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Title:
Investigation on the Hermitian matrix expression subject to some consistent equations

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# INVESTIGATION ON THE HERMITIAN MATRIX EXPRESSION SUBJECT TO SOME CONSISTENT EQUATIONS 

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#### Abstract

In this paper, we study the extremal ranks and inertias of the Hermitian matrix expression $$
f(X, Y)=C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-A_{4} X A_{4}^{*}
$$ where $C_{4}$ is Hermitian, $*$ denotes the conjugate transpose, $X$ and $Y$ satisfy the following consistent system of matrix equations $A_{3} Y=$ $C_{3}, A_{1} X=C_{1}, X B_{1}=D_{1}, A_{2} X A_{2}^{*}=C_{2}, X=X^{*}$. As consequences, we get the necessary and sufficient conditions for the above expression $f(X, Y)$ to be (semi) positive, (semi) negative. The relations between the Hermitian part of the solution to the matrix equation $A_{3} Y=C_{3}$ and the Hermitian solution to the system of matrix equations $A_{1} X=C_{1}, X B_{1}=D_{1}, A_{2} X A_{2}^{*}=C_{2}$ are also characterized. Moreover, we give the necessary and sufficient conditions for the solvability to the following system of matrix equations $A_{3} Y=C_{3}, A_{1} X=C_{1}, X B_{1}=D_{1}, A_{2} X A_{2}^{*}=C_{2}, X=$ $X^{*}, B_{4} Y+\left(B_{4} Y\right)^{*}+A_{4} X A_{4}^{*}=C_{4}$ and provide an expression of the general solution to this system when it is solvable.


Keywords: Linear matrix equation, Moore-Penrose inverse, rank, inertia.
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## 1. Introduction

It is well known that the linear matrix expressions and their special cases-linear matrix equations are fundamental objects in matrix theory and applications. In recent 30 years, many authors addressed to the

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research on extremal ranks and inertias of matrix expressions and obtained many useful applications. In 2011, Liu and Tian [9] provided the formulas for the extremal ranks and inertias of

$$
A-B X C-(B X C)^{*} .
$$

Chu [3], Liu and Tian [10] considered the extremal ranks and inertias of

$$
\begin{equation*}
A-B X B^{*}-C Y C^{*} \tag{1.1}
\end{equation*}
$$

In 2011, Tian [15] derived the maximal and minimal ranks and inertias of

$$
\begin{equation*}
f(X, Y)=C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-A_{4} X A_{4}^{*} . \tag{1.2}
\end{equation*}
$$

Note that the rank and inertia are not only important in matrix theory but also powerful tools in investigating the solvability of some matrix equations. One purpose of this work is to get the necessary and sufficient conditions for the consistence of some new matrix equations by investigating the rank and inertia of some Hermitian matrix expressions with restrictions.

Study on the solvability conditions and the general solution to linear matrix equations is active in recent years (e.g. [4], [7], [17]- [22]). The well-known Lyapunov matrix equation

$$
\begin{equation*}
B X+(B X)^{*}=A \tag{1.3}
\end{equation*}
$$

has played an important role in contemporary mathematics, such as system theory, stability analysis, optimal control, model reduction, etc. (e.g., [12, 13]). In 1998, Braden [1] gave the general solution to (1.3). In 2007, Djordjević [5] considered the explicit solution to the equation (1.3) for linear bounded operators on Hilbert spaces. Moreover, Cao [2] investigated the general solution to

$$
\begin{equation*}
B X C+(B X C)^{*}=A . \tag{1.4}
\end{equation*}
$$

$\mathrm{Xu}[20]$ obtained an explicit expression of the solution to the operator equation (1.4). Liu [9] considered the general solutions to the matrix equation (1.4). In 2012, Wang and He [17] studied some necessary and sufficient conditions for the consistence of the matrix equation

$$
\begin{equation*}
A_{1} X+\left(A_{1} X\right)^{*}+B_{1} Y C_{1}+\left(B_{1} Y C_{1}\right)^{*}=E_{1} \tag{1.5}
\end{equation*}
$$

and presented an expression of the general solution to (1.5).

Hermitian solutions to some matrix equations were also investigated by many authors. Khatri and Mitra [8] considered the Hermitian solutions to

$$
\begin{equation*}
A_{3} Y=C_{3}, \tag{1.6}
\end{equation*}
$$

and

$$
A Y A^{*}=B
$$

respectively. In 2010, Wang and Wu [16] considered the solvability conditions and the Hermitian solution to the following system

$$
\begin{equation*}
A_{1} X=C_{1}, X B_{1}=D_{1}, A_{2} X A_{2}^{*}=C_{2}, A_{4} X A_{4}^{*}=C_{4} \tag{1.7}
\end{equation*}
$$

for adjointable operators between Hilbert $\mathcal{C}^{*}$-modules.
Note that the matrix equation (1.3) and the system of matrix equations (1.7) are special cases of

$$
\left\{\begin{array}{l}
A_{3} Y=C_{3}, A_{1} X=C_{1}, X B_{1}=D_{1},  \tag{1.8}\\
A_{2} X A_{2}^{*}=C_{2}, X=X^{*}, \\
B_{4} Y+\left(B_{4} Y\right)^{*}+A_{4} X A_{4}^{*}=C_{4} .
\end{array}\right.
$$

To our knowledge, there has been little results about this system. The purpose of this work is to give the solvability conditions and the expressions of the general solution to (1.8). In order to obtain the necessary and sufficient conditions for the consistence of the system (1.8), we give the extremal ranks and inertias of the matrix expression (1.2) subject to the matrix equations (1.6) and

$$
\begin{equation*}
A_{1} X=C_{1}, X B_{1}=D_{1}, A_{2} X A_{2}^{*}=C_{2}, X=X^{*} \tag{1.9}
\end{equation*}
$$

The paper is organized as follows. In Section 2, we consider the extremal ranks and inertias of (1.2), where $Y$ and $X$ satisfy (1.6) and (1.9), respectively. We then obtain the relations between the Hermitian part of the solution to (1.6) and the Hermitian solution to (1.9). In section 3 , we give the solvability conditions for (1.8) and the expression of the solution to (1.8) when it is solvable. As a special case of the system (1.8), we derive some new solvability conditions for the system (1.7) and a new expression of the general solution to (1.7), which extend the main result in [16].

Throughout this paper, we denote the field of complex numbers by $\mathbb{C}$, the set of all $m \times n$ matrices over $\mathbb{C}$ by $\mathbb{C}^{m \times n}$, the set of all $m \times m$ Hermitian matrices by $\mathbb{C}_{h}^{m \times m}$. The symbols $I_{n}, A^{*}$ and $\mathcal{R}(A)$ stand for the $n \times n$ identity matrix, the conjugate transpose, the column space of
a complex matrix $A$, respectively. The Moore-Penrose inverse $A^{\dagger}$ of $A$, is the unique matrix $A^{\dagger}$, such that
(i) $A A^{\dagger} A=A$, (ii) $A^{\dagger} A A^{\dagger}=A^{\dagger}$, (iii) $\left(A A^{\dagger}\right)^{*}=A A^{\dagger}$, (iv) $\left(A^{\dagger} A\right)^{*}=A^{\dagger} A$.

If a Hermitian matrix $X$ satisfies the equality (i), then it is called a Hermitian $g$ inverse of $A$ and is denoted by $A_{h}^{-}$. Moreover, $L_{A}$ and $R_{A}$ stand for the two projectors $L_{A}=I-A^{\dagger} A, R_{A}=I-A A^{\dagger}$. The eigenvalues of a Hermitian matrix $A \in \mathbb{C}^{n \times n}$ are real and the inertia of $A$ is defined to be the triplet

$$
\mathbb{I}_{n}(A)=\left\{i_{+}(A), i_{-}(A), i_{0}(A)\right\},
$$

where $i_{+}(A), i_{-}(A)$ and $i_{0}(A)$ stand for the numbers of positive, negative and zero eigenvalues of $A$, respectively. For two Hermitian matrices $A$ and $B$ of the same size, we say $A \geq B(A \leq B)$ in the Löwner partial ordering if $A-B$ is positive (negative) semidefinite. The Hermitian part of $X$ is defined as $H(X)=\left(X+X^{*}\right) / 2$.

## 2. Extremal ranks and inertias of (1.2) subject to (1.6) and (1.9) with applications

In this section, we give the extremal ranks and inertias of (1.2) subject to (1.6) and (1.9). Then we characterize the relations between the Hermitian part of the solution to (1.6) and the Hermitian solution to (1.9). We also consider the extremal ranks and inertias of Hermitian Schur complement of a given Hermitian matrix, which extend the known results in [9].

Lemma 2.1. [8] Let $A_{3}$ and $C_{3}$ be given. Then the following statements are equivalent:
(a) Equation (1.6) is consistent.

$$
\begin{equation*}
R_{A_{3}} C_{3}=0 . \tag{b}
\end{equation*}
$$

(c)

$$
r\left[\begin{array}{ll}
A_{3} & C_{3} \tag{2.2}
\end{array}\right]=r\left(A_{3}\right) .
$$

In this case, the general solution can be written as

$$
\begin{equation*}
Y=A_{3}^{\dagger} C_{3}+L_{A_{3}} U, \tag{2.3}
\end{equation*}
$$

where $U$ is arbitrary matrix over $\mathbb{C}$ with appropriate size.

Lemma 2.2. ( [14]) Let $A_{1}, B_{1}, C_{1}, D_{1}, A_{2}$, and $C_{2}=C_{2}^{*}$ be given. Set

$$
\begin{array}{r}
E_{1}=\left[\begin{array}{l}
A_{1} \\
B_{1}^{*}
\end{array}\right], \quad F_{1}=\left[\begin{array}{l}
C_{1} \\
D_{1}^{*}
\end{array}\right], M=A_{2} L_{E_{1}}, \\
Q=C_{2}-A_{2}\left[E_{1}^{\dagger} F_{1}+F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*}-E_{1}^{\dagger} E_{1} F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*}\right] A_{2}^{*} .
\end{array}
$$

Then the following statements are equivalent:
(a) System (1.9) is consistent.
(b)

$$
\begin{equation*}
E_{1} F_{1}^{*}=F_{1} E_{1}^{*}, R_{E_{1}} F_{1}=0, R_{M} Q=0 \tag{2.4}
\end{equation*}
$$

(c)

$$
E_{1} F_{1}^{*}=F_{1} E_{1}^{*}, r\left[\begin{array}{ll}
E_{1} & F_{1}
\end{array}\right]=r\left(E_{1}\right), r\left[\begin{array}{cc}
A_{2} & C_{2}  \tag{2.5}\\
E_{1} & F_{1} A_{2}^{*}
\end{array}\right]=r\left[\begin{array}{l}
A_{2} \\
E_{1}
\end{array}\right] .
$$

In this case, the general Hermitian solution to (1.9) can be expressed as

$$
\begin{align*}
X= & E_{1}^{\dagger} F_{1}+F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*}-E_{1}^{\dagger} E_{1} F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*}+L_{E_{1}} M^{\dagger} Q\left(M^{\dagger}\right)^{*} L_{E_{1}}  \tag{2.6}\\
& +L_{E_{1}} L_{M} V L_{E_{1}}+\left(L_{E_{1}} L_{M} V L_{E_{1}}\right)^{*},
\end{align*}
$$

where $V$ is an arbitrary Hermitian matrix over $\mathbb{C}$ with appropriate size.
Lemma 2.3. ([11]) Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times k}, C \in \mathbb{C}^{l \times n}, D \in \mathbb{C}^{m \times p}, E \in$ $\mathbb{C}^{q \times n}, Q \in \mathbb{C}^{m_{1} \times k}$, and $P \in \mathbb{C}^{l \times n_{1}}$ be given. Then
(a) $r(A)+r\left(R_{A} B\right)=r(B)+r\left(R_{B} A\right)=r\left[\begin{array}{ll}A & B\end{array}\right]$.
(b) $r(A)+r\left(C L_{A}\right)=r(C)+r\left(A L_{C}\right)=r\left[\begin{array}{l}A \\ C\end{array}\right]$.
(c) $r(B)+r(C)+r\left(R_{B} A L_{C}\right)=r\left[\begin{array}{ll}A & B \\ C & 0\end{array}\right]$.
(d) $r(P)+r(Q)+r\left[\begin{array}{cc}A & B L_{Q} \\ R_{P} C & 0\end{array}\right]=r\left[\begin{array}{ccc}A & B & 0 \\ C & 0 & P \\ 0 & Q & 0\end{array}\right]$.
(e) $r\left[\begin{array}{cc}R_{B} A L_{C} & R_{B} D \\ E L_{C} & 0\end{array}\right]+r(B)+r(C)=r\left[\begin{array}{ccc}A & D & B \\ E & 0 & 0 \\ C & 0 & 0\end{array}\right]$.

Lemma 2.4. ([15]) Let $A \in \mathbb{C}_{h}^{m \times m}, B \in \mathbb{C}^{m \times n}, C \in \mathbb{C}_{h}^{n \times n}, Q \in \mathbb{C}^{m \times n}$, $P \in \mathbb{C}^{p \times n}$ be given, and $T \in \mathbb{C}^{m \times m}$ be nonsingular. Then
(a) $i_{ \pm}\left(T A T^{*}\right)=i_{ \pm}(A)$,
(b) $i_{ \pm}\left[\begin{array}{cc}A & 0 \\ 0 & C\end{array}\right]=i_{ \pm}(A)+i_{ \pm}(C)$,
(c) $i_{ \pm}\left[\begin{array}{cc}0 & Q \\ Q^{*} & 0\end{array}\right]=r(Q)$,
(d) $i_{ \pm}\left[\begin{array}{cc}A & B L_{P} \\ L_{P} B^{*} & 0\end{array}\right]+r(P)=i_{ \pm}\left[\begin{array}{ccc}A & B & 0 \\ B^{*} & 0 & P^{*} \\ 0 & P & 0\end{array}\right]$.

Lemma 2.5. ([23]) Let $p(X, Y)=A-B X-(B X)^{*}-C Y D-(C Y D)^{*}$, where $A, B, C$, and $D$ are given with appropriate sizes, and let

$$
\begin{gathered}
M_{1}=\left[\begin{array}{ccc}
A & B & C \\
B^{*} & 0 & 0 \\
C^{*} & 0 & 0
\end{array}\right], M_{2}=\left[\begin{array}{ccc}
A & B & D^{*} \\
B^{*} & 0 & 0 \\
D & 0 & 0
\end{array}\right], M_{3}=\left[\begin{array}{cccc}
A & B & C & D^{*} \\
B^{*} & 0 & 0 & 0
\end{array}\right], \\
M_{4}=\left[\begin{array}{cccc}
A & B & C & D^{*} \\
B^{*} & 0 & 0 & 0 \\
C^{*} & 0 & 0 & 0
\end{array}\right], M_{5}=\left[\begin{array}{cccc}
A & B & C & D^{*} \\
B^{*} & 0 & 0 & 0 \\
D & 0 & 0 & 0
\end{array}\right] .
\end{gathered}
$$

Then we have the identities:

$$
\begin{gathered}
\max _{X, Y} r[p(X, Y)]=\min \left\{m, r\left(M_{1}\right), r\left(M_{2}\right), r\left(M_{3}\right)\right\}, \\
\min _{X, Y} r[p(X, Y)]= \\
2 r\left(M_{3}\right)-2 r(B)+\max \left\{u_{+}+u_{-}, v_{+}+v_{-}, u_{+}+v_{-}, u_{-}+v_{+}\right\}, \\
\max _{X, Y} i_{ \pm}[p(X, Y)]=\min \left\{i_{ \pm}\left(M_{1}\right), i_{ \pm}\left(M_{2}\right)\right\}, \\
\min _{X, Y} i_{ \pm}[p(X, Y)]= \\
r\left(M_{3}\right)-r(B)+\max \left\{i_{ \pm}\left(M_{1}\right)-r\left(M_{4}\right), i_{ \pm}\left(M_{2}\right)-r\left(M_{5}\right)\right\},
\end{gathered}
$$

where

$$
u_{ \pm}=i_{ \pm}\left(M_{1}\right)-r\left(M_{4}\right), \quad v_{ \pm}=i_{ \pm}\left(M_{2}\right)-r\left(M_{5}\right) .
$$

For convenience of representation, we adopt the following notations:

$$
\begin{gathered}
S_{1}=\left\{Y \mid A_{3} Y=C_{3}\right\} \\
S_{2}=\left\{X=X^{*} \mid A_{1} X=C_{1}, X B_{1}=D_{1}, A_{2} X A_{2}^{*}=C_{2}\right\} .
\end{gathered}
$$

Now we give the main theorem of this paper.
Theorem 2.6. Let $A_{1}, C_{1}, B_{1}, D_{1}, A_{2}, C_{2}=C_{2}^{*}, A_{3}, C_{3}, A_{4}, B_{4}$, and $C_{4}=$ $C_{4}^{*} \in \mathbb{C}^{m \times m}$ be given, and let $E_{1}, F_{1}, M, Q$ be as in Lemma 2.2. Assume that (1.6) and (1.9) are consistent, respectively. Then,
(a) The maximal rank of (1.2) is

$$
\begin{equation*}
\max _{Y \in S_{1}, X \in S_{2}} r[f(X, Y)]=\min \left\{m, k_{1}, k_{2}, k_{3}\right\} . \tag{2.7}
\end{equation*}
$$

(b) The minimal rank of (1.2) is

$$
\begin{align*}
& \min _{Y \in S_{1}, X \in S_{2}} r[f(X, Y)]=  \tag{2.8}\\
& 2 k_{3}-2 k_{5}+\max \left\{s_{+}+s_{-}, t_{+}+t_{-}, s_{+}+t_{-}, s_{-}+t_{+}\right\}
\end{align*}
$$

(c) The maximal inertia of (1.2) is

$$
\begin{equation*}
\max _{Y \in S_{1}, X \in S_{2}} i_{ \pm}[f(X, Y)]=\min \left\{i_{ \pm}\left(N_{1}\right), i_{ \pm}\left(N_{2}\right)\right\} . \tag{2.9}
\end{equation*}
$$

(d) The minimal inertia of (1.2) is

$$
\begin{align*}
& \min _{Y \in S_{1}, X \in S_{2}} i_{ \pm}[f(X, Y)]=  \tag{2.10}\\
& k_{3}-k_{5}+\max \left\{i_{ \pm}\left(N_{1}\right)-k_{4},-i_{\mp}\left(N_{2}\right)\right\},
\end{align*}
$$

where

$$
\begin{aligned}
& k_{1}=r {\left[\begin{array}{cccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & 0 & \frac{1}{2} A_{4} F_{1}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 & 0 \\
A_{4}^{*} & 0 & 0 & 0 & A_{2}^{*} & E_{1}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{2} & 0 & -C_{2} & -\frac{1}{2} A_{2} F_{1}^{*} \\
\frac{1}{2} F_{1} A_{4}^{*} & 0 & E_{1} & 0 & -\frac{1}{2} F_{1} A_{2}^{*} & 0
\end{array}\right]-2 r\left[\begin{array}{l}
A_{2} \\
E_{1}
\end{array}\right]-2 r\left(A_{3}\right), } \\
& k_{2}=r\left[\begin{array}{ccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & \frac{1}{2} A_{4} F_{1}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 \\
A_{4}^{*} & 0 & 0 & 0 & E_{1}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 \\
\frac{1}{2} F_{1} A_{4}^{*} & 0 & E_{1} & 0 & 0
\end{array}\right]-2 r\left(E_{1}\right)-2 r\left(A_{3}\right),
\end{aligned}
$$

$$
\begin{aligned}
& k_{3}=r\left[\begin{array}{cccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} \\
C_{3} & A_{3} & 0 & 0 \\
F_{1} A_{4}^{*} & 0 & E_{1} & 0
\end{array}\right]-r\left(E_{1}\right)-2 r\left(A_{3}\right), \\
& k_{4}=r\left[\begin{array}{cccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & 0 & 0 \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 & 0 \\
A_{4}^{*} & 0 & 0 & 0 & A_{2}^{*} & E_{1}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 & 0 \\
F_{1} A_{4}^{*} & 0 & E_{1} & 0 & 0 & 0
\end{array}\right]-r\left[\begin{array}{l}
A_{2} \\
E_{1}
\end{array}\right]-2 r\left(A_{3}\right)-r\left(E_{1}\right), \\
& k_{5}=r\left[\begin{array}{l}
A_{3} \\
B_{3}
\end{array}\right]-r\left(A_{3}\right), \\
& i_{ \pm}\left(N_{1}\right)=i_{ \pm}\left[\begin{array}{cccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & 0 & \frac{1}{2} A_{4} F_{1}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 & 0 \\
A_{4}^{*} & 0 & 0 & 0 & A_{2}^{*} & E_{1}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{2} & 0 & -C_{2} & -\frac{1}{2} A_{2} F_{1}^{*} \\
\frac{1}{2} F_{1} A_{4}^{*} & 0 & E_{1} & 0 & -\frac{1}{2} F_{1} A_{2}^{*} & 0
\end{array}\right] \\
& -r\left[\begin{array}{l}
A_{2} \\
E_{1}
\end{array}\right]-r\left(A_{3}\right), \\
& i_{ \pm}\left(N_{2}\right)=i_{ \pm}\left[\begin{array}{ccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & \frac{1}{2} A_{4} F_{1}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 \\
A_{4}^{*} & 0 & 0 & 0 & E_{1}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 \\
\frac{1}{2} F_{1} A_{4}^{*} & 0 & E_{1} & 0 & 0
\end{array}\right]-r\left(E_{1}\right)-r\left(A_{3}\right), \\
& s_{ \pm}=i_{ \pm}\left(N_{1}\right)-k_{4}, \quad t_{ \pm}=-i_{\mp}\left(N_{2}\right) .
\end{aligned}
$$

Proof. By Lemma 2.1 and Lemma 2.2, the general solutions to (1.6) and (1.9) can be expressed as

$$
\begin{gather*}
Y=Y_{0}+L_{A_{3}} U  \tag{2.11}\\
X=X_{0}+L_{E_{1}} L_{M} V L_{E_{1}}+\left(L_{E_{1}} L_{M} V L_{E_{1}}\right)^{*} \tag{2.12}
\end{gather*}
$$

where $Y_{0}$ and $X_{0}$ are special solutions to (1.6) and (1.9), respectively, $U$ and $V$ are arbitrary matrices over $\mathbb{C}$ with appropriate sizes. Let
$A=C_{4}-A_{4} X_{0} A_{4}^{*}-B_{4} Y_{0}-\left(B_{4} Y_{0}\right)^{*}, B=B_{4} L_{A_{3}}, C=A_{4} L_{E_{1}} L_{M}, D=$ $L_{E_{1}} A_{4}^{*}$. Substituting (2.11) and (2.12) into (1.2) gives

$$
\begin{equation*}
\underset{Y \in S_{1}, X \in S_{2}}{f(X, Y)}=\underset{U, V}{h(U, V)}=A-B U-(B U)^{*}-C V D-(C V D)^{*} . \tag{2.13}
\end{equation*}
$$

Applying Lemma 2.5 to (2.13) yields

$$
\begin{align*}
& \min _{U, V} r[h(U, V)]=  \tag{2.15}\\
& 2 r\left(N_{3}\right)-2 r(B)+\max \left\{s_{+}+s_{-}, t_{+}+t_{-}, s_{+}+t_{-}, s_{-}+t_{+}\right\},
\end{align*}
$$

$$
\begin{equation*}
\max _{U, V} r[h(U, V)]=\min \left\{m, r\left(N_{1}\right), r\left(N_{2}\right), r\left(N_{3}\right)\right\}, \tag{2.14}
\end{equation*}
$$

$$
\begin{equation*}
\min _{U, V} i_{ \pm}[h(U, V)]= \tag{2.17}
\end{equation*}
$$

$$
r\left(N_{3}\right)-r(B)+\max \left\{i_{ \pm}\left(N_{1}\right)-r\left(N_{4}\right), i_{ \pm}\left(N_{2}\right)-r\left(N_{5}\right)\right\},
$$

where

$$
\begin{gathered}
N_{1}=\left[\begin{array}{ccc}
A & B & C \\
B^{*} & 0 & 0 \\
C^{*} & 0 & 0
\end{array}\right], N_{2}=\left[\begin{array}{ccc}
A & B & D^{*} \\
B^{*} & 0 & 0 \\
D & 0 & 0
\end{array}\right], N_{3}=\left[\begin{array}{cccc}
A & B & C & D^{*} \\
B^{*} & 0 & 0 & 0
\end{array}\right], \\
N_{4}=\left[\begin{array}{cccc}
A & B & C & D^{*} \\
B^{*} & 0 & 0 & 0 \\
C^{*} & 0 & 0 & 0
\end{array}\right], N_{5}=\left[\begin{array}{cccc}
A & B & C & D^{*} \\
B^{*} & 0 & 0 & 0 \\
D & 0 & 0 & 0
\end{array}\right], \\
s_{ \pm}=i_{ \pm}\left(N_{1}\right)-r\left(N_{4}\right), \quad t_{ \pm}=i_{ \pm}\left(N_{2}\right)-r\left(N_{5}\right) .
\end{gathered}
$$

Note that $r\left(N_{2}\right)=r\left(N_{5}\right)$. Applying Lemma 2.3, we have

$$
\begin{aligned}
& k_{1}:=r\left(N_{1}\right)=r\left[\begin{array}{ccc}
A & B & C \\
B^{*} & 0 & 0 \\
C^{*} & 0 & 0
\end{array}\right]=r\left[\begin{array}{ccc}
A & B_{4} L_{A_{3}} & A_{4} L_{E_{1}} L_{M} \\
R_{A_{3}^{*}} B_{4}^{*} & 0 & 0 \\
R_{M^{*}} R_{E^{*}} A_{4}^{*} & 0 & 0
\end{array}\right] \\
& =r\left[\begin{array}{ccccc}
A & B_{4} & A_{4} L_{E_{1}} & 0 & 0 \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 \\
R_{E^{*}} A_{4}^{*} & 0 & 0 & 0 & M^{*} \\
0 & A_{3} & 0 & 0 & 0 \\
0 & 0 & M & 0 & 0
\end{array}\right]-2 r\left(A_{3}\right)-2 r(M) \\
& =r\left[\begin{array}{cccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & 0 & \frac{1}{2} A_{4} F_{1}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 & 0 \\
A_{4}^{*} & 0 & 0 & 0 & A_{2}^{*} & E_{1}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{2} & 0 & -C_{2} & -\frac{1}{2} A_{2} F_{1}^{*} \\
\frac{1}{2} F_{1} A_{4}^{*} & 0 & E_{1} & 0 & -\frac{1}{2} F_{1} A_{2}^{*} & 0
\end{array}\right]-2 r\left[\begin{array}{l}
A_{2} \\
E_{1}
\end{array}\right]-2 r\left(A_{3}\right) .
\end{aligned}
$$

Similarly, we can get the results of the ranks of $N_{2}-N_{4}, B$ and inertias of $N_{1}$ and $N_{2}$ by using Lemma 2.3 and Lemma 2.4.

Corollary 2.7. Let $A_{1}, C_{1}, B_{1}, D_{1}, A_{2}, C_{2}, A_{3}, C_{3}, A_{4}, B_{4}, C_{4}, E_{1}, F_{1}, k_{1}-$ $k_{6}, s, t, N_{1}$, and $N_{2}$ be the same as in Theorem 2.6. Assume that (1.6) and (1.9) are consistent, respectively. Then,
(a) There exist $Y \in S_{1}$ and $X \in S_{2}$ such that $C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-$ $A_{4} X A_{4}^{*}>0$ if and only if

$$
i_{+}\left(N_{1}\right) \geq m, i_{+}\left(N_{2}\right) \geq m
$$

(b) There exist $Y \in S_{1}$ and $X \in S_{2}$ such that $C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-$ $A_{4} X A_{4}^{*}<0$ if and only if

$$
i_{-}\left(N_{1}\right) \geq m, i_{-}\left(N_{2}\right) \geq m
$$

(c) There exist $Y \in S_{1}$ and $X \in S_{2}$ such that $C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-$ $A_{4} X A_{4}^{*} \geq 0$ if and only if

$$
k_{3}-k_{5}+i_{-}\left(N_{1}\right)-k_{4} \leq 0, k_{3}-k_{5}-i_{+}\left(N_{2}\right) \leq 0
$$

(d) There exist $Y \in S_{1}$ and $X \in S_{2}$ such that $C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-$ $A_{4} X A_{4}^{*} \leq 0$ if and only if

$$
k_{3}-k_{5}+i_{+}\left(N_{1}\right)-k_{4} \geq 0, k_{3}-k_{5}-i_{-}\left(N_{2}\right) \geq 0
$$

(e) $C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-A_{4} X A_{4}^{*}>0$ for all $Y \in S_{1}$ and $X \in S_{2}$ if and only if

$$
k_{3}-k_{5}+i_{+}\left(N_{1}\right)-k_{4}=m, \text { or } k_{3}-k_{5}-i_{-}\left(N_{2}\right)=m .
$$

(f) $C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-A_{4} X A_{4}^{*}<0$ for all $Y \in S_{1}$ and $X \in S_{2}$ if and only if

$$
k_{3}-k_{5}+i_{-}\left(N_{1}\right)-k_{4}=m, \text { or } k_{3}-k_{5}-i_{+}\left(N_{2}\right)=m .
$$

(g) $C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-A_{4} X A_{4}^{*} \geq 0$ for all $Y \in S_{1}$ and $X \in S_{2}$ if and only if

$$
i_{-}\left(N_{1}\right)=0 \quad \text { or } \quad i_{-}\left(N_{2}\right)=0 .
$$

(h) $C_{4}-B_{4} Y-\left(B_{4} Y\right)^{*}-A_{4} X A_{4}^{*} \leq 0$ for all $Y \in S_{1}$ and $X \in S_{2}$ if and only if

$$
i_{+}\left(N_{1}\right)=0 \quad \text { or } \quad i_{+}\left(N_{2}\right)=0 .
$$

Next, we reveal the relations between the Hermitian part of the solution to (1.6) and the Hermitian solution to (1.9). Let $A_{4}=I_{m}$, $B_{4}=-I_{m} / 2, C_{4}=0$, put

$$
\begin{gathered}
u_{1}=2 m-2 r\left[\begin{array}{l}
A_{2} \\
E_{1}
\end{array}\right]-2 r\left(A_{3}\right)+ \\
r\left[\begin{array}{cccc}
0 & 2 A_{3}^{*} & A_{2}^{*} & E_{1}^{*}+ \\
2 A_{3} & 2 A_{3} C_{3}^{*}+2 C_{3} A_{3}^{*} & 0 & A_{3}^{*} F_{1}^{*} \\
A_{2} & 0 & -C_{2} & -\frac{1}{2} A_{2} F_{1}^{*} \\
E_{1} & F_{1}^{*} A_{3}^{*} & -\frac{1}{2} F_{1} A_{2}^{*} & 0
\end{array}\right], \\
u_{2}=2 m+r\left[\begin{array}{ccc}
0 & 2 A_{3}^{*} & E_{1}^{*} \\
2 A_{3} & 2 A_{3} C_{3}^{*}+2 C_{3} A_{3}^{*} & A_{3} F_{1}^{*} \\
0 & F_{1} A_{3}^{*} & 0
\end{array}\right]-2 r\left(E_{1}\right)-2 r\left(A_{3}\right), \\
u_{3}=2 m+r\left[\begin{array}{cc}
2 A_{3} & 2 A_{3} C_{3}^{*} \\
E_{1} & 0
\end{array}\right]-r\left(E_{1}\right)-2 r\left(A_{3}\right), \\
u_{4}=2 m-r\left[\begin{array}{l}
A_{2} \\
E_{1}
\end{array}\right]-2 r\left(A_{3}\right)-r\left(E_{1}\right)+ \\
r\left[\begin{array}{cc}
0 & 2 A_{3}^{*} \\
2 A_{3} & 2 A_{3} C_{3}^{*}+2 C_{3} A_{3}^{*} \\
2 F_{1} A_{3}^{*} & 0 \\
E_{1}^{*} & 0 \\
2
\end{array}\right], \\
k_{5}=r\left[\begin{array}{l}
A_{3} \\
B_{3}
\end{array}\right]-r\left(A_{3}\right),
\end{gathered}
$$

$$
\begin{gathered}
i_{ \pm}\left(M_{1}\right)=m-r\left[\begin{array}{l}
A_{2} \\
E_{1}
\end{array}\right]-r\left(A_{3}\right)+ \\
i_{ \pm}\left[\begin{array}{cccc}
0 & 2 A_{3}^{*} & A_{2}^{*} & E_{1}^{*} \\
2 A_{3} & 2 A_{3} C_{3}^{*}+2 C_{3} A_{3}^{*} & 0 & A_{3}^{*} F_{1}^{*} \\
A_{2} & 0 & -C_{2} & -\frac{1}{2} A_{2} F_{1}^{*} \\
E_{1} & F_{1}^{*} A_{3}^{*} & -\frac{1}{2} F_{1} A_{2}^{*} & 0
\end{array}\right], \\
i_{ \pm}\left(M_{2}\right)=m+i_{ \pm}\left[\begin{array}{ccc}
0 & 2 A_{3}^{*} & E_{1}^{*} \\
2 A_{3} & 2 A_{3} C_{3}^{*}+2 C_{3} A_{3}^{*} & A_{3} F_{1}^{*} \\
0 & F_{1} A_{3}^{*} & 0
\end{array}\right]-r\left(E_{1}\right)-r\left(A_{3}\right), \\
p_{ \pm}=i_{ \pm}\left(M_{1}\right)-k_{4}, \quad q_{ \pm}=-i_{\mp}\left(M_{2}\right) .
\end{gathered}
$$

Then we get the next corollary.
Corollary 2.8. Assume that (1.6) and (1.9) are consistent, respectively. Then,
(a) The maximal rank of $\frac{1}{2}\left(Y+Y^{*}\right)-X$ is

$$
\max _{Y \in S_{1}, X \in S_{2}} r[f(X, Y)]=\min \left\{m, u_{1}, u_{2}, u_{3}\right\} .
$$

(b) The minimal rank of $\frac{1}{2}\left(Y+Y^{*}\right)-X$ is

$$
\begin{array}{r}
\min _{Y \in S_{1}, X \in S_{2}} r[f(X, Y)]=2 u_{3}-2 k_{5}+ \\
\max \left\{p_{+}+p_{-}, q_{+}+q_{-}, p_{+}+q_{-}, p_{-}+q_{+}\right\} .
\end{array}
$$

(c) The maximal inertia of $\frac{1}{2}\left(Y+Y^{*}\right)-X$ is

$$
\max _{Y \in S_{1}, X \in S_{2}} i_{ \pm}[f(X, Y)]=\min \left\{i_{ \pm}\left(M_{1}\right), i_{ \pm}\left(M_{2}\right)\right\}
$$

(d) The minimal inertia of $\frac{1}{2}\left(Y+Y^{*}\right)-X$ is

$$
\min _{Y \in S_{1}, X \in S_{2}} i_{ \pm}[f(X, Y)]=u_{3}-k_{5}+\max \left\{i_{ \pm}\left(M_{1}\right)-u_{4},-i_{\mp}\left(M_{2}\right)\right\}
$$

Therefore:
(e) There exist $Y \in S_{1}$ and $X \in S_{2}$ such that $\frac{1}{2}\left(Y+Y^{*}\right)>X$ if and only if

$$
i_{+}\left(M_{1}\right) \geq m, i_{+}\left(M_{2}\right) \geq m
$$

(f) There exist $Y \in S_{1}$ and $X \in S_{2}$ such that $\frac{1}{2}\left(Y+Y^{*}\right)<X$ if and only if

$$
i_{-}\left(M_{1}\right) \geq m, i_{-}\left(M_{2}\right) \geq m .
$$

(g) There exist $Y \in S_{1}$ and $X \in S_{2}$ such that $\frac{1}{2}\left(Y+Y^{*}\right) \geq X$ if and only if

$$
u_{3}-k_{5}+i_{-}\left(M_{1}\right)-u_{4} \leq 0, u_{3}-k_{5}-i_{+}\left(M_{2}\right) \leq 0
$$

(h) There exist $Y \in S_{1}$ and $X \in S_{2}$ such that $\frac{1}{2}\left(Y+Y^{*}\right) \leq X$ if and only if

$$
u_{3}-k_{5}+i_{+}\left(M_{1}\right)-u_{4} \geq 0, u_{3}-k_{5}-i_{-}\left(M_{2}\right) \geq 0
$$

(i) $\frac{1}{2}\left(Y+Y^{*}\right)>X$ for all $Y \in S_{1}$ and $X \in S_{2}$ if and only if

$$
u_{3}-k_{5}+i_{+}\left(M_{1}\right)-u_{4}=m, \text { or } u_{3}-k_{5}-i_{-}\left(M_{2}\right)=m
$$

(j) $\frac{1}{2}\left(Y+Y^{*}\right)<X$ for all $Y \in S_{1}$ and $X \in S_{2}$ if and only if

$$
u_{3}-k_{5}+i_{-}\left(M_{1}\right)-u_{4}=m, \text { or } u_{3}-k_{5}-i_{+}\left(M_{2}\right)=m
$$

(k) $\frac{1}{2}\left(Y+Y^{*}\right) \geq X$ for all $Y \in S_{1}$ and $X \in S_{2}$ if and only if

$$
i_{-}\left(M_{1}\right)=0 \quad \text { or } \quad i_{-}\left(M_{2}\right)=0
$$

(l) $\frac{1}{2}\left(Y+Y^{*}\right) \leq X$ for all $Y \in S_{1}$ and $X \in S_{2}$ if and only if

$$
i_{+}\left(M_{1}\right)=0 \quad \text { or } \quad i_{+}\left(M_{2}\right)=0
$$

(m)There exist $Y \in S_{1}$ and $X \in S_{2}$ such that $\frac{1}{2}\left(Y+Y^{*}\right)=X$ if and only if

$$
\begin{gathered}
2 u_{3}+p_{+}+p_{-} \leq 2 k_{5}, \text { or } 2 u_{3}+q_{+}+q_{-} \leq 2 k_{5} \\
\text { or } \quad 2 u_{3}+p_{+}+q_{-} \leq 2 k_{5}, \text { or } \quad 2 u_{3}+p_{-}+q_{+} \leq 2 k_{5}
\end{gathered}
$$

For a $2 \times 2$ block Hermitian matrix $M=\left[\begin{array}{cc}A & B \\ B^{*} & D\end{array}\right]$, the $S_{A}=D-$ $B A_{h}^{-} B$ is in fact a linear matrix function $D-B^{*} X B$ subject to the Hermitian solution of the matrix equation $A X A=A$. In Theorem 2.6, let $A_{1}, C_{1}, B_{1}, D_{1}, A_{3}, C_{3}$ vanish, and let $C_{4}=D \in \mathbb{C}_{h}^{m \times m}, A_{2}=C_{2}=$ $A=A^{*}, A_{4}=B$. Then we have the following result.

Corollary 2.9. Let $A \in \mathbb{C}_{h}^{m \times m}, B \in \mathbb{C}^{m \times n}$ and $D \in \mathbb{C}_{h}^{n \times n}$ be given. Let

$$
N=\left[\begin{array}{ccc}
A & 0 & B \\
0 & B^{*} & D
\end{array}\right]
$$

Then the maximal and minimal values of the rank, positive and negative signatures of the Schur complement $D-B^{*} A_{h}^{-} B$ are given by

$$
\begin{aligned}
& \max r\left(D-B^{*} A_{h}^{-} B\right)=\min \left\{r\left[B^{*}, D\right], r(M)-r(A)\right\} \\
& \max i_{ \pm}\left(D-B^{*} A_{h}^{-} B\right)=i_{ \pm}(M)-i_{ \pm}(A) \\
& \min r\left(D-B^{*} A_{h}^{-} B\right)=r(A)+2 r\left[B^{*}, D\right]+r(M)-2 r(N) \\
& \min i_{ \pm}\left(D-B^{*} A_{h}^{-} B\right)=i_{ \pm}(A)+r\left[B^{*}, D\right]+i_{ \pm}(M)-r(N)
\end{aligned}
$$

Remark 2.10. It is the main result of Theorem 5.3 in [9].

## 3. The solvability conditions and the expression of the general solution to (1.8)

Now we consider the solvability conditions and the explicit expression of the general solution to (1.8). We begin with a lemma which plays an important role in the development in this section.

Lemma 3.1. ([17]) Let $A_{1} \in \mathbb{C}^{m \times n_{1}}, B_{1} \in \mathbb{C}^{m \times n_{2}}, C_{1} \in \mathbb{C}^{q \times m}$, and $E_{1} \in \mathbb{C}_{h}^{m \times m}$ be given. Let $A=R_{A_{1}} B_{1}, B=C_{1} R_{A_{1}}, E=R_{A_{1}} E_{1} R_{A_{1}}, M=$ $R_{A} B^{*}, N=A^{*} L_{B}, S=B^{*} L_{M}$. Then the following statements are equivalent:
(a) Equation (1.5) is consistent.
(b)

$$
R_{M} R_{A} E=0, \quad R_{A} E R_{A}=0, \quad L_{B} E L_{B}=0
$$

(c)

$$
\begin{aligned}
& r\left[\begin{array}{cccc}
E_{1} & B_{1} & C_{1}^{*} & A_{1} \\
A_{1}^{*} & 0 & 0 & 0
\end{array}\right]=r\left[\begin{array}{lll}
B_{1} & C_{1}^{*} & A_{1}
\end{array}\right]+r\left(A_{1}\right) \\
& r\left[\begin{array}{ccc}
E_{1} & B_{1} & A_{1} \\
A_{1}^{*} & 0 & 0 \\
B_{1}^{*} & 0 & 0
\end{array}\right]=2 r\left[\begin{array}{ll}
B_{1} & A_{1}
\end{array}\right] \\
& r\left[\begin{array}{ccc}
E_{1} & C_{1}^{*} & A_{1} \\
A_{1}^{*} & 0 & 0 \\
C_{1} & 0 & 0
\end{array}\right]=2 r\left[\begin{array}{ll}
C_{1}^{*} & A_{1}
\end{array}\right]
\end{aligned}
$$

In this case, the general solution of equation (1.5) can be expressed as

$$
\begin{aligned}
Y= & \frac{1}{2}\left[A^{\dagger} E B^{\dagger}-A^{\dagger} B^{*} M^{\dagger} E B^{\dagger}-A^{\dagger} S\left(B^{\dagger}\right)^{*} E N^{\dagger} A^{*} B^{\dagger}+A^{\dagger} E\left(M^{\dagger}\right)^{*}\right. \\
& \left.+\left(N^{\dagger}\right)^{*} E B^{\dagger} S^{\dagger} S\right]+L_{A} V_{1}+V_{2} R_{B}+U_{1} L_{S} L_{M}+R_{N} U_{2}^{*} L_{M} \\
& -A^{\dagger} S U_{2} R_{N} A^{*} B^{\dagger}
\end{aligned}
$$

$$
\begin{aligned}
X= & A_{1}^{\dagger}\left[E_{1}-B_{1} Y C_{1}-\left(B_{1} Y C_{1}\right)^{*}\right]-\frac{1}{2} A_{1}^{\dagger}\left[E_{1}-B_{1} Y C_{1}-\left(B_{1} Y C_{1}\right)^{*}\right] A_{1} A_{1}^{\dagger} \\
& -A_{1}^{\dagger} W_{1} A_{1}^{*}+W_{1}^{*} A_{1} A_{1}^{\dagger}+L_{A_{1}} W_{2},
\end{aligned}
$$

where $U_{1}, U_{2}, V_{1}, V_{2}, W_{1}, W_{2}$ are arbitrary matrices over $\mathbb{C}$ with appropriate sizes..

Now we present the fundamental theorem of this paper.
Theorem 3.2. Let $A_{1}, B_{1}, C_{1}, D_{1}, A_{2}, C_{2}, A_{3}, C_{3}, A_{4}, B_{4}, C_{4}, E_{1}, F_{1}, M$ and $Q$ be the same as in Theorem 2.6. Set

$$
\begin{gathered}
\alpha=E_{1}^{\dagger} F_{1}+F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*}-E_{1}^{\dagger} E_{1} F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*}+L_{E_{1}} M^{\dagger} Q\left(M^{\dagger}\right)^{*} L_{E_{1}}, \\
B=B_{4} L_{A_{3}}, C=A_{4} L_{E_{1}} L_{M}, D=L_{E_{1}} A_{4}^{*}, \\
A=C_{4}-A_{4} \alpha A_{4}^{*}-B_{4} A_{3}^{\dagger} C_{3}-\left(B_{4} A_{3}^{\dagger} C_{3}\right)^{*}, G=R_{B} C, \\
H=D R_{B}, E=R_{B} A R_{B}, P=R_{G} H^{*}, N=G^{*} L_{H}, S=H^{*} L_{P} .
\end{gathered}
$$

Then the following statements are equivalent:
(a) System (1.8) is consistent.
(b) The equalities in (2.1), (2.4) hold, and

$$
R_{P} R_{G} E=0, R_{G} E R_{G}=0, L_{H} E L_{H}=0
$$

(c) The equalities in (2.2) and (2.5) hold, and

$$
\begin{gathered}
r\left[\begin{array}{cccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} \\
C_{3} & A_{3} & 0 & 0 \\
F_{1} A_{4}^{*} & 0 & E_{1} & 0
\end{array}\right]=r\left[\begin{array}{cc}
B_{4} & A_{4} \\
0 & E_{1} \\
A_{3} & 0
\end{array}\right]+r\left[\begin{array}{l}
B_{4} \\
A_{3}
\end{array}\right], \\
r\left[\begin{array}{ccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & 0 \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 \\
A_{4}^{*} & 0 & 0 & 0 & A_{2}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 \\
0 & 0 & A_{2} & 0 & -C_{4} F_{1}^{*} \\
\frac{E_{1}^{*}}{2} \\
\frac{1}{2} F_{1} A_{4}^{*} & 0 & E_{1} & 0 & -\frac{1}{2} F_{1} A_{2}^{*} \\
-\frac{1}{2} A_{2} F_{1}^{*} \\
0
\end{array}\right]=2 r\left[\begin{array}{cc}
A_{4} & B_{4} \\
A_{2} & 0 \\
E_{1} & 0 \\
0 & A_{3}
\end{array}\right], \\
r\left[\begin{array}{ccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & \frac{1}{2} A_{4} F_{1}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 \\
A_{4}^{*} & 0 & 0 & 0 & E_{1}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 \\
\frac{1}{2} F_{1} A_{4}^{*} & 0 & E_{1} & 0 & 0
\end{array}\right]=2 r\left[\begin{array}{cc}
A_{4} & B_{4} \\
E_{1} & 0 \\
0 & A_{3}
\end{array}\right] .
\end{gathered}
$$

In this case, the general solution to system (1.8) can be expressed as

$$
\begin{gathered}
X=E_{1}^{\dagger} F_{1}+F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*}-E_{1}^{\dagger} E_{1} F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*}+L_{E_{1}} M^{\dagger} Q\left(M^{\dagger}\right)^{*} L_{E_{1}}+ \\
L_{E_{1}} L_{M} V L_{E_{1}}+\left(L_{E_{1}} L_{M} V L_{E_{1}}\right)^{*}, \\
Y=A_{3}^{\dagger} C_{3}+L_{A_{3}} U,
\end{gathered}
$$

where

$$
\begin{aligned}
V= & \frac{1}{2}\left[G^{\dagger} E H^{\dagger}-G^{\dagger} H^{*} P^{\dagger} E H^{\dagger}-G^{\dagger} S\left(H^{\dagger}\right)^{*} E N^{\dagger} G^{*} H^{\dagger}+G^{\dagger} E\left(P^{\dagger}\right)^{*}\right. \\
+ & \left.\left(N^{\dagger}\right)^{*} E H^{\dagger} S^{\dagger} S\right]+L_{G} V_{1}+V_{2} R_{H}+U_{1} L_{S} L_{P}+R_{N} U_{2}^{*} L_{P} \\
- & G^{\dagger} S U_{2} R_{N} G^{*} H^{\dagger}, \\
U= & B^{\dagger}\left[A-C Y D-(C Y D)^{*}\right]-\frac{1}{2} B^{\dagger}\left[A-C V D-(C V D)^{*}\right] B B^{\dagger} \\
& -B^{\dagger} W_{1} B^{*}+W_{1}^{*} B B^{\dagger}+L_{B} W_{2},
\end{aligned}
$$

where $U_{1}, U_{2}, V_{1}, V_{2}, W_{1}, W_{2}$ are arbitrary matrices over $\mathbb{C}$ with appropriate sizes.

Proof. $(b) \Longleftrightarrow(c)$ : Clearly, $(2.1) \Longleftrightarrow(2.2),(2.4) \Longleftrightarrow(2.5)$. By Lemma 2.3 and the block Gaussion elimination, we obtain

$$
\begin{aligned}
& R_{P} R_{G} E=0 \Longleftrightarrow r\left(R_{P} R_{G} E\right)=0 \\
& \Longleftrightarrow r\left[\begin{array}{cccc}
A & C & D^{*} & B \\
B^{*} & 0 & 0 & 0
\end{array}\right]=r\left[\begin{array}{lll}
C & D^{*} & B
\end{array}\right]+r(B) \\
& \Longleftrightarrow \\
& \Longleftrightarrow\left[\begin{array}{cccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} \\
C_{3} & A_{3} & 0 & 0 \\
F_{1} A_{4}^{*} & 0 & E_{1} & 0
\end{array}\right]-2 r\left(A_{3}\right)-r\left(E_{1}\right) \\
& \quad=r\left[\begin{array}{cc}
B_{4} & A_{4} \\
0 & E_{1} \\
A_{3} & 0
\end{array}\right]+r\left[\begin{array}{l}
B_{4} \\
A_{3}
\end{array}\right]-2 r\left(A_{3}\right)-r\left(E_{1}\right) \\
& \Longleftrightarrow r\left[\begin{array}{ccc}
C_{4} & B_{4} & A_{4} \\
B_{4}^{*} & 0 & 0 \\
C_{3}^{*} \\
C_{3} & A_{3} & 0 \\
F_{1} A_{4}^{*} & 0 & E_{1}
\end{array}\right]=r\left[\begin{array}{cc}
B_{4} & A_{4} \\
0 & E_{1} \\
A_{3} & 0
\end{array}\right]+r\left[\begin{array}{l}
B_{4} \\
A_{3}
\end{array}\right] .
\end{aligned}
$$

Similarly, we can get

$$
\begin{aligned}
& R_{G} E R_{G}=0 \Longleftrightarrow {\left[\begin{array}{cccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & 0 & \frac{1}{2} A_{4} F_{1}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 & 0 \\
A_{4}^{*} & 0 & 0 & 0 & A_{2}^{*} & E_{1}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{2} & 0 & -C_{2} & -\frac{1}{2} A_{2} F_{1}^{*} \\
\frac{1}{2} F_{1} A_{4}^{*} & 0 & E_{1} & 0 & -\frac{1}{2} F_{1} A_{2}^{*} & 0
\end{array}\right] } \\
&=2 r\left[\begin{array}{cc}
A_{4} & B_{4} \\
A_{2} & 0 \\
E_{1} & 0 \\
0 & A_{3}
\end{array}\right], \\
& L_{H} E L_{H}=0 \Longleftrightarrow r\left[\begin{array}{ccccc}
C_{4} & B_{4} & A_{4} & C_{3}^{*} & \frac{1}{2} A_{4} F_{1}^{*} \\
B_{4}^{*} & 0 & 0 & A_{3}^{*} & 0 \\
A_{4}^{*} & 0 & 0 & 0 & E_{1}^{*} \\
C_{3} & A_{3} & 0 & 0 & 0 \\
\frac{1}{2} F_{1} A_{4}^{*} & 0 & E_{1} & 0 & 0
\end{array}\right]=2 r\left[\begin{array}{cc}
A_{4} & B_{4} \\
E_{1} & 0 \\
0 & A_{3}
\end{array}\right] .
\end{aligned}
$$

$(a) \Longleftrightarrow(b):$ We separate the equations in system (1.8) into three groups

$$
\begin{gather*}
A_{3} Y=C_{3}  \tag{3.1}\\
A_{1} X=C_{1}, X B_{1}=D_{1}, A_{2} X A_{2}^{*}=C_{2}, X=X^{*}  \tag{3.2}\\
B_{4} Y+\left(B_{4} Y\right)^{*}+A_{4} X A_{4}^{*}=C_{4} \tag{3.3}
\end{gather*}
$$

It follows from Lemma 2.1 and 2.2 that equation (3.1) and system (3.2) are consistent if and only if the equalities in (2.1) and (2.4) hold, respectively. The general solutions to equation (3.1) and system (3.2) can be expressed as (2.3) and (2.6), respectively. Substituting (2.3) and (2.6) into (3.3) gives

$$
\begin{equation*}
B U+(B U)^{*}+C V D+(C V D)^{*}=A . \tag{3.4}
\end{equation*}
$$

Hence, the system (1.8) is consistent if and only if equations (3.1), (3.2) and (3.4) are consistent, respectively. By Lemma 3.1, we know that equation (3.4) is consistent if and only if

$$
R_{P} R_{G} E=0, R_{G} E R_{G}=0, L_{H} E L_{H}=0 .
$$

In this case, the general solution to equation is (3.4).

In Theorem 3.2, if $A_{3}$ and $B_{4}$ vanish, then we obtain the general Hermitian solution to (1.7).

Corollary 3.3. Let $A_{1}, B_{1}, C_{1}, D_{1}, A_{2}, A_{4}, C_{2}=C_{2}^{*}$, and $C_{4}=C_{4}^{*}$ be given, and let $E_{1}, F_{1}, M, Q, \alpha$ be the same as in Theorem 3.2. Put
$B=A_{4} L_{E_{1}} L_{M}, C=L_{E_{1}} A_{4}^{*}, A=C_{4}-A_{4} \alpha A_{4}^{*}, P=R_{B} C^{*}, N=B^{*} L_{C}$, $S=C^{*} L_{P}$. Then the following statements are equivalent:
(a) System (1.7) has a Hermitian solution.
(b) The equalities in (2.4) hold and

$$
R_{P} R_{B} A=0, R_{B} A R_{B}=0, L_{C} A L_{C}=0
$$

(c) The equalities in (2.5) hold and

$$
\begin{gathered}
r\left[\begin{array}{cc}
C_{4} & A_{4} \\
F_{1} A_{4}^{*} & E_{1}
\end{array}\right]=r\left[\begin{array}{l}
A_{4} \\
E_{1}
\end{array}\right], \\
r\left[\begin{array}{cccc}
C_{4} & A_{4} & 0 & \frac{1}{2} A_{4} F_{1}^{*} \\
A_{4}^{*} & 0 & A_{2}^{*} & E_{1}^{*} \\
0 & A_{2} & -C_{2} & -\frac{1}{2} A_{2} F_{1}^{*} \\
\frac{1}{2} F_{1} A_{4}^{*} & E_{1} & -\frac{1}{2} F_{1} A_{2}^{*} & 0
\end{array}\right]=2 r\left[\begin{array}{l}
A_{4} \\
A_{2} \\
E_{1}
\end{array}\right], \\
r\left[\begin{array}{ccc}
C_{4} & A_{4} & \frac{1}{2} A_{4} F_{1}^{*} \\
A_{4}^{*} & 0 & E_{1}^{*} \\
\frac{1}{2} F_{1} A_{4}^{*} & E_{1} & 0
\end{array}\right]=2 r\left[\begin{array}{l}
A_{4} \\
E_{1}
\end{array}\right] .
\end{gathered}
$$

In this case, the general Hermitian solution of system (1.7) can be expressed as

$$
\begin{aligned}
X= & E_{1}^{\dagger} F_{1}+F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*}-E_{1}^{\dagger} E_{1} F_{1}^{*}\left(E_{1}^{\dagger}\right)^{*} \\
& +L_{E_{1}} M^{\dagger} Q\left(M^{\dagger}\right)^{*} L_{E_{1}}+L_{E_{1}} L_{M} V L_{E_{1}}+\left(L_{E_{1}} L_{M} V L_{E_{1}}\right)^{*},
\end{aligned}
$$

where

$$
\begin{array}{r}
V=\frac{1}{2}\left[B^{\dagger} A C^{\dagger}+B^{\dagger} A\left(P^{\dagger}\right)^{*}+\left(N^{\dagger}\right)^{*} A C^{\dagger} S^{\dagger} S-B^{\dagger} C^{*} P^{\dagger} A C^{\dagger}-B^{\dagger} S\left(C^{\dagger}\right)^{*}\right. \\
\left.A N^{\dagger} B^{*} C^{\dagger}\right]+L_{B} U_{1}+U_{2} R_{C}+U_{3} L_{S} L_{P}+R_{N} U_{4}^{*} L_{P}-B^{\dagger} S U_{4} R_{N} B^{*} C^{\dagger}, \\
\text { and } U_{1}, U_{2}, U_{3}, U_{4} \text { are arbitrary matrices over } \mathbb{C} \text { with appropriate sizes. }
\end{array}
$$

Remark 3.4. Wang and Wu [16] derived the solvability conditions and the expression of the general Hermitian solution to (1.7) for adjointable operators between Hilbert $\mathcal{C}^{*}$-modules. We give some new necessary and
sufficient conditions for the solvability to the system (1.7). Our expression of the Hermitian solution is also true for Hilbert $\mathcal{C}^{*}$-modules, which has different form compared to the expression in [16].

## 4. Conclusion

In this paper, we have derived the extremal ranks and inertias of the matrix function (1.2) subject to (1.6) and (1.9). Using the results in (1.2), we have characterized the relations between the Hermitian part of the solution to the matrix equation $A_{3} Y=C_{3}$ and the Hermitian solution to the system of matrix equations $A_{1} X=C_{1}, X B_{1}=D_{1}, A_{2} X A_{2}^{*}=$ $C_{2}$. Moreover, we have established some necessary and sufficient conditions for the existence of the general solution to (1.8). The expression of such a solution to (1.8) has also been given when its solvability conditions are satisfied. As a special case of the system (1.8), we have given some new solvability conditions for the system (1.7) and a new expression of the general solution to (1.7), which extend the main result in [16].

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## References

[1] H. W. Braden, The equation $A^{T} X \pm X^{T} A=B$, SIAM J. Matrix Anal. Appl. 20 (1998), no. 2, 295-302.
[2] W. S. Cao, Solvability of a quaternion matrix equation, Appl. Math. J. Chinese Univ. Ser. B. 17 (2002), no. 4, 490-498.
[3] D. L. Chu and Y. S. Hung, Inertia and rank characterizations of some matrix expressions, SIAM J. Matrix Anal. Appl. 31 (2009), no. 3, 1187-1226.
[4] M. Dehghan and M. Hajarian, The general coupled matrix equations over generalized bisymmetric matrices, Linear Algebra Appl. 432 (2010), no. 6, 1531-1552.
[5] D. S. Djordjević, Explicit solution of the operator equation $A^{*} X \pm X^{*} A=B, J$. Comput. Appl. Math. 200 (2007), no. 2, 701-704.
[6] Z. H. He, Q. W. Wang, A real quaternion matrix equation with applications, Linear Multilinear Algebra. 61 (2013), no. 6, 725-740.
[7] J. Grö $\beta$, Explicit solutions to the matrix inverse problem $A X=B$, Linear Algebra Appl. 289 (1999), no. 1-3, 131-134.
[8] C. G. Khatri and S. K. Mitra, Hermitian and nonnegative definite solutions of linear matrix equations, SIAM J. Appl. Math. 31 (1976), no. 4, 579-585.
[9] Y. H. Liu and Y. G. Tian, Max-min problems on the ranks and inertias of the matrix expressions $A-B X C \pm(B X C)^{*}$, J. Optim. Theory Appl. 148 (2011), no. 3, 593-622.
[10] Y. H. Liu and Y. G. Tian, A simultaneous decomposition of a matrix triplet with applications, Numer. Linear Algebra Appl. 18 (2011), no. 1, 69-85.
[11] G. Marsaglia and G. P. H. Styan, Equalities and inequalities for ranks of matrices, Linear Multilinear Algebra 2 (1974) 269-292.
[12] F. X. Piao, Q. L. Zhang and Z. F. Wang, The solution to matrix equation $A X+X^{T} C=B, J$. Franklin Inst. 344 (2007), no. 8, 1056-1062.
[13] D. C. Sorensen and A. C. Antoulas, The Sylvester equation and approximate balanced reduction, Linear Algebra Appl. 351 (2002) 671-700.
[14] Y. G. Tian, Equalities and inequalities for inertias of Hermitian matrices with applications, Linear Algebra Appl. 433 (2010), no. 1, 263-296.
[15] Y. G. Tian, Maximization and minimization of the rank and inertia of the Hermitian matrix expression $A-B X-(B X)^{*}$ with applications, Linear Algebra Appl. 434 (2011), no. 10, 2109-2139.
[16] Q. W. Wang and Z. C. Wu, Common Hermitian solutions to some operator equations on Hilbert $\mathcal{C}^{*}$-modules, Linear Algebra Appl. 432 (2010), no. 12, 31593171.
[17] Q. W. Wang and Z. H. He, Some matrix equations with applications, Linear and Multilinear Algebra 60 (2012), no. 11-12, 1327-1353.
[18] Q. W. Wang and C. Z. Dong, The general solution to a system of adjointable operator equations over the Hilbert $C^{*}$-modules, Oper. Matrices 5 (2011), no. 2, 333-350.
[19] D. Xie, X. Hu and Y. P. Sheng, The solvability conditions for the inverse eigenproblems of symmetric and generalized centro-symmetric matrices and their approximations, Linear Algebra Appl. 418 (2006), no. 1, 142-152.
[20] Q. X. Xu, L. J. Sheng and Y. Y. Gu, The solution to some operator equations, Linear Algebra Appl. 429 (2008), no. 8-9, 1997-2024.
[21] S. Yuan, A. Liao and Y. Lei, Inverse eigenvalue problems of tridiagonal symmetric matrices and tridiagonal bisymmetric matrices, Comput. Math. Appl. 55 (2008), no. 11, 2521-2532.
[22] S. Yuan, A. Liao and Y. Lei, Least squares Hermitian solution of the matrix equation $(A X B, C X D)=(E, F)$ with the least norm over the skew field of quaternions, Math. Comput. Modelling 48 (2008), no. 1-2, 91-100.
[23] F. X. Zhang, Y. Li, J. L. Zhao, Common Hermitian least squares solutions of matrix equations $A_{1} X A_{2^{*}}=B_{1}$ and $A_{2} X A_{2^{*}}=B_{2}$ subject to inequality restrictions, Comput. Math. Appl. 62 (2011), no. 6, 2424-2433.
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