \in \mathcal{V}$;

(v_2) for all $u, v \in \mathcal{V}$, there is a $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$;

(v_3) $u + v \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$, whenever $u, v \in \mathcal{V}$ and $\alpha > 0$.

For every $a \in \mathcal{C}$ and $v \in \mathcal{V}$, we define,

$$v(a) = \{b \in \mathcal{C} \ : \ b \leq a + v\}, \quad \text{resp.} \quad (a)v = \{b \in \mathcal{C} \ : \ a \leq b + v\},$$

to be a neighborhood of a in the upper, resp. lower, topologies on $\mathcal{C}$. Their common refinement is called symmetric topology. We denote the neighborhoods of the symmetric topology as $v(a) \cap (a)v$ or $v(a)v$ for $a \in \mathcal{C}$ and $v \in \mathcal{V}$. We call $(\mathcal{C}, \mathcal{V})$ a full locally convex cone, and each subcone of $\mathcal{C}$, not necessarily containing $\mathcal{V}$, is called a locally convex cone. For technical reasons, we require the elements of a locally convex cone to be bounded below; i.e., for every $a \in \mathcal{C}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$, for some $\rho > 0$. An element a of $(\mathcal{C}, \mathcal{V})$ is called bounded if it is also upper bounded; i.e., for every $v \in \mathcal{V}$, there is a $\rho > 0$ such that $a \leq \rho v$.

For locally convex cones $\mathcal{C}$ and $\mathcal{D}$, with (abstract) 0-neighborhood systems $\mathcal{V}$ and $\mathcal{W}$, respectively, $T : \mathcal{C} \to \mathcal{D}$ is called uniformly continuous (u-continuous) if for every $w \in \mathcal{W}$, there is a $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$, whenever $a \leq b + v$.

Endowed with the (abstract) 0-neighborhood system $\boldsymbol{\varepsilon} = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. The u-continuous linear functionals on the locally convex cone $(\mathcal{C}, \mathcal{V})$ form a cone with the usual addition and scalar multiplication of functions. This cone is called the dual cone of $\mathcal{C}$ and denoted by $\mathcal{C}^*$.

For a locally convex cone $(\mathcal{C}, \mathcal{V})$, the polar v° of $v \in \mathcal{V}$ consists of all linear functionals μ on $\mathcal{C}$ satisfying $\mu(a) \leq \mu(b) + 1$, whenever $a \leq b + v$, for $a, b \in \mathcal{C}$. We have $\mathcal{C}^* = \cup\{v^\circ : v \in \mathcal{V}\}$.

Here, in Section 2 we study the bounded sets and bounded operators in locally convex cones. We give counterexamples for some properties of the topological vector spaces which are not satisfied for the locally convex cones. In section 3, we verify some relations between bounded sets and barrels.

2. **Bounded sets and bounded operators**

Definition 2.1. A set E in a locally convex cone $(\mathcal{C}, \mathcal{V})$ is called *bounded* if every symmetric neighborhood of 0 absorbs E; that is, for each $v \in \mathcal{V}$, there is a $\lambda > 0$ such that $E \subseteq \lambda v(0)v$.

This definition is equivalent to the one in [4]: $E \subset (\mathcal{C}, \mathcal{V})$ is bounded if for every $v \in \mathcal{V}$, there is a $\lambda > 0$ such that

$$0 \leq a + \lambda v \quad \text{and} \quad a \leq \lambda v,$$

for all $a \in E$.

If we consider $\tilde{v} = \{(a, b) \in \mathcal{C} \times \mathcal{C} : a \leq b + v\}$, we can state the definition of a bounded set by $\tilde{v}$: for every $v \in \mathcal{V}$, there is a $\lambda > 0$ such that

$$(0, a)\ ,\ (a, 0) \in \lambda \tilde{v},$$

for all $a \in E$ (cf. [3]).

$E \subseteq (\mathcal{C}, \mathcal{V})$ is called *internally bounded* if for every $v \in \mathcal{V}$, there is a $\lambda > 0$ such that

$$a \leq b + \lambda v \quad \text{or} \quad (a, b) \in \lambda \tilde{v}$$

for all $a, b \in E$ (cf. [4]).

Proposition 2.2. *In a locally convex cone $(\mathcal{C}, \mathcal{V})$, we have,*

(i) *subsets of (internally) bounded sets are (internally) bounded;*
(ii) *finite unions of (internally) bounded sets are (internally) bounded;*
(iii) *finite sums and positive scalar multiples of (internally) bounded sets are (internally) bounded.*

Proof. Parts (i) and (ii) are clear, as is the boundedness of scalar multiples of bounded sets. Let $E_1, \ldots, E_n \subseteq \mathcal{C}$ be bounded, and $v \in \mathcal{V}$. Let $e_1 + \cdots + e_n \in E_1 + \cdots + E_n$. For each $i = 1, \ldots, n$, there is a $\lambda_i > 0$ such that

$$0 \leq e_i + \lambda_i v \quad \text{and} \quad e_i \leq \lambda_i v,$$

for all $e_i \in E_i$. Put $\lambda = \lambda_1 + \cdots + \lambda_n$. We have,

$$0 \leq e_1 + \cdots + e_n + \lambda v \quad \text{and} \quad e_1 + \cdots + e_n \leq \lambda v.$$

□

Proposition 2.3. *Let $(\mathcal{C}, \mathcal{V})$ be a locally convex cone and E be a subset of $\mathcal{C}$. If E is (internally) bounded, then $\overline{E}$, the closure of E (w.r.t. the symmetric topology), is also (internally) bounded.*

Proof. Let E be bounded, $x \in \overline{E}$ and $v \in \mathcal{V}$. There is an $e \in E$ such that $e \in v(x)v$; that is,

$$e \leq x + v \quad \text{and} \quad x \leq e + v.$$

On the other hand, since E is bounded, then there is a $\lambda > 0$ such that

$$0 \leq e + \lambda v \quad \text{and} \quad e \leq \lambda v.$$

So, we have,

$$0 \leq x + (\lambda + 1)v \quad \text{and} \quad x \leq (\lambda + 1)v.$$

This means that $\overline{E}$ is bounded. The internally bounded case is similarly proved. □

For a set E in the locally convex cone $(\mathcal{C}, \mathcal{V})$ which contains 0, boundedness and internally boundedness is the same. Indeed, if E is bounded, then E is internally bounded, since by the boundedness of E, for every $v \in \mathcal{V}$, there is a $\lambda > 0$ such that

$$0 \leq e + \lambda v \quad \text{and} \quad e \leq \lambda v,$$

for all $e \in E$. Now, if $x, y \in E$ are arbitrary, then we have,

$$0 \leq y + \lambda v \quad \text{and} \quad x \leq \lambda v.$$

Thus, we have,

$$x \leq y + 2\lambda v,$$

for all $x, y \in E$; i.e., E is internally bounded. On the other hand, if E is internally bonded and contains 0, for every $v \in \mathcal{V}$, there is a $\lambda > 0$ such that

$$x \le y + \lambda v,$$

for all $x, y \in E$. If we put $x = 0$, then $0 \le y + \lambda v$, for all $y \in E$. Also, by putting $y = 0$, we have $x \le \lambda v$, for all $x \in E$. So,

$$0 \le e + \lambda v \quad \text{and} \quad e \le \lambda v,$$

for all $e \in E$, and this shows that E is bounded. But, it is not necessary that each internally bounded set is bounded. For example, $E = \{+\infty\}$ is internally bounded in $\overline{\mathbb{R}}$ with (abstract) 0-neighborhood system $\boldsymbol{\varepsilon} = \{\epsilon > 0 \ : \ \epsilon \in \mathbb{R}\}$, but it is not bounded. On the other hand, we have the following result.

Theorem 2.4. *If E is an internally bounded set in the locally convex cone $(\mathcal{C}, \mathcal{V})$ and $\overline{E}$ (the closure of E) has a bounded element, then E is bounded.*

Proof. Let $v \in \mathcal{V}$ be arbitrary and $x \in \overline{E}$ be a bounded element. There is a $\lambda > 0$ such that

$$0 \le x + \lambda v \quad \text{and} \quad x \le \lambda v.$$

On the other hand, since $x \in \overline{E}$, then there exists $y \in E$ such that

$$y \le x + v \quad \text{and} \quad x \le y + v.$$

Now, let $e \in E$ be arbitrary. Since E is internally bounded, then there is a $\lambda' > 0$ such that

$$y \le e + \lambda' v \quad \text{and} \quad e \le y + \lambda' v.$$

We have,

$$0 \le e + (\lambda + \lambda' + 1)v \quad \text{and} \quad e \le (\lambda + \lambda' + 1)v.$$

□

A sequence $\{x_n\}$ in the locally convex cone $(\mathcal{C}, \mathcal{V})$ *converges* to an element $x \in \mathcal{C}$, with respect to the symmetric topology, if for every $v \in \mathcal{V}$, there is a positive integer n_0 such that

$$x_n \le x + v \quad \text{and} \quad x \le x_n + v,$$

for all $n > n_0$. In this case, we write $x_n \to x$.

The convergence of nets in the locally convex cones is defined similarly.

Proposition 2.5. *If $\{x_n\}$ is a sequence in $\mathcal{C}$ such that $x_n \to x$ and x_n is bounded for a sufficiently large n, then x is bounded.*

Proof. Let $v \in \mathcal{V}$ be arbitrary. There is a positive integer n_0 such that

$$x_n \leq x + v \quad \text{and} \quad x \leq x_n + v,$$

for all $n > n_0$. Let $n > n_0$ be such that x_n is bounded. There is a $\lambda > 0$ such that

$$0 \leq x_n + \lambda v \quad \text{and} \quad x_n \leq \lambda v.$$

We have,

$$0 \leq x + (\lambda + 1)v \quad \text{and} \quad x \leq (\lambda + 1)v.$$

□

Remark 2.6. It is straightforward to see that if $x_n \to x$ and x_n is bounded for some sufficiently large n, then x_m is bounded for every $m > n$.

Let $(\mathcal{C}, \mathcal{V})$ be a locally convex cone. A sequence $\{x_n\}$ in $\mathcal{C}$ is a *Cauchy sequence* if for every $v \in \mathcal{V}$ there corresponds a positive integer n_0 such that $x_n \leq x_m + v$, whenever $m > n_0$ and $n > n_0$. It is clear that each convergent sequence is Cauchy.

Proposition 2.7. *If $\{x_n\}$ is a Cauchy sequence in $\mathcal{C}$, then there is an n_0 such that the set $\{x_n \ : \ n > n_0\}$ is internally bounded.*

Proof. Straightforward. □

Corollary 2.8. *If $\{x_n\}$ is a sequence in the locally convex cone $(\mathcal{C}, \mathcal{V})$ which is convergent to a bounded element, then there is an n_0 such that $E = \{x_n \ : \ n > n_0\}$ is (internally) bounded.*

Proof. By Proposition 2.7, E is internally bounded and by Proposition 2.4 is bounded. □

Remark 2.9. (i) If we consider $\overline{\mathbb{R}}$ with (abstract) 0-neighborhood system $\mathcal{V} = \{+\infty\}$, then for each $v \in \mathcal{V}$, we will have $v(0)v = \overline{\mathbb{R}}$; that is, $\overline{\mathbb{R}}$ and (so) every subcone of $\overline{\mathbb{R}}$ is bounded. On the other hand, we may have many unbounded elements in locally convex cones. For example, if we consider $Conv(\mathbb{R}) = \{A \subseteq \mathbb{R} \ : \ A \ \text{ is convex}\}$ with set inclusion as preorder and $\mathcal{V} = \{(-\alpha, \alpha) \ : \ \alpha > 0\}$ as (abstract) 0-neighborhood system, then all elements like $(-\infty, \alpha)$ and $(\alpha, +\infty)$, where $\alpha \in \mathbb{R}$, are unbounded in $Conv(\mathbb{R})$.

(ii) In locally convex cones, it is not necessary that compact sets and so the precompact sets be bounded. For instance, $\{0, +\infty\}$ is a compact (and so precompact) set in $(\overline{\mathbb{R}}, \boldsymbol{\varepsilon})$, but it is not (internally) bounded.

Proposition 2.10. *Let $(\mathcal{C}, \mathcal{V})$ be a locally convex cone and E be a subset of $\mathcal{C}$. The following properties are equivalent:*

(i) *E is bounded.*

(ii) *If $\{x_n\}$ is a sequence in E and $\{\alpha_n\}$ is a sequence of (nonnegative) scalars such that $\alpha_n \to 0$ as $n \to +\infty$, then $\alpha_n x_n \to 0$ as $n \to +\infty$.*

Proof. Suppose E is bounded and $v \in \mathcal{V}$. There is a $\lambda > 0$ such that $E \subseteq \lambda v(0)v$. If $x_n \in E$ and $\alpha_n \to 0$, then there exists an n_0 such that $\alpha_n \lambda < 1$, for all $n > n_0$. Since $\frac{1}{\lambda}E \subseteq v(0)v$, then $\alpha_n x_n \in v(0)v$, for all $n > n_0$. Thus, $\alpha_n x_n \to 0$.

Conversely, if E is not bounded, then there is a $v \in \mathcal{V}$ such that $E \nsubseteq nv(0)v$, for all $n \in \mathbb{N}$. Choose $x_n \in E$ such that $x_n \notin nv(0)v$. Then, $\frac{1}{n}x_n \nrightarrow 0$, but $\frac{1}{n} \to 0$. □

Now, we verify some properties of bounded operators.

Suppose T is a linear mapping of locally convex cone $\mathcal{C}$ into another locally convex cone $\mathcal{D}$. We shall say that T is *bounded* (*internally bounded*) if T maps bounded (*internally bounded*) subsets of $\mathcal{C}$ into bounded (*internally bounded*) subsets of $\mathcal{D}$.

Proposition 2.11. *Let $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ be locally convex cones and $T : \mathcal{C} \to \mathcal{D}$ be a linear operator. If T is (internally) bounded and $x_n \to x$ (under the symmetric topology) and x is bounded, then $\{Tx_n : n > n_0\}$ is bounded for some n_0.*

Proof. By Corollary 2.8, there is an n_0 such that $\{x_n \ : \ n > n_0\}$ is bounded and then $\{Tx_n \ : \ n > n_0\}$ is bounded. □

Proposition 2.12. *Let $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ be locally convex cones and T be a linear operator from $\mathcal{C}$ into $\mathcal{D}$. Among the three properties of T*

listed below, the implications

$$(a) \to (b) \to (c),$$

hold:

(a) *T is bounded in some neighborhood of* 0 *in the symmetric topology.*
(b) *T is continuous at* 0*.*
(c) *T is bounded.*

Furthermore, if $\mathcal{C}$ has a bounded neighborhood of 0, then $(c) \to (a)$.

Proof. We prove only $(a) \to (b)$. The other implications are easy. Let $w \in \mathcal{W}$ be arbitrary. By the assumption (a), there is a $v \in \mathcal{V}$ such that $T(v(0)v)$ is bounded in $(\mathcal{D}, \mathcal{W})$. Then, there is a $\lambda > 0$ such that $T(v(0)v) \subseteq \lambda w(0)w$; i.e., $T(\frac{v}{\lambda}(0)\frac{v}{\lambda}) \subseteq \lambda w(0)w$. This shows that T is continuous at 0. □

Corollary 2.13. *If T is u-continuous, then T is (internally) bounded.*

Proof. U-continuity implies continuity with respect to the symmetric topology. □

Let T be a linear mapping from the cone $\mathcal{C}$ into another cone $\mathcal{D}$. The set,

$$T^{-1}(0) = \{p \in \mathcal{C} \ : \ Tp = 0\} = \mathcal{N}(T),$$

is a subcone of $\mathcal{C}$, called the *null cone* of T.

Proposition 2.14. *Let $(\mathcal{C}, \mathcal{V})$ be a locally convex cone and T be a linear functional on $\mathcal{C}$ such that $Tp \neq 0$, for some $p \in \mathcal{C}$. Among the three properties of T listed below, the implications,*

$$(a) \to (b) \to (c),$$

hold:

(a) *T is continuous on $\mathcal{C}$.*
(b) *The null cone $\mathcal{N}(T)$ is closed.*
(c) *$\mathcal{N}(T)$ is not dense in $\mathcal{C}$.*

Proof. Straightforward. □

Remark 2.15. (i) For topological vector spaces, continuity at 0 implies continuity at the other points, but for locally convex cones, this is not true. We give a counter example next.

Let $\mathcal{C}$ be the cone of all finite convex subsets of $\overline{\mathbb{R}}$. We can consider $\mathcal{C}$ as a subcone of $Conv(\overline{\mathbb{R}})$. Indeed, $\mathcal{C} = \{\{a\}\ ,\ \{a, +\infty\}\ :\ a \in \overline{\mathbb{R}}\}$. We consider the preorder of $Conv(\overline{\mathbb{R}})$ as $A \leq B$ if and only if for every $a \in A$, there is a $b \in B$ such that $a \leq b$, and $\mathcal{V} = \{\{\epsilon\}\ :\ \epsilon > 0 \text{ and } \epsilon \in \mathbb{R}\}$ is an (abstract) 0-neighborhood system for $\mathcal{C}$. Let $T : \mathcal{C} \to \overline{\mathbb{R}}$ be defined as $T(A) = \inf(A)$. T is a linear functional which is continuous at $\{0\}$ with respect to the symmetric topology. Indeed, if $\epsilon > 0$ is arbitrary, by considering $0 < \delta \leq \epsilon$, we have,

$$\{\delta\}(\{0\})\{\delta\} = \{\{a\}\ :\ 0 \leq a + \delta \text{ and } a \leq \delta\},$$

and then,

$$T(\{\delta\}(\{0\})\{\delta\}) \subseteq \epsilon(0)\epsilon.$$

But T is not continuous at $\{0, +\infty\}$, since

$$\{\delta\}(\{0, +\infty\})\{\delta\} = \{\{a, +\infty\}\ :\ a \in \overline{\mathbb{R}}\},$$

for all $\delta > 0$ and

$$T(\{\delta\}(\{0, +\infty\})\{\delta\}) = \overline{\mathbb{R}}$$

which is not a subset of $1(0)1$ (for $\epsilon = 1$).

(ii) If T is bounded, or T is continuous (only) at 0, then it is not necessary that $\mathcal{N}(T)$ be closed. For, if we consider $(\mathcal{C}, \mathcal{V})$ and T as in (i), then T is continuous at 0, and so bounded by Proposition 2.12, but $\mathcal{N}(T) = \{\{0\}\ ,\ \{0, +\infty\}\}$ is not closed in $\mathcal{C}$. Indeed, for all $a \in \overline{\mathbb{R}}$, $\{a, +\infty\}$ is in the closure of $\mathcal{N}(T)$ (w.r.t. the symmetric topology).

Also, $\mathcal{N}(T)$ is not dense in $\mathcal{C}$, for this example. Hence, $(c) \to (a)$ is not true in Proposition 2.14.

Remark 2.16. Let $(\mathcal{C}, \mathcal{V})$ be the projective limit of the locally convex cones $(\mathcal{C}_\gamma, \mathcal{V}_\gamma)_{\gamma\in\Gamma}$ by the mappings g_γ as studied in [3]. We saw that a subset A of $\mathcal{C}$ is bounded if and only if each $g_\gamma(A)$ is bounded ([3], Proposition 2.5). Also, we gave the product of the locally convex cones $(\mathcal{C}_\gamma, \mathcal{V}_\gamma)$ as an example of the projective limit ([3], Example 2.8). Hence, $\times A_\gamma \subseteq \times\mathcal{C}_\gamma$ is bounded if and only if each A_γ is bounded. Also, in [3] we have defined and studied the inductive limit of a family of locally convex cones $(\mathcal{C}_\gamma, \mathcal{V}_\gamma)_{\gamma\in\Gamma}$ by the mappings f_γ. Since $f_\gamma : \mathcal{C}_\gamma \to \mathcal{C}$ is u-continuous for each $\gamma \in \Gamma$, hence if $A_\gamma \subseteq \mathcal{C}_\gamma$ is bounded, then $f_\gamma(A_\gamma) \subseteq \mathcal{C}$ is bounded. The quotient cone is introduced and studied in [2] separately, and as an example of inductive limit in [3]. So, the quotient of a bounded set is bounded.

In the remainder of this section, we investigate the image of a bounded set by a continuous homogeneous function.

Theorem 2.17. *Let $(\mathcal{C}, \mathcal{V})$ and $(\mathcal{D}, \mathcal{W})$ be locally convex cones. Let $f : \mathcal{C} \to \mathcal{D}$ (not necessarily linear) be continuous at 0 and*

$$f(\lambda x) = \lambda^r f(x),$$

for some $r > 0$, for all $\lambda > 0$ and $x \in \mathcal{C}$. If B is a bounded subset of $\mathcal{C}$, then the image of B under f is a bounded subset of $\mathcal{D}$.

Proof. Let $w \in \mathcal{W}$. There is a $v \in \mathcal{V}$ such that $f(v(0)v) \subseteq w(0)w$ (it is clear that $f(0) = 0$). Since B is bounded, then there is a $\lambda > 0$ such that $B \subseteq \lambda v(0)v$. Then,

$$f(B) \subseteq f(\lambda v(0)v) = \lambda^r f(v(0)v) \subseteq \lambda^r w(0)w,$$

that is, $f(B)$ is bounded. □

3. **Boundedness and barreledness**

In [4], a barrel and a barreled cone have been defined as follows.

Definition 3.1. Let $(\mathcal{C}, \mathcal{V})$ be a locally convex cone. A *barrel* is a convex subset B of $\mathcal{C}^2$ with the following properties:

- $(B1)$ For every $b \in \mathcal{C}$, there is a $v \in \mathcal{V}$ such that for every $a \in v(b)v$, there is a $\lambda > 0$ such that $(a, b) \in \lambda B$.
- $(B2)$ For all $a, b \in \mathcal{C}$ such that $(a, b) \notin B$, there is a $\mu \in \mathcal{C}^*$ such that $\mu(c) \le \mu(d) + 1$, for all $(c, d) \in B$ and $\mu(a) > \mu(b) + 1$.

Definition 3.2. A locally convex cone $(\mathcal{C}, \mathcal{V})$ is said to be barreled if for every barrel $B \subseteq \mathcal{C}^2$ and every $b \in \mathcal{C}$ there is a $v \in \mathcal{V}$ and a $\lambda > 0$ such that $(a, b) \in \lambda B$, for all $a \in v(b)v$.

Also, an upper-barreled cone is defined in [3] as follows.

Definition 3.3. Let $(\mathcal{C}, \mathcal{V})$ be a locally convex cone. $\mathcal{C}$ is called *upper-barreled* if for every barrel $B \subseteq \mathcal{C}^2$, there is a $v \in \mathcal{V}$ such that $\tilde{v} \subseteq B$.

An upper-barreled cone is a barreled cone but the converse is not true (see [3], for more conditions).

Let $(\mathcal{C},\mathcal{V})$ and $(\mathcal{D},\mathcal{W})$ be two locally convex cones. Then, $\mathcal{C}\times\mathcal{D}=\{(a,b):a\in\mathcal{C},b\in\mathcal{D}\}$ is a pre-ordered cone with

$$(a,b)\le(a',b') \quad \text{if and only if} \quad a\le a' \quad \text{and} \quad b\le b'.$$

We consider $\mathcal{V}\times\mathcal{W}=\{(v,w):v\in\mathcal{V},w\in W\}$ as an (abstract) 0-neighborhood system for $\mathcal{C}\times\mathcal{D}$.

In the locally convex cone $(\mathcal{C},\mathcal{V})$, it is not necessary that a barrel, as defined above, be bounded in $\mathcal{C}\times\mathcal{C}$. This is in every locally convex cone $(\mathcal{C},\mathcal{V})$, which is tightly covered by its bounded elements; i.e., for all $a,b\in\mathcal{C}$ and $v\in\mathcal{V}$ and $a\notin v(b)$ (or $a\not\le b+v$), there is some bounded element $a'\in\mathcal{C}$ such that $a'\preceq a$ and $a'\notin v(b)$ (see II.2.13 of [1]), $\tilde{v}$ is a barrel (cf. [3]), and so $\tilde{1}=\{(a,b)\ :\ a\le b+1\}$ is a barrel in $(\overline{\mathbb{R}},\varepsilon)$, but clearly it is not bounded in $(\overline{\mathbb{R}}\times\overline{\mathbb{R}},\varepsilon\times\varepsilon)$. On the other hand, as described following Lemma 2.2 in [4], in a barreled locally convex cone, $E\times\{e\}\subseteq\lambda B$ holds for every internally bounded set E, every $e\in E$ and every barrel B with some $\lambda>0$. Also, for bounded sets we have the following result.

Theorem 3.4. *Let $(\mathcal{C},\mathcal{V})$ be a locally convex cone, and E a bounded set of $\mathcal{C}$. Then, we have:*

(i) *If $\mathcal{C}$ is barreled, then every barrel absorbs $E\times\{0\}$.*
(ii) *If $\mathcal{C}$ is upper-barreled, then every barrel absorbs $E\times\{0\}$ and $\{0\}\times E$.*

Proof. Let $B\subseteq\mathcal{C}^2$ be a barrel. For (i), since $(\mathcal{C},\mathcal{V})$ is barreled, then there exist a neighborhood $v\in\mathcal{V}$ and $\lambda>0$ such that $(a,0)\in\lambda B$, for all $a\in v(0)v$. On the other hand, E is bounded, and thus, there exists $\lambda'>0$ such that $E\subseteq\lambda' v(0)v$. We have $E\times\{0\}\subseteq(\lambda+\lambda')B$ (B is convex). For (ii), from upper-barreledness of $(\mathcal{C},\mathcal{V})$, there exists a $v\in\mathcal{V}$ such that $\tilde{v}\subseteq B$. Also, from boundedness of E, there exists $\eta>0$ such that

$$(0,a)\in\eta\tilde{v} \quad \text{and} \quad (a,0)\in\eta\tilde{v},$$

for all $a\in E$; that is,

$$E\times\{0\}\subseteq\eta B \quad \text{and} \quad \{0\}\times E\subseteq\eta B.$$

□

We do not know whether generally in a locally convex cone $(\mathcal{C}, \mathcal{V})$, a barrel absorbs $E \times \{0\}$ or $\{0\} \times E$, where E is a bounded set in $\mathcal{C}$.

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