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INVESTIGATION ON THE HERMITIAN MATRIX EXPRESSION SUBJECT TO SOME CONSISTENT EQUATIONS

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Abstract. In this paper, we study the extremal ranks and inertias of the Hermitian matrix expression

$$f(X,Y) = C_4 - B_4Y - (B_4Y)^* - A_4XA_4^*,$$

where C_4 is Hermitian, $*$ denotes the conjugate transpose, X and Y satisfy the following consistent system of matrix equations $A_3Y = C_3, A_1X = C_1, XB_1 = D_1, A_2XA_2^* = C_2, X = X^*$. As consequences, we get the necessary and sufficient conditions for the above expression $f(X,Y)$ to be (semi) positive, (semi) negative. The relations between the Hermitian part of the solution to the matrix equation $A_3Y = C_3$ and the Hermitian solution to the system of matrix equations $A_1X = C_1, XB_1 = D_1, A_2XA_2^* = C_2$ are also characterized. Moreover, we give the necessary and sufficient conditions for the solvability to the following system of matrix equations $A_3Y = C_3, A_1X = C_1, XB_1 = D_1, A_2XA_2^* = C_2, X = X^*, B_4Y + (B_4Y)^* + A_4XA_4^* = C_4$ and provide an expression of the general solution to this system when it is solvable.
Keywords: Linear matrix equation, Moore-Penrose inverse, rank, inertia.
MSC(2010): Primary: 15A24, 15A09, 15A03.

1. Introduction

It is well known that the linear matrix expressions and their special cases–linear matrix equations are fundamental objects in matrix theory and applications. In recent 30 years, many authors addressed to the

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research on extremal ranks and inertias of matrix expressions and obtained many useful applications. In 2011, Liu and Tian [9] provided the formulas for the extremal ranks and inertias of

$$A - BXC - (BXC)^*.$$

Chu [3], Liu and Tian [10] considered the extremal ranks and inertias of

$$A - BXB^* - CYC^*. \tag{1.1}$$

In 2011, Tian [15] derived the maximal and minimal ranks and inertias of

$$f(X,Y) = C_4 - B_4Y - (B_4Y)^* - A_4XA_4^*. \tag{1.2}$$

Note that the rank and inertia are not only important in matrix theory but also powerful tools in investigating the solvability of some matrix equations. One purpose of this work is to get the necessary and sufficient conditions for the consistence of some new matrix equations by investigating the rank and inertia of some Hermitian matrix expressions with restrictions.

Study on the solvability conditions and the general solution to linear matrix equations is active in recent years (e.g. [4], [7], [17]- [22]). The well-known Lyapunov matrix equation

$$BX + (BX)^* = A \tag{1.3}$$

has played an important role in contemporary mathematics, such as system theory, stability analysis, optimal control, model reduction, etc. (e.g., [12, 13]). In 1998, Braden [1] gave the general solution to (1.3). In 2007, Djordjević [5] considered the explicit solution to the equation (1.3) for linear bounded operators on Hilbert spaces. Moreover, Cao [2] investigated the general solution to

$$BXC + (BXC)^* = A. \tag{1.4}$$

Xu [20] obtained an explicit expression of the solution to the operator equation (1.4). Liu [9] considered the general solutions to the matrix equation (1.4). In 2012, Wang and He [17] studied some necessary and sufficient conditions for the consistence of the matrix equation

$$A_1X + (A_1X)^* + B_1YC_1 + (B_1YC_1)^* = E_1 \tag{1.5}$$

and presented an expression of the general solution to (1.5).

Hermitian solutions to some matrix equations were also investigated by many authors. Khatri and Mitra [8] considered the Hermitian solutions to

$$A_3Y = C_3, \tag{1.6}$$

and

$$AYA^* = B,$$

respectively. In 2010, Wang and Wu [16] considered the solvability conditions and the Hermitian solution to the following system

$$A_1X = C_1, XB_1 = D_1, A_2XA_2^* = C_2, A_4XA_4^* = C_4 \tag{1.7}$$

for adjointable operators between Hilbert $\mathcal{C}^*$-modules.

Note that the matrix equation (1.3) and the system of matrix equations (1.7) are special cases of

$$\begin{cases} A_3Y = C_3, A_1X = C_1, XB_1 = D_1, \\ A_2XA_2^* = C_2, X = X^*, \\ B_4Y + (B_4Y)^* + A_4XA_4^* = C_4. \end{cases} \tag{1.8}$$

To our knowledge, there has been little results about this system. The purpose of this work is to give the solvability conditions and the expressions of the general solution to (1.8). In order to obtain the necessary and sufficient conditions for the consistence of the system (1.8), we give the extremal ranks and inertias of the matrix expression (1.2) subject to the matrix equations (1.6) and

$$A_1X = C_1, XB_1 = D_1, A_2XA_2^* = C_2, X = X^*. \tag{1.9}$$

The paper is organized as follows. In Section 2, we consider the extremal ranks and inertias of (1.2), where Y and X satisfy (1.6) and (1.9), respectively. We then obtain the relations between the Hermitian part of the solution to (1.6) and the Hermitian solution to (1.9). In section 3, we give the solvability conditions for (1.8) and the expression of the solution to (1.8) when it is solvable. As a special case of the system (1.8), we derive some new solvability conditions for the system (1.7) and a new expression of the general solution to (1.7), which extend the main result in [16].

Throughout this paper, we denote the field of complex numbers by $\mathbb{C}$, the set of all $m \times n$ matrices over $\mathbb{C}$ by $\mathbb{C}^{m\times n}$, the set of all $m \times m$ Hermitian matrices by $\mathbb{C}_h^{m\times m}$. The symbols I_n, A^* and $\mathcal{R}(A)$ stand for the $n \times n$ identity matrix, the conjugate transpose, the column space of a complex matrix A, respectively. The Moore-Penrose inverse $A^{\dagger}$ of A, is the unique matrix $A^{\dagger}$, such that

(i) $AA^{\dagger}A = A$, (ii) $A^{\dagger}AA^{\dagger} = A^{\dagger}$, (iii) $(AA^{\dagger})^{*} = AA^{\dagger}$, (iv) $(A^{\dagger}A)^{*} = A^{\dagger}A$.

If a Hermitian matrix X satisfies the equality (i), then it is called a Hermitian g inverse of A and is denoted by A_{h}^{-}. Moreover, L_A and R_A stand for the two projectors $L_A = I - A^{\dagger}A$, $R_A = I - AA^{\dagger}$. The eigenvalues of a Hermitian matrix $A \in \mathbb{C}^{n\times n}$ are real and the inertia of A is defined to be the triplet

$$\mathbb{I}_n(A) = \{i_+(A), i_-(A), i_0(A)\},$$

where $i_+(A), i_-(A)$ and $i_0(A)$ stand for the numbers of positive, negative and zero eigenvalues of A, respectively. For two Hermitian matrices A and B of the same size, we say $A \geq B$ $(A \leq B)$ in the Löwner partial ordering if $A-B$ is positive (negative) semidefinite. The Hermitian part of X is defined as $H(X) = (X + X^{*})/2$.

2. Extremal ranks and inertias of (1.2) subject to (1.6) and (1.9) with applications

In this section, we give the extremal ranks and inertias of (1.2) subject to (1.6) and (1.9). Then we characterize the relations between the Hermitian part of the solution to (1.6) and the Hermitian solution to (1.9). We also consider the extremal ranks and inertias of Hermitian Schur complement of a given Hermitian matrix, which extend the known results in [9].

Lemma 2.1. *[8] Let A_3 and C_3 be given. Then the following statements are equivalent:*
(a) Equation (1.6) is consistent.
(b)

$$R_{A_3}C_3 = 0. \tag{2.1}$$

(c)

$$r\begin{bmatrix} A_3 & C_3 \end{bmatrix} = r(A_3). \tag{2.2}$$

In this case, the general solution can be written as

$$Y = A_3^{\dagger}C_3 + L_{A_3}U, \tag{2.3}$$

where U is arbitrary matrix over $\mathbb{C}$ with appropriate size.

Lemma 2.2. *([14]) Let A_1, B_1, C_1, D_1, A_2, and $C_2 = C_2^*$ be given. Set*

$$E_1 = \begin{bmatrix} A_1 \\ B_1^* \end{bmatrix},\ F_1 = \begin{bmatrix} C_1 \\ D_1^* \end{bmatrix},\ M = A_2 L_{E_1},$$
$$Q = C_2 - A_2[E_1^{\dagger} F_1 + F_1^*(E_1^{\dagger})^* - E_1^{\dagger} E_1 F_1^* (E_1^{\dagger})^*] A_2^*.$$

Then the following statements are equivalent:
(a) System (1.9) is consistent.
(b)

$$E_1 F_1^* = F_1 E_1^*,\ R_{E_1} F_1 = 0,\ R_M Q = 0. \tag{2.4}$$

(c)

$$E_1 F_1^* = F_1 E_1^*,\ r\begin{bmatrix} E_1 & F_1 \end{bmatrix} = r(E_1),\ r\begin{bmatrix} A_2 & C_2 \\ E_1 & F_1 A_2^* \end{bmatrix} = r\begin{bmatrix} A_2 \\ E_1 \end{bmatrix}. \tag{2.5}$$

In this case, the general Hermitian solution to (1.9) can be expressed as

$$\begin{aligned} X =& E_1^{\dagger} F_1 + F_1^*(E_1^{\dagger})^* - E_1^{\dagger} E_1 F_1^* (E_1^{\dagger})^* + L_{E_1} M^{\dagger} Q (M^{\dagger})^* L_{E_1} \\ &+ L_{E_1} L_M V L_{E_1} + (L_{E_1} L_M V L_{E_1})^*, \end{aligned} \tag{2.6}$$

where V is an arbitrary Hermitian matrix over $\mathbb{C}$ with appropriate size.

Lemma 2.3. *([11]) Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{m\times k}, C \in \mathbb{C}^{l\times n}, D \in \mathbb{C}^{m\times p}, E \in \mathbb{C}^{q\times n}, Q \in \mathbb{C}^{m_1\times k}$, and $P \in \mathbb{C}^{l\times n_1}$ be given. Then*

(a) $r(A) + r(R_A B) = r(B) + r(R_B A) = r\begin{bmatrix} A & B \end{bmatrix}.$

(b) $r(A) + r(C L_A) = r(C) + r(A L_C) = r\begin{bmatrix} A \\ C \end{bmatrix}.$

(c) $r(B) + r(C) + r(R_B A L_C) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$

(d) $r(P) + r(Q) + r\begin{bmatrix} A & B L_Q \\ R_P C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & P \\ 0 & Q & 0 \end{bmatrix}.$

(e) $r\begin{bmatrix} R_B A L_C & R_B D \\ E L_C & 0 \end{bmatrix} + r(B) + r(C) = r\begin{bmatrix} A & D & B \\ E & 0 & 0 \\ C & 0 & 0 \end{bmatrix}.$

Lemma 2.4. *([15]) Let $A \in \mathbb{C}_h^{m\times m}, B \in \mathbb{C}^{m\times n}, C \in \mathbb{C}_h^{n\times n}, Q \in \mathbb{C}^{m\times n}$, $P \in \mathbb{C}^{p\times n}$ be given, and $T \in \mathbb{C}^{m\times m}$ be nonsingular. Then*

$$(a)\ i_{\pm}(TAT^*) = i_{\pm}(A),$$

$$(b)\ i_{\pm}\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} = i_{\pm}(A) + i_{\pm}(C),$$

$$(c)\ i_{\pm}\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q),$$

$$(d)\ i_{\pm}\begin{bmatrix} A & BL_P \\ L_PB^* & 0 \end{bmatrix} + r(P) = i_{\pm}\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix}.$$

Lemma 2.5. *([23]) Let $p(X,Y) = A - BX - (BX)^* - CYD - (CYD)^*$, where A, B, C, and D are given with appropriate sizes, and let*

$$M_1 = \begin{bmatrix} A & B & C \\ B^* & 0 & 0 \\ C^* & 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} A & B & D^* \\ B^* & 0 & 0 \\ D & 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 \end{bmatrix}, M_5 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix}.$$

Then we have the identities:

$$\max_{X,Y} r\left[p(X,Y)\right] = \min\left\{m,\ r(M_1),\ r(M_2),\ r(M_3)\right\},$$

$$\min_{X,Y} r\left[p(X,Y)\right] =$$
$$2r(M_3) - 2r(B) + \max\{u_+ + u_-,\ v_+ + v_-,\ u_+ + v_-,\ u_- + v_+\},$$

$$\max_{X,Y} i_{\pm}\left[p(X,Y)\right] = \min\left\{i_{\pm}(M_1), i_{\pm}(M_2)\right\},$$

$$\min_{X,Y} i_{\pm}\left[p(X,Y)\right] =$$
$$r(M_3) - r(B) + \max\left\{i_{\pm}(M_1) - r(M_4), i_{\pm}(M_2) - r(M_5)\right\},$$

where

$$u_{\pm} = i_{\pm}(M_1) - r(M_4), \qquad v_{\pm} = i_{\pm}(M_2) - r(M_5).$$

For convenience of representation, we adopt the following notations:

$$S_1 = \left\{ Y \middle| \ A_3 Y = C_3 \ \right\},$$

$$S_2 = \left\{ X = X^* \middle| \ A_1 X = C_1,\ XB_1 = D_1,\ A_2 X A_2^* = C_2 \ \right\}.$$

Now we give the main theorem of this paper.

Theorem 2.6. *Let $A_1, C_1, B_1, D_1, A_2, C_2 = C_2^*, A_3, C_3, A_4, B_4,$ and $C_4 = C_4^* \in \mathbb{C}^{m\times m}$ be given, and let E_1, F_1, M, Q be as in Lemma 2.2. Assume that (1.6) and (1.9) are consistent, respectively. Then,*
(a) The maximal rank of (1.2) is

$$\max_{Y\in S_1, X\in S_2} r\left[f(X,Y)\right] = \min\left\{m,\ k_1,\ k_2,\ k_3\right\}. \tag{2.7}$$

(b) The minimal rank of (1.2) is

$$\min_{Y\in S_1, X\in S_2} r\left[f(X,Y)\right] = 2k_3 - 2k_5 + \max\{s_+ + s_-,\ t_+ + t_-,\ s_+ + t_-,\ s_- + t_+\}. \tag{2.8}$$

(c) The maximal inertia of (1.2) is

$$\max_{Y\in S_1, X\in S_2} i_{\pm}\left[f(X,Y)\right] = \min\left\{i_{\pm}(N_1), i_{\pm}(N_2)\right\}. \tag{2.9}$$

(d) The minimal inertia of (1.2) is

$$\min_{Y\in S_1, X\in S_2} i_{\pm}\left[f(X,Y)\right] = k_3 - k_5 + \max\left\{i_{\pm}(N_1) - k_4, -i_{\mp}(N_2)\right\}, \tag{2.10}$$

where

$$k_1 = r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & 0 & \frac{1}{2}A_4F_1^* \\ B_4^* & 0 & 0 & A_3^* & 0 & 0 \\ A_4^* & 0 & 0 & 0 & A_2^* & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & -C_2 & -\frac{1}{2}A_2F_1^* \\ \frac{1}{2}F_1A_4^* & 0 & E_1 & 0 & -\frac{1}{2}F_1A_2^* & 0 \end{bmatrix} - 2r\begin{bmatrix} A_2 \\ E_1 \end{bmatrix} - 2r(A_3),$$

$$k_2 = r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & \frac{1}{2}A_4F_1^* \\ B_4^* & 0 & 0 & A_3^* & 0 \\ A_4^* & 0 & 0 & 0 & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 \\ \frac{1}{2}F_1A_4^* & 0 & E_1 & 0 & 0 \end{bmatrix} - 2r(E_1) - 2r(A_3),$$

$$k_3 = r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* \\ B_4^* & 0 & 0 & A_3^* \\ C_3 & A_3 & 0 & 0 \\ F_1A_4^* & 0 & E_1 & 0 \end{bmatrix} - r(E_1) - 2r(A_3),$$

$$k_4 = r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & 0 & 0 \\ B_4^* & 0 & 0 & A_3^* & 0 & 0 \\ A_4^* & 0 & 0 & 0 & A_2^* & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 & 0 \\ F_1A_4^* & 0 & E_1 & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A_2 \\ E_1 \end{bmatrix} - 2r(A_3) - r(E_1),$$

$$k_5 = r\begin{bmatrix} A_3 \\ B_3 \end{bmatrix} - r(A_3),$$

$$\begin{aligned} i_{\pm}(N_1) =& i_{\pm}\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & 0 & \frac{1}{2}A_4F_1^* \\ B_4^* & 0 & 0 & A_3^* & 0 & 0 \\ A_4^* & 0 & 0 & 0 & A_2^* & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & -C_2 & -\frac{1}{2}A_2F_1^* \\ \frac{1}{2}F_1A_4^* & 0 & E_1 & 0 & -\frac{1}{2}F_1A_2^* & 0 \end{bmatrix} \\ &- r\begin{bmatrix} A_2 \\ E_1 \end{bmatrix} - r(A_3), \end{aligned}$$

$$i_{\pm}(N_2) = i_{\pm}\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & \frac{1}{2}A_4F_1^* \\ B_4^* & 0 & 0 & A_3^* & 0 \\ A_4^* & 0 & 0 & 0 & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 \\ \frac{1}{2}F_1A_4^* & 0 & E_1 & 0 & 0 \end{bmatrix} - r(E_1) - r(A_3),$$

$$s_{\pm} = i_{\pm}(N_1) - k_4, \qquad t_{\pm} = -i_{\mp}(N_2).$$

Proof. By Lemma 2.1 and Lemma 2.2, the general solutions to (1.6) and (1.9) can be expressed as

$$Y = Y_0 + L_{A_3}U, \tag{2.11}$$

$$X = X_0 + L_{E_1}L_MVL_{E_1} + (L_{E_1}L_MVL_{E_1})^*, \tag{2.12}$$

where Y_0 and X_0 are special solutions to (1.6) and (1.9), respectively, U and V are arbitrary matrices over $\mathbb{C}$ with appropriate sizes. Let

$A = C_4 - A_4 X_0 A_4^* - B_4 Y_0 - (B_4 Y_0)^*,\ B = B_4 L_{A_3},\ C = A_4 L_{E_1} L_M,\ D = L_{E_1} A_4^*$. Substituting (2.11) and (2.12) into (1.2) gives

$$\underset{Y \in S_1, X \in S_2}{f(X,Y)} = \underset{U,V}{h(U,V)} = A - BU - (BU)^* - CVD - (CVD)^*. \tag{2.13}$$

Applying Lemma 2.5 to (2.13) yields

$$\max_{U,V} r\left[h(U,V)\right] = \min\left\{m,\ r(N_1),\ r(N_2),\ r(N_3)\right\}, \tag{2.14}$$

$$\begin{aligned} \min_{U,V} r\left[h(U,V)\right] = \\ 2r(N_3) - 2r(B) + \max\{s_+ + s_-,\ t_+ + t_-,\ s_+ + t_-,\ s_- + t_+\}, \end{aligned} \tag{2.15}$$

$$\max_{U,V} i_\pm\left[h(U,V)\right] = \min\left\{i_\pm(N_1), i_\pm(N_2)\right\}, \tag{2.16}$$

$$\begin{aligned} \min_{U,V} i_\pm\left[h(U,V)\right] = \\ r(N_3) - r(B) + \max\left\{i_\pm(N_1) - r(N_4), i_\pm(N_2) - r(N_5)\right\}, \end{aligned} \tag{2.17}$$

where

$$N_1 = \begin{bmatrix} A & B & C \\ B^* & 0 & 0 \\ C^* & 0 & 0 \end{bmatrix}, N_2 = \begin{bmatrix} A & B & D^* \\ B^* & 0 & 0 \\ D & 0 & 0 \end{bmatrix}, N_3 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \end{bmatrix},$$

$$N_4 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 \end{bmatrix}, N_5 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix},$$

$$s_\pm = i_\pm(N_1) - r(N_4), \qquad t_\pm = i_\pm(N_2) - r(N_5).$$

Note that $r(N_2)=r(N_5)$. Applying Lemma 2.3, we have

$$k_1:=r(N_1)=r\begin{bmatrix} A & B & C \\ B^* & 0 & 0 \\ C^* & 0 & 0 \end{bmatrix}=r\begin{bmatrix} A & B_4L_{A_3} & A_4L_{E_1}L_M \\ R_{A_3^*}B_4^* & 0 & 0 \\ R_{M^*}R_{E^*}A_4^* & 0 & 0 \end{bmatrix}$$
$$=r\begin{bmatrix} A & B_4 & A_4L_{E_1} & 0 & 0 \\ B_4^* & 0 & 0 & A_3^* & 0 \\ R_{E^*}A_4^* & 0 & 0 & 0 & M^* \\ 0 & A_3 & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 \end{bmatrix}-2r(A_3)-2r(M)$$
$$=r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & 0 & \frac{1}{2}A_4F_1^* \\ B_4^* & 0 & 0 & A_3^* & 0 & 0 \\ A_4^* & 0 & 0 & 0 & A_2^* & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & -C_2 & -\frac{1}{2}A_2F_1^* \\ \frac{1}{2}F_1A_4^* & 0 & E_1 & 0 & -\frac{1}{2}F_1A_2^* & 0 \end{bmatrix}-2r\begin{bmatrix} A_2 \\ E_1 \end{bmatrix}-2r(A_3).$$

Similarly, we can get the results of the ranks of N_2-N_4, B and inertias of N_1 and N_2 by using Lemma 2.3 and Lemma 2.4.

Corollary 2.7. *Let $A_1,C_1,B_1,D_1,A_2,C_2,A_3,C_3,A_4,B_4,C_4,E_1,F_1,k_1-k_6,s,t,N_1$, and N_2 be the same as in Theorem 2.6. Assume that (1.6) and (1.9) are consistent, respectively. Then,*
(a) There exist $Y\in S_1$ and $X\in S_2$ such that $C_4-B_4Y-(B_4Y)^*-A_4XA_4^*>0$ if and only if

$$i_+(N_1)\geq m, i_+(N_2)\geq m.$$

(b) There exist $Y\in S_1$ and $X\in S_2$ such that $C_4-B_4Y-(B_4Y)^*-A_4XA_4^*<0$ if and only if

$$i_-(N_1)\geq m, i_-(N_2)\geq m.$$

(c) There exist $Y\in S_1$ and $X\in S_2$ such that $C_4-B_4Y-(B_4Y)^*-A_4XA_4^*\geq 0$ if and only if

$$k_3-k_5+i_-(N_1)-k_4\leq 0,\ k_3-k_5-i_+(N_2)\leq 0.$$

(d) There exist $Y\in S_1$ and $X\in S_2$ such that $C_4-B_4Y-(B_4Y)^*-A_4XA_4^*\leq 0$ if and only if

$$k_3-k_5+i_+(N_1)-k_4\geq 0,\ k_3-k_5-i_-(N_2)\geq 0.$$

(e) $C_4 - B_4Y - (B_4Y)^* - A_4XA_4^* > 0$ *for all* $Y \in S_1$ *and* $X \in S_2$ *if and only if*

$$k_3 - k_5 + i_+(N_1) - k_4 = m, \text{ or } k_3 - k_5 - i_-(N_2) = m.$$

(f) $C_4 - B_4Y - (B_4Y)^* - A_4XA_4^* < 0$ *for all* $Y \in S_1$ *and* $X \in S_2$ *if and only if*

$$k_3 - k_5 + i_-(N_1) - k_4 = m, \text{ or } k_3 - k_5 - i_+(N_2) = m.$$

(g) $C_4 - B_4Y - (B_4Y)^* - A_4XA_4^* \geq 0$ *for all* $Y \in S_1$ *and* $X \in S_2$ *if and only if*

$$i_-(N_1) = 0 \quad or \quad i_-(N_2) = 0.$$

(h)$C_4 - B_4Y - (B_4Y)^* - A_4XA_4^* \leq 0$ *for all* $Y \in S_1$ *and* $X \in S_2$ *if and only if*

$$i_+(N_1) = 0 \quad or \quad i_+(N_2) = 0.$$

Next, we reveal the relations between the Hermitian part of the solution to (1.6) and the Hermitian solution to (1.9). Let $A_4 = I_m$, $B_4 = -I_m/2, C_4 = 0$, put

$$u_1 = 2m - 2r\begin{bmatrix} A_2 \\ E_1 \end{bmatrix} - 2r(A_3) +$$

$$r\begin{bmatrix} 0 & 2A_3^* & A_2^* & E_1^* + \\ 2A_3 & 2A_3C_3^* + 2C_3A_3^* & 0 & A_3^*F_1^* \\ A_2 & 0 & -C_2 & -\frac{1}{2}A_2F_1^* \\ E_1 & F_1^*A_3^* & -\frac{1}{2}F_1A_2^* & 0 \end{bmatrix},$$

$$u_2 = 2m + r\begin{bmatrix} 0 & 2A_3^* & E_1^* \\ 2A_3 & 2A_3C_3^* + 2C_3A_3^* & A_3F_1^* \\ 0 & F_1A_3^* & 0 \end{bmatrix} - 2r(E_1) - 2r(A_3),$$

$$u_3 = 2m + r\begin{bmatrix} 2A_3 & 2A_3C_3^* \\ E_1 & 0 \end{bmatrix} - r(E_1) - 2r(A_3),$$

$$u_4 = 2m - r\begin{bmatrix} A_2 \\ E_1 \end{bmatrix} - 2r(A_3) - r(E_1) +$$

$$r\begin{bmatrix} 0 & 2A_3^* & A_2^* & E_1^* \\ 2A_3 & 2A_3C_3^* + 2C_3A_3^* & 0 & 0 \\ E_1 & 2F_1A_3^* & 0 & 0 \end{bmatrix},$$

$$k_5 = r\begin{bmatrix} A_3 \\ B_3 \end{bmatrix} - r(A_3),$$

$$i_{\pm}(M_1) = m - r\begin{bmatrix} A_2 \\ E_1 \end{bmatrix} - r(A_3) +$$
$$i_{\pm}\begin{bmatrix} 0 & 2A_3^* & A_2^* & E_1^* \\ 2A_3 & 2A_3C_3^* + 2C_3A_3^* & 0 & A_3^*F_1^* \\ A_2 & 0 & -C_2 & -\frac{1}{2}A_2F_1^* \\ E_1 & F_1^*A_3^* & -\frac{1}{2}F_1A_2^* & 0 \end{bmatrix},$$

$$i_{\pm}(M_2) = m + i_{\pm}\begin{bmatrix} 0 & 2A_3^* & E_1^* \\ 2A_3 & 2A_3C_3^* + 2C_3A_3^* & A_3F_1^* \\ 0 & F_1A_3^* & 0 \end{bmatrix} - r(E_1) - r(A_3),$$

$$p_{\pm} = i_{\pm}(M_1) - k_4, \qquad q_{\pm} = -i_{\mp}(M_2).$$

Then we get the next corollary.

Corollary 2.8. *Assume that (1.6) and (1.9) are consistent, respectively. Then,*
(a) The maximal rank of $\frac{1}{2}(Y+Y^*) - X$ *is*

$$\max_{Y\in S_1, X\in S_2} r\left[f(X,Y)\right] = \min\left\{m,\ u_1,\ u_2,\ u_3\right\}.$$

(b) The minimal rank of $\frac{1}{2}(Y+Y^*) - X$ *is*

$$\min_{Y\in S_1, X\in S_2} r\left[f(X,Y)\right] = 2u_3 - 2k_5 +$$
$$\max\{p_+ + p_-,\ q_+ + q_-,\ p_+ + q_-,\ p_- + q_+\}.$$

(c) The maximal inertia of $\frac{1}{2}(Y+Y^*) - X$ *is*

$$\max_{Y\in S_1, X\in S_2} i_{\pm}\left[f(X,Y)\right] = \min\left\{i_{\pm}(M_1), i_{\pm}(M_2)\right\}.$$

(d) The minimal inertia of $\frac{1}{2}(Y+Y^*) - X$ *is*

$$\min_{Y\in S_1, X\in S_2} i_{\pm}\left[f(X,Y)\right] = u_3 - k_5 + \max\left\{i_{\pm}(M_1) - u_4, -i_{\mp}(M_2)\right\}.$$

Therefore:
(e) There exist $Y \in S_1$ *and* $X \in S_2$ *such that* $\frac{1}{2}(Y+Y^*) > X$ *if and only if*

$$i_+(M_1) \geq m, i_+(M_2) \geq m.$$

(f) There exist $Y \in S_1$ *and* $X \in S_2$ *such that* $\frac{1}{2}(Y+Y^*) < X$ *if and only if*

$$i_-(M_1) \geq m, i_-(M_2) \geq m.$$

(g) There exist $Y \in S_1$ and $X \in S_2$ such that $\frac{1}{2}(Y+Y^*) \geq X$ if and only if

$$u_3 - k_5 + i_-(M_1) - u_4 \leq 0,\ u_3 - k_5 - i_+(M_2) \leq 0.$$

(h) There exist $Y \in S_1$ and $X \in S_2$ such that $\frac{1}{2}(Y+Y^*) \leq X$ if and only if

$$u_3 - k_5 + i_+(M_1) - u_4 \geq 0,\ u_3 - k_5 - i_-(M_2) \geq 0.$$

(i) $\frac{1}{2}(Y+Y^*) > X$ for all $Y \in S_1$ and $X \in S_2$ if and only if

$$u_3 - k_5 + i_+(M_1) - u_4 = m,\ \text{or}\ u_3 - k_5 - i_-(M_2) = m.$$

(j) $\frac{1}{2}(Y+Y^*) < X$ for all $Y \in S_1$ and $X \in S_2$ if and only if

$$u_3 - k_5 + i_-(M_1) - u_4 = m,\ \text{or}\ u_3 - k_5 - i_+(M_2) = m.$$

(k) $\frac{1}{2}(Y+Y^*) \geq X$ for all $Y \in S_1$ and $X \in S_2$ if and only if

$$i_-(M_1) = 0 \quad \text{or} \quad i_-(M_2) = 0.$$

(l)$\frac{1}{2}(Y+Y^*) \leq X$ for all $Y \in S_1$ and $X \in S_2$ if and only if

$$i_+(M_1) = 0 \quad \text{or} \quad i_+(M_2) = 0.$$

(m)There exist $Y \in S_1$ and $X \in S_2$ such that $\frac{1}{2}(Y+Y^*) = X$ if and only if

$$2u_3 + p_+ + p_- \leq 2k_5,\ \text{or}\ 2u_3 + q_+ + q_- \leq 2k_5,$$

$$\text{or} \quad 2u_3 + p_+ + q_- \leq 2k_5,\ \text{or} \quad 2u_3 + p_- + q_+ \leq 2k_5.$$

For a 2×2 block Hermitian matrix $M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$, the $S_A = D - BA_h^- B$ is in fact a linear matrix function $D - B^*XB$ subject to the Hermitian solution of the matrix equation $AXA = A$. In Theorem 2.6, let $A_1, C_1, B_1, D_1, A_3, C_3$ vanish, and let $C_4 = D \in \mathbb{C}_h^{m\times m}$, $A_2 = C_2 = A = A^*, A_4 = B$. Then we have the following result.

Corollary 2.9. *Let $A \in \mathbb{C}_h^{m\times m}, B \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}_h^{n\times n}$ be given. Let*

$$N = \begin{bmatrix} A & 0 & B \\ 0 & B^* & D \end{bmatrix}.$$

Then the maximal and minimal values of the rank, positive and negative signatures of the Schur complement $D - B^*A_h^- B$ *are given by*

$$\max r(D - B^*A_h^- B) = min\{r[B^*, D], r(M) - r(A)\},$$
$$\max i_\pm(D - B^*A_h^- B) = i_\pm(M) - i_\pm(A),$$
$$\min r(D - B^*A_h^- B) = r(A) + 2r[B^*, D] + r(M) - 2r(N),$$
$$\min i_\pm(D - B^*A_h^- B) = i_\pm(A) + r[B^*, D] + i_\pm(M) - r(N).$$

Remark 2.10. *It is the main result of Theorem 5.3 in [9].*

3. The solvability conditions and the expression of the general solution to (1.8)

Now we consider the solvability conditions and the explicit expression of the general solution to (1.8). We begin with a lemma which plays an important role in the development in this section.

Lemma 3.1. *([17]) Let* $A_1 \in \mathbb{C}^{m\times n_1}, B_1 \in \mathbb{C}^{m\times n_2}, C_1 \in \mathbb{C}^{q\times m}$, *and* $E_1 \in \mathbb{C}_h^{m\times m}$ *be given. Let* $A = R_{A_1}B_1, B = C_1R_{A_1}, E = R_{A_1}E_1R_{A_1}, M = R_AB^*, N = A^*L_B, S = B^*L_M$. *Then the following statements are equivalent:*
(a) Equation (1.5) is consistent.
(b)

$$R_MR_AE = 0, \quad R_AER_A = 0, \quad L_BEL_B = 0.$$

(c)

$$r\begin{bmatrix} E_1 & B_1 & C_1^* & A_1 \\ A_1^* & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & C_1^* & A_1 \end{bmatrix} + r(A_1),$$
$$r\begin{bmatrix} E_1 & B_1 & A_1 \\ A_1^* & 0 & 0 \\ B_1^* & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} B_1 & A_1 \end{bmatrix},$$
$$r\begin{bmatrix} E_1 & C_1^* & A_1 \\ A_1^* & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} C_1^* & A_1 \end{bmatrix}.$$

In this case, the general solution of equation (1.5) can be expressed as

$$\begin{aligned} Y =& \frac{1}{2}[A^\dagger EB^\dagger - A^\dagger B^*M^\dagger EB^\dagger - A^\dagger S(B^\dagger)^*EN^\dagger A^*B^\dagger + A^\dagger E(M^\dagger)^* \\ &+ (N^\dagger)^*EB^\dagger S^\dagger S] + L_AV_1 + V_2R_B + U_1L_SL_M + R_NU_2^*L_M \\ &- A^\dagger SU_2R_NA^*B^\dagger, \end{aligned}$$

$$\begin{aligned}X =& A_1^\dagger[E_1 - B_1YC_1 - (B_1YC_1)^*] - \frac{1}{2}A_1^\dagger[E_1 - B_1YC_1 - (B_1YC_1)^*]A_1A_1^\dagger \\ &- A_1^\dagger W_1A_1^* + W_1^*A_1A_1^\dagger + L_{A_1}W_2,\end{aligned}$$

where $U_1, U_2, V_1, V_2, W_1, W_2$ are arbitrary matrices over $\mathbb{C}$ with appropriate sizes..

Now we present the fundamental theorem of this paper.

Theorem 3.2. *Let $A_1, B_1, C_1, D_1, A_2, C_2, A_3, C_3, A_4, B_4, C_4, E_1, F_1, M$ and Q be the same as in Theorem 2.6. Set*

$$\alpha = E_1^\dagger F_1 + F_1^*(E_1^\dagger)^* - E_1^\dagger E_1F_1^*(E_1^\dagger)^* + L_{E_1}M^\dagger Q(M^\dagger)^*L_{E_1},$$

$$B = B_4L_{A_3},\ C = A_4L_{E_1}L_M,\ D = L_{E_1}A_4^*,$$

$$A = C_4 - A_4\alpha A_4^* - B_4A_3^\dagger C_3 - (B_4A_3^\dagger C_3)^*,\ G = R_BC,$$

$$H = DR_B,\ E = R_BAR_B,\ P = R_GH^*,\ N = G^*L_H,\ S = H^*L_P.$$

Then the following statements are equivalent:
(a) System (1.8) is consistent.
(b) The equalities in (2.1), (2.4) hold, and

$$R_PR_GE = 0,\ R_GER_G = 0,\ L_HEL_H = 0.$$

(c) The equalities in (2.2) and (2.5) hold, and

$$r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* \\ B_4^* & 0 & 0 & A_3^* \\ C_3 & A_3 & 0 & 0 \\ F_1A_4^* & 0 & E_1 & 0 \end{bmatrix} = r\begin{bmatrix} B_4 & A_4 \\ 0 & E_1 \\ A_3 & 0 \end{bmatrix} + r\begin{bmatrix} B_4 \\ A_3 \end{bmatrix},$$

$$r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & 0 & \frac{1}{2}A_4F_1^* \\ B_4^* & 0 & 0 & A_3^* & 0 & 0 \\ A_4^* & 0 & 0 & 0 & A_2^* & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & -C_2 & -\frac{1}{2}A_2F_1^* \\ \frac{1}{2}F_1A_4^* & 0 & E_1 & 0 & -\frac{1}{2}F_1A_2^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A_4 & B_4 \\ A_2 & 0 \\ E_1 & 0 \\ 0 & A_3 \end{bmatrix},$$

$$r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & \frac{1}{2}A_4F_1^* \\ B_4^* & 0 & 0 & A_3^* & 0 \\ A_4^* & 0 & 0 & 0 & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 \\ \frac{1}{2}F_1A_4^* & 0 & E_1 & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} A_4 & B_4 \\ E_1 & 0 \\ 0 & A_3 \end{bmatrix}.$$

In this case, the general solution to system (1.8) can be expressed as

$$X = E_1^{\dagger}F_1 + F_1^{*}(E_1^{\dagger})^{*} - E_1^{\dagger}E_1F_1^{*}(E_1^{\dagger})^{*} + L_{E_1}M^{\dagger}Q(M^{\dagger})^{*}L_{E_1} + L_{E_1}L_MVL_{E_1} + (L_{E_1}L_MVL_{E_1})^{*},$$

$$Y = A_3^{\dagger}C_3 + L_{A_3}U,$$

where

$$V = \frac{1}{2}[G^{\dagger}EH^{\dagger} - G^{\dagger}H^{*}P^{\dagger}EH^{\dagger} - G^{\dagger}S(H^{\dagger})^{*}EN^{\dagger}G^{*}H^{\dagger} + G^{\dagger}E(P^{\dagger})^{*} + (N^{\dagger})^{*}EH^{\dagger}S^{\dagger}S] + L_GV_1 + V_2R_H + U_1L_SL_P + R_NU_2^{*}L_P - G^{\dagger}SU_2R_NG^{*}H^{\dagger},$$

$$U = B^{\dagger}[A - CYD - (CYD)^{*}] - \frac{1}{2}B^{\dagger}[A - CVD - (CVD)^{*}]BB^{\dagger} - B^{\dagger}W_1B^{*} + W_1^{*}BB^{\dagger} + L_BW_2,$$

where $U_1, U_2, V_1, V_2, W_1, W_2$ *are arbitrary matrices over* $\mathbb{C}$ *with appropriate sizes.*

Proof. $(b) \Longleftrightarrow (c)$: Clearly, $(2.1) \Longleftrightarrow (2.2), (2.4) \Longleftrightarrow (2.5)$. By Lemma 2.3 and the block Gaussion elimination, we obtain

$$\begin{aligned}
&R_PR_GE = 0 \Longleftrightarrow r(R_PR_GE) = 0\\
&\Longleftrightarrow r\begin{bmatrix} A & C & D^{*} & B \\ B^{*} & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C & D^{*} & B \end{bmatrix} + r(B)\\
&\Longleftrightarrow r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^{*} \\ B_4^{*} & 0 & 0 & A_3^{*} \\ C_3 & A_3 & 0 & 0 \\ F_1A_4^{*} & 0 & E_1 & 0 \end{bmatrix} - 2r(A_3) - r(E_1)\\
&= r\begin{bmatrix} B_4 & A_4 \\ 0 & E_1 \\ A_3 & 0 \end{bmatrix} + r\begin{bmatrix} B_4 \\ A_3 \end{bmatrix} - 2r(A_3) - r(E_1)
\end{aligned}$$

$$\Longleftrightarrow r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^{*} \\ B_4^{*} & 0 & 0 & A_3^{*} \\ C_3 & A_3 & 0 & 0 \\ F_1A_4^{*} & 0 & E_1 & 0 \end{bmatrix} = r\begin{bmatrix} B_4 & A_4 \\ 0 & E_1 \\ A_3 & 0 \end{bmatrix} + r\begin{bmatrix} B_4 \\ A_3 \end{bmatrix}.$$

Similarly, we can get

$$R_GER_G=0 \Longleftrightarrow r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & 0 & \frac{1}{2}A_4F_1^* \\ B_4^* & 0 & 0 & A_3^* & 0 & 0 \\ A_4^* & 0 & 0 & 0 & A_2^* & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & -C_2 & -\frac{1}{2}A_2F_1^* \\ \frac{1}{2}F_1A_4^* & 0 & E_1 & 0 & -\frac{1}{2}F_1A_2^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A_4 & B_4 \\ A_2 & 0 \\ E_1 & 0 \\ 0 & A_3 \end{bmatrix},$$

$$L_HEL_H=0 \Longleftrightarrow r\begin{bmatrix} C_4 & B_4 & A_4 & C_3^* & \frac{1}{2}A_4F_1^* \\ B_4^* & 0 & 0 & A_3^* & 0 \\ A_4^* & 0 & 0 & 0 & E_1^* \\ C_3 & A_3 & 0 & 0 & 0 \\ \frac{1}{2}F_1A_4^* & 0 & E_1 & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} A_4 & B_4 \\ E_1 & 0 \\ 0 & A_3 \end{bmatrix}.$$

$(a) \Longleftrightarrow (b)$: We separate the equations in system (1.8) into three groups

$$A_3Y=C_3, \tag{3.1}$$

$$A_1X=C_1,\ XB_1=D_1,\ A_2XA_2^*=C_2,\ X=X^*, \tag{3.2}$$

$$B_4Y+(B_4Y)^*+A_4XA_4^*=C_4. \tag{3.3}$$

It follows from Lemma 2.1 and 2.2 that equation (3.1) and system (3.2) are consistent if and only if the equalities in (2.1) and (2.4) hold, respectively. The general solutions to equation (3.1) and system (3.2) can be expressed as (2.3) and (2.6), respectively. Substituting (2.3) and (2.6) into (3.3) gives

$$BU+(BU)^*+CVD+(CVD)^*=A. \tag{3.4}$$

Hence, the system (1.8) is consistent if and only if equations (3.1), (3.2) and (3.4) are consistent, respectively. By Lemma 3.1, we know that equation (3.4) is consistent if and only if

$$R_PR_GE=0,\ R_GER_G=0,\ L_HEL_H=0.$$

In this case, the general solution to equation is (3.4).

In Theorem 3.2, if A_3 and B_4 vanish, then we obtain the general Hermitian solution to (1.7).

Corollary 3.3. *Let $A_1, B_1, C_1, D_1, A_2, A_4, C_2 = C_2^*$, and $C_4 = C_4^*$ be given, and let E_1, F_1, M, Q, α be the same as in Theorem 3.2. Put*

$B = A_4 L_{E_1} L_M, C = L_{E_1} A_4^*,\ A = C_4 - A_4 \alpha A_4^*, P = R_B C^*,\ N = B^* L_C,$

$S = C^* L_P$. *Then the following statements are equivalent:*
(a) System (1.7) has a Hermitian solution.
(b) The equalities in (2.4) hold and

$$R_P R_B A = 0,\ R_B A R_B = 0,\ L_C A L_C = 0.$$

(c) The equalities in (2.5) hold and

$$r\begin{bmatrix} C_4 & A_4 \\ F_1 A_4^* & E_1 \end{bmatrix} = r\begin{bmatrix} A_4 \\ E_1 \end{bmatrix},$$

$$r\begin{bmatrix} C_4 & A_4 & 0 & \frac{1}{2}A_4 F_1^* \\ A_4^* & 0 & A_2^* & E_1^* \\ 0 & A_2 & -C_2 & -\frac{1}{2}A_2 F_1^* \\ \frac{1}{2}F_1 A_4^* & E_1 & -\frac{1}{2}F_1 A_2^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A_4 \\ A_2 \\ E_1 \end{bmatrix},$$

$$r\begin{bmatrix} C_4 & A_4 & \frac{1}{2}A_4 F_1^* \\ A_4^* & 0 & E_1^* \\ \frac{1}{2}F_1 A_4^* & E_1 & 0 \end{bmatrix} = 2r\begin{bmatrix} A_4 \\ E_1 \end{bmatrix}.$$

In this case, the general Hermitian solution of system (1.7) can be expressed as

$$\begin{aligned} X =& E_1^\dagger F_1 + F_1^*(E_1^\dagger)^* - E_1^\dagger E_1 F_1^*(E_1^\dagger)^* \\ &+ L_{E_1} M^\dagger Q (M^\dagger)^* L_{E_1} + L_{E_1} L_M V L_{E_1} + (L_{E_1} L_M V L_{E_1})^*, \end{aligned}$$

where

$V = \dfrac{1}{2}[B^\dagger A C^\dagger + B^\dagger A (P^\dagger)^* + (N^\dagger)^* A C^\dagger S^\dagger S - B^\dagger C^* P^\dagger A C^\dagger - B^\dagger S (C^\dagger)^*$

$A N^\dagger B^* C^\dagger] + L_B U_1 + U_2 R_C + U_3 L_S L_P + R_N U_4^* L_P - B^\dagger S U_4 R_N B^* C^\dagger,$

and U_1, U_2, U_3, U_4 are arbitrary matrices over $\mathbb{C}$ with appropriate sizes.

Remark 3.4. *Wang and Wu [16] derived the solvability conditions and the expression of the general Hermitian solution to (1.7) for adjointable operators between Hilbert $\mathcal{C}^*$-modules. We give some new necessary and* *sufficient conditions for the solvability to the system (1.7). Our expression of the Hermitian solution is also true for Hilbert $\mathcal{C}^*$-modules, which has different form compared to the expression in [16].*

4. Conclusion

In this paper, we have derived the extremal ranks and inertias of the matrix function (1.2) subject to (1.6) and (1.9). Using the results in (1.2), we have characterized the relations between the Hermitian part of the solution to the matrix equation $A_3Y = C_3$ and the Hermitian solution to the system of matrix equations $A_1X = C_1, XB_1 = D_1, A_2XA_2^* = C_2$. Moreover, we have established some necessary and sufficient conditions for the existence of the general solution to (1.8). The expression of such a solution to (1.8) has also been given when its solvability conditions are satisfied. As a special case of the system (1.8), we have given some new solvability conditions for the system (1.7) and a new expression of the general solution to (1.7), which extend the main result in [16].

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