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AN EXTENSION THEOREM FOR FINITE POSITIVE MEASURES ON SURFACES OF FINITE DIMENSIONAL UNIT BALLS IN HILBERT SPACES

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Abstract. A consistency criteria is given for a certain class of finite positive measures on the surfaces of the finite dimensional unit balls in a real separable Hilbert space. It is proved, through a Kolmogorov type existence theorem, that the class induces a unique positive measure on the surface of the unit ball in the Hilbert space. As an application, this will naturally accomplish the work of Kanter on the existence and uniqueness of the spectral measures of finite dimensional stable random vectors to the infinite dimensional ones. The approach presented here is direct and different from the functional analysis approach in the work of Kuelbs and Linde and the indirect approach of Tortrat and Dettweiler.
Keywords: Kolmogorove existence theorem, separable Hilbert space, stable distribution, spectral measure.
MSC(2010): Primary: 60E07; Secondary: 60E10, 46G12.

1. Introduction

In this work we assume that H is a real separable Hilbert space and $\eta = \{\Gamma_n,\ n \geq 1\}$ is a sequence of finite positive measures, η is defined on the Borel sets in S_{n-1}, the surface of the n dimensional unit ball in H. We provide a consistency criteria for Γ and then establish a Kolmogorov type extension theorem, showing that η induces a unique positive measure on the surface of the unit ball in H. The surfaces $\{S_{n-1},\ n \geq 1\}$ do not possess the cylindrical structure which is essential in the classical

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Kolmogorov Extension Theorem. This, perhaps, caused [12] and [3] to take an indirect approach in defining measures on infinite sectors having vertices at origin; of course they treated more general spaces. Our approach is direct, in the sense that we work with the surfaces directly. Our extension theorem is not a consequence of the Kolmogorov Extension Theorem, but it is in its style. The existence and the uniqueness of the spectral measure of a stable random vector taking values in a real separable Hilbert space will be an immediate consequence of our extension theorem, due to the work of [5] which provides the class η. The approach presented here benefits from utilization of first principles in measure theory, rather than advanced functional analysis tools as in [6] and [8], that provide the existence, and to the indirect approach of A. Tortrat [12] and E. Dettweiler [3], where no clear consistency condition for γ, as presented here, is presented. Moreover there is no need to produce and apply Levy representation for stable measures on abstract spaces, as in [6] and [8]. This work also provides an interesting application for the celebrated result of [1] on the tail behavior of the infinite dimensional stable distributions.

Let us provide some notions and notations below. Let H be a real separable Hilbert space with a base $\{e_i, i \in \mathbb{N}\}$, $\mathbb{N}$ is the set of natural numbers. The inner product and the norm correspondingly are denoted by $< . >$ and $\|.\|$. The Borel field on H, the smallest σ-field containing open sets in H, is denoted by $\mathcal{B}$. Also let $S = \{s \in H : \ ||s|| = 1\}$ be the surface of the unit ball in H, and $\mathcal{S}$ denote the Borel field on S; $\mathcal{S} = S \cap \mathcal{B}$. A random vector in H is a Borel measurable mapping on a probability space $(\Omega, \mathcal{F}, P)$ taking its values in H. If X is a random vector in H, then $\mu = P^{-1}X$ is a finite measure on $\mathcal{B}$ which is called the distribution of X. The characteristic function of X, denoted by ϕ_X, is defined to be the Fourier transform of the distribution μ; so for $t \in H$, $\phi_X(t)$ is the expected value of $e^{i<X,t>}$, given by

$$\phi_X(t) = Ee^{i<X,t>} = \int_H e^{i<x,t>} d\mu(x).$$

A random vector X, or correspondingly its distribution μ, is said to be stable if for every $k = 1, 2, \cdots$, $\mu^{*k} = T_{a_k}\mu * \delta_{b^k}$ for some positive constant a_k and vector b^k in H, where $*$ stands for the convolution of two distributions on H, $\mu^{*k} = \mu^{*(k-1)} * \mu,\ \mu^{*1} = \mu$, the δ_{b^k} is the distribution with unit mass at b^k, and $T_a\mu(B) = \mu(\{(1/a)b,\ b \in B\}),\ a > 0$. Equivalently, X is stable if its characteristic function satisfies $[\phi_X(t)]^k =$

$\phi_X(a_k t)e^{i<t,b^k>},\ t \in H,\ k = 1, 2, \cdots$. If $\mu(B) = \mu(-B),\ B \in \mathcal{B}$, then μ, and also X, is said to be symmetric. Without loss of generality, as long as it concerns the existence and the uniqueness of the spectral measures, we may assume $b^k = 0,\ k = 1, 2, \cdots$. In this case X is called strictly α-stable. A note will be provided for the general case.

Every x in H admits the series representation $x = \sum_{i=1}^{\infty} < x, e_i > e_i$. We refer to $x_{i_1,\cdots,i_n} = (< x, e_{i_1} >, ..., < x, e_{i_n} >)$ as a finite dimensional sub-vector of x. For a simplicity in notations, $x_{1,\cdots,n}$ is denoted by x_n. It easily follows that if X is a stable random vector in H with constants $a_k,\ k = 1, 2, \cdots$, then each finite dimensional sub-vector of X is a finite dimensional stable random vector with the same $a_k,\ k = 1, 2, \cdots$. Consequently, $a_k = k^{1/\alpha}$, [4]. The parameter α is called the index. The converse is also true; If finite dimensional sub-vectors are stable with the same index, then the random vector will be stable with the same index. The characteristic function of $X_{i_1,\cdots,i_n}$ is identified from ϕ_X through $\phi_{X_{i_1,\cdots,i_n}}(t) = \phi_X(t'_{i_1,\cdots,i_n}),\ t \in \mathbb{R}^n$, where $t'_{i_1,\cdots,i_n} \in H$ has all of its coordinates zero except the i_1-th,$\cdots, i_n$-th that are the same as those of t respectively. The continuity of the norm and the Bounded Convergence Theorem imply that ϕ_X is also identified by the characteristic functions of the finite dimensional sub-vectors. Since $X_n \equiv X_{1,\cdots,n}$ is a strictly α stable random vector in R^n, with $b_n^k = 0$,

$$\phi_{X_n}(t) = e^{-\int_{S_{n-1}} |s\cdot t|^\alpha d\Gamma_n(s) + iC(\alpha,\Gamma_n,t)} \qquad t \in \Re^n, \tag{1.1}$$

$$C(\alpha, \Gamma_n, t) = \begin{cases} \tan(\frac{\pi\alpha}{2}) \int_{S_{n-1}} (t\cdot s)|t\cdot s|^{\alpha-1}\Gamma_n(ds) & \alpha \neq 1 \\ \frac{2}{\pi}\int_{S_{n-1}} (t\cdot s)\ln|t\cdot s|\Gamma_n(ds) & \alpha = 1, \end{cases}$$

where Γ_n is a finite measure on $S_{n-1} = \{s = (s_1, \cdots, s_n) \in \mathbb{R}^n : |s|^2 = \sum_{i=1}^{n} s_i^2 = 1\}$ and $s \cdot t = s_1 t_1 + \cdots + s_n t_n$. The characterization (1.1) was first proved by Levy [7], see also [10]. Kanter proved that the measure Γ is indeed unique [5]. We will intensively use (1.1) and Kanter's result in our approach for the generalization to the real separable Hilbert spaces.

2. **The Existence and the Uniqueness**

Let X be a random vector in a real separable Hilbert space H. Suppose $S_{n-1},\ \Gamma_n,\ X_n,\ n \geq 1$, are as in (1.1). It easily follows from the uniqueness of Γ_n that

$$\Gamma_k(B_k) = \int_{T^{-1}_{n,k}(B_k)} |s_k|^\alpha d\Gamma_n(s), \quad B_k \in \mathcal{S}_k \tag{2.1}$$

where $T_{n,k} : S_{n-1} - O_{n,k} \longrightarrow S_{k-1}$, such that $O_{n,k} = \{s = (s_1, ..., s_n) \in S_{n-1} |\ |s_k| = (\sum_{i=1}^{k} s_i^2)^{1/2} = 0\}$ and $T_{n,k}(s) = \frac{s_k}{|s_k|}$, $1 \leq k \leq n$, [9].

Let $S = \{s \in H : \ ||s|| = 1\}$ be the surface of the unit ball in H, and $O_n = \{s \in S : \ |s_n| = 0\}$. Define the $T_n : S - O_n \longrightarrow S_{n-1}$ by $T_n(s) = \frac{s_n}{|s_n|}$. Clearly T_n is continuous on its domain. On the other hand O_n is a Borel measurable set in S. Thus it follows that $T_n^{-1}(B_n)$ is a Borel measurable set in S for every $B_n \in \mathcal{S}_{n-1}$. It is also a closed set in S whenever B_n is closed in S_{n-1}. Thus clearly $\sigma(\mathcal{F}_0) \subseteq \mathcal{S}$, where $\mathcal{F}_0 = \bigcup_{n=1}^\infty T_n^{-1}(\mathcal{S}_{n-1})$. The following lemma shows that the converse is also true.

Lemma 2.1. *The σ-field $\mathcal{S}$ coincides with the σ-field generated by $\mathcal{F}_0$, $\mathcal{S} = \sigma(\mathcal{F}_0)$.*

Proof. Let B be a compact set in $\mathcal{S}$ and B_n be a compact set in $\mathcal{S}_{n-1}$. First we prove that there is a k for which $B = \bigcap_{n=k}^{\infty} T_n^{-1}(T_n(B))$. First of all there is k for which $T_n(B)$, $n \geq k$, is well defined. Otherwise $B \cap O_n \neq \emptyset$, $n = 1, 2, \cdots$. Let $s^n \in B \cap O_n$, then $\{s^n\}$ will be a bounded sequence in B, having a convergence subsequence $\{s^{n_i}\}$ that converges to an $s \in B$, $\|s\| = 1$. But $s_k = \lim_{i\to\infty} s_k^{n_i} = 0$, $k = 1, 2, \cdots$ which will be a contradiction. Next we show that $T_n(B)$ is indeed a closed set in S_{n-1}. This can be furnished by showing that $T_n(B)$ contains its limit points. If $T_n(s^\nu)$ converges to an $s^n \in S_{n-1}$, $\nu \to \infty$, then s^ν will have a convergence subsequence converging to an $s \in B$. Thus continuity of T_n will imply that $s^n = T_n(s)$. These two observations imply that $\bigcap_{n=k}^{\infty} T_n^{-1}(T_n(B))$ is measurable in $\mathcal{S}$ and $B \subseteq \bigcap_{n=k}^{\infty} T_n^{-1}(T_n(B))$. We will show the converse of the inclusion is also true.
Let $s \in \bigcap_{n=k}^{\infty} T_n^{-1}(T_n(B))$, then $T_n(s) \in T_n(B)$, $\quad n \geq k$. It follows that there exists a sequence $\{x^n\} \in B$ such that $T_n(s) = T_n(x^n)$, $n \geq k$; which implies $\frac{s_n}{|s_n|} = \frac{x_n^n}{|x_n^n|}$. Since $\{x^n\}$ is a bounded sequence, it contains a convergent subsequence $\{x^{n_k}\}$. Let x^{n_i} be a bounded subsequence of

x^n that converges to an x in B. Thus

$$\frac{s_{n_i}}{|s_{n_i}|} = \frac{x_{n_i}^{n_i}}{|x_{n_i}^{n_i}|} \to x, \quad n_i \to \infty.$$

On the other hand $|s_{n_i}| \to 1,\ n_i \to \infty$. Thus $s = x$; and we have proved that $B = \bigcap_{n=k}^{\infty} T_n^{-1}(T_n(B))$. This immediately implies that $\mathcal{S} = \sigma(\mathcal{F}_0)$ and the proof is complete. □

The following theorem is the main result of this article. It is an extension theorem for positive finite measures satisfying the consistency condition (2.1).

Theorem 2.2. *Assume that S, S_{n-1} and T_n are as in Lemma 2.1, and Γ_n, finite positive measures on $\mathcal{S}_{n-1}$, $n \geq 1$, respectively. Suppose that Γ_n, $n \geq 1$ satisfy (2.1) and*

$$\sup_n \Gamma_n(S_{n-1}) < \infty. \tag{2.2}$$

Then there is a unique finite positive measure Γ on $\mathcal{S}$ satisfying

$$\Gamma_n(B_n) = \int_{T_n^{-1}(B_n)} |s_n|^\alpha d\Gamma(s), \quad B_n \in \mathcal{S}_{n-1}, \quad n \geq 1. \tag{2.3}$$

Proof. Define

$$\Gamma_n^*(T_n^{-1}(B_n)) = \Gamma_n(B_n), \quad B_n \in \mathcal{S}_{n-1}. \tag{2.4}$$

Then clearly Γ_n^* is a finite Borel measure on $T_n^{-1}(\mathcal{S}_{n-1})$. Now define

$$\Gamma(T_n^{-1}(B_n)) = \int_{T_n^{-1}(B_n)} |s_n|^{-\alpha} d\Gamma_n^*(s), \quad B_n \in \mathcal{S}_{n-1}. \tag{2.5}$$

We will show that the Γ possesses the following properties:

(i) Γ is a well defined finite measure on $\mathcal{F}_0$;

(ii) Γ can be uniquely extended to a finite measure on $\mathcal{S}$;

(iii) The extension Γ satisfies (2.3), and is the only one.

First we note that $T_k = T_{n,k}T_n$ and that $\Gamma_n^* T_n^{-1}$ and Γ_n agree on $\mathcal{S}_{n-1}$ embedded in $\mathcal{S}$. Also by equations (2.4) and (2.5),

$$\Gamma_k(B_k) = \int_{T_k^{-1}(B_k)} |s_k|^\alpha d\Gamma(s), \quad k < n, B_k \in \mathcal{S}_k.$$

Now define

$$\gamma_n(T_{n,k}^{-1}(B_k)) = \int_{T_{n,k}^{-1}(B_k)} |s_k|^\alpha d\Gamma_n(s),$$

and

$$\gamma(T_k^{-1}(B_k)) = \int_{T_k^{-1}(B_k)} |s_k|^\alpha d\Gamma(s).$$

It follows from (2.1) that $\gamma_n(T_{n,k}^{-1}(B_k)) = \gamma(T_k^{-1}(B_k)) = \gamma(T_n^{-1}(T_{n,k}^{-1}(B_k)))$ and hence $\gamma_n = \gamma T_n^{-1}$ on $T_{n,k}^{-1}(\mathcal{S}_k)$. Then for each $k < n$,

$$\begin{aligned}
\Gamma_n(T_{n,k}^{-1}(B_k)) &= \int_{T_{n,k}^{-1}(B_k)} |s_k|^{-\alpha} d\gamma_n(s) \\
&= \int_{T_{n,k}^{-1}(B_k)} |s_k|^{-\alpha} d\gamma T_n^{-1}(s) \\
&= \int_{T_n^{-1}(T_{n,k}^{-1}(B_k))} (|s_n|/|s_k|)^\alpha d\gamma(s) \\
&= \int_{T_k^{-1}(B_k)} |s_n|^\alpha d\Gamma(s).
\end{aligned}$$

But Fatou's lemma implies that

$$\liminf_n \Gamma_n(T_{n,k}^{-1}(B_k)) \geq \int_{T_k^{-1}(B_k)} \liminf_n |s_n|^\alpha d\Gamma(s) = \Gamma(T_k^{-1}(B_k)).$$

Therefore Γ is finite on $\mathcal{F}_0$. Also

$$\begin{aligned}
\Gamma_k^*(T_k^{-1}(B_k')) &= \Gamma_k(B_k') \\
&= \int_{T_{n,k}^{-1}(B_k')} |s_k|^\alpha d\Gamma_n(s) \\
&= \int_{T_{n,k}^{-1}(B_k')} |s_k|^\alpha d\Gamma_n^* T_n^{-1}(s) \\
&= \int_{T_n^{-1}(T_{n,k}^{-1}(B_k'))} |\frac{s_k}{|s_n|}|^\alpha d\Gamma_n^*(s) \\
&= \int_{T_k^{-1}(B_k')} \frac{|s_k|^\alpha}{|s_n|^\alpha} d\Gamma_n^*(s),
\end{aligned}$$

where the third equality follows from the change of variable $T_n(s) = \frac{s^n}{|s^n|}$. Thus $d\Gamma_k^*(s) = \frac{|s_k|^\alpha}{|s_n|^\alpha} d\Gamma_n^*(s)$ on $T_k^{-1}(\mathcal{S}_{k-1}), \quad k \leq n$.

For (i) suppose $T_n^{-1}(B_n) = T_k^{-1}(B'_k),\ \ k < n$. Then

$$\begin{aligned}\Gamma(T_k^{-1}(B'_k)) &= \int_{T_k^{-1}(B'_k)} |s_k|^{-\alpha} d\Gamma^*_k(s) \\ &= \int_{T_n^{-1}(B_n)} |s_n|^{-\alpha} d\Gamma^*_n(s) \\ &= \Gamma(T_n^{-1}(B_n)),\end{aligned}$$

giving that Γ is well defined on $\mathcal{F}_0$. Also $T_k^{-1} = T_n^{-1}T_{n,k}^{-1}$ implies that $T_k^{-1}(\mathcal{S}_{k-1}) \subseteq T_n^{-1}(\mathcal{S}_{n-1}),\ k \leq n$; which in turn implies that Γ is finitely additive on $\mathcal{F}_0$. The class $\mathcal{F}_0$ should be enlarged to become a field. This can be done by adding the sets S and $O_n,\ n = 1, 2, \cdots$ to it. The set function Γ can be extended to the new class, say $\mathcal{F}_{00}$. Indeed since $S = \cup_n T_n^{-1}(S_{n-1})$, and $\{T_n^{-1}(S_{n-1})\}$ is increasing, we define Γ on S to be

$$\Gamma(S) = \lim_{n \to +\infty} \Gamma(T_n^{-1}(S_{n-1})),$$

which, because of (2.2), is finite. Also for $n = 1, 2, \cdots$, let $\Gamma(O_n) = \Gamma(S) - \Gamma(T_n^{-1}(S_{n-1}))$. Thus we have established a finitely additive set function on the field $\mathcal{F}_{00}$ of cylinders in S. A classical argument as in the proof of the Kolmogorov Existence Theorem, [2] page 490, will imply that Γ is indeed countably additive on $\mathcal{F}_{00}$.
For (ii), since by Lemma 2.1, $\mathcal{S} = \sigma(\mathcal{F}_0)$ and $\mathcal{F}_{00}$ is a field, the classical extension theorem implies that Γ has a unique extension on $\mathcal{S}$. Part (iii) follows from the fact that $\Gamma(ds) = |s_n|^{-\alpha}\Gamma^*_n(ds)$ on $T_n^{-1}(B_n)$. The proof of the theorem is complete. $\square$

Theorem 2.3. *Let X be an α-stable random vector on a real separable Hilbert space H for which $b^k = 0,\ k = 1, 2, \cdots$, then there exists a unique finite Borel measure Γ on $S = \{s \in H :\ \|s\| = 1\}$ such that*

$$\phi_X(t) = e^{-\int_S |<s,t>|^\alpha d\Gamma(s) + iC(\alpha,\Gamma,t)}, \qquad t \in H; \tag{2.6}$$

$$C(\alpha, \Gamma, t) = \begin{cases} \tan(\frac{\pi\alpha}{2} \int_S <t, s> | <t, s> |^{\alpha-1}\Gamma(ds) & \alpha \neq 1 \\ \frac{2}{\pi} \int_S <t, s> \ln | <t, s> |\Gamma(ds) & \alpha = 1, \end{cases}$$

the measure Γ is called the spectral measure of the α-stable random vector X and is unique for $0 < \alpha < 2$.

Proof. The proof rests on Theorem 2.2. The condition (2.2) is satisfied due to the tail behavior of the distribution of X established in [1]. Indeed for every $n \geq 1$,

$$\Gamma_n(S_{n-1}) = (C_\alpha)^{-1} \lim_{t\to\infty} t^\alpha P(\|X_n\| > t) < (C_\alpha)^{-1} \sup_t t^\alpha P(\|X\| > t) < \infty,$$

where $C_\alpha = (\int_0^\infty x^{-\alpha} \sin x \, dx)^{-1}$; For the equality see [11], page 197. The formula (2.6) follows from the facts that

$$\phi_X(t) = \lim_n e^{-\int_{S_{n-1}} |<t_n,s_n>|^\alpha d\Gamma_n(s_n) + iC_n(\alpha,\Gamma_n,t_n)}, \quad t \in H,$$

$$\int_{S_{n-1}} | < t_n, s_n > |^\alpha d\Gamma_n(s_n) = \int_S | < T_n^{'}(t), s > |^\alpha d\Gamma(s),$$

and

$$C_n(\alpha, \Gamma_n, t_n) = C(\alpha, \Gamma, T_n'(t)),$$

where $T_n'(t) = (t_n, 0, 0, ...)$.

The uniqueness of Γ for X follows from the uniqueness of spectral measures for stable random vectors in $\mathbb{R}^n$, and Theorem 2.2. The proof is complete. $\square$

Remark 2.4. *A modification will be needed in Theorem 2.2 if the condition $b^k = 0,\ k = 1, 2, \cdots$ is not satisfied. Indeed let X be an α-stable random vector in H, then the characteristic function of the random vector X_n is given by*

$$\phi_{X_n}(t) = e^{-\int_{S_{n-1}} |s\cdot t|^\alpha d\Gamma_n(s) + iC_n(\alpha,\Gamma_n,t) + it.\gamma_n} \qquad t \in \mathbb{R}^n,$$

where $\gamma_n \in \mathbb{R}^n$. The construction of the spectral measure in Theorem 2.2 provides an α-stable random vector Y on H with $\phi_{Y_n}(t) = e^{-it\cdot\gamma_n}\phi_{X_n}(t),\ t \in \mathbb{R}^n,\ n = 1, 2, \cdots$. So it follows that for every $k = 1, 2, \cdots$,

$$\begin{aligned}
\phi_{Y_n}(t) &= \left[\phi_{Y_n}(k^{\frac{1}{\alpha}} t)\right]^{1/k} \\
&= e^{-ik^{-1+1/\alpha}\gamma_n\cdot t} \left[\phi_{X_n}(k^{\frac{1}{\alpha}} t)\right]^{1/k} \\
&= e^{-ik^{-1+1/\alpha}\gamma_n\cdot t - ik^{-1}b^k\cdot t}\phi_{X_n}(t) \\
&= e^{-ik^{-1+1/\alpha}\gamma_n\cdot t - ik^{-1}b^k\cdot t - \int_{S_{n-1}} |s\cdot t|^\alpha d\Gamma_n(s) + iC_n(\alpha,\Gamma_n,t) + it.\gamma_n} \\
&= e^{-ik^{-1+1/\alpha}\gamma_n\cdot t - ik^{-1}b_n^k\cdot t + it.\gamma_n}\phi_{Y_n}(t), \quad t \in \mathbb{R}^n.
\end{aligned}$$

Therefore it follows that for a fixed k, $b_n^k = (k - k^{1/\alpha})\gamma_n,\ n = 1, 2, \cdots$. Thus $b^k = (k - k^{1/\alpha})\gamma,\ k = 1, 2, \cdots$, for a $\gamma \in H$. Consequently

$\phi_X(t) = e^{i<\gamma,t>}\phi_Y(t),\ t \in H$, *giving that*

$$\phi_X(t) = e^{i<\gamma,t> - \int_S |<s,t>|^\alpha d\Gamma(s) + iC(\alpha,\Gamma,t)}, \qquad t \in H.$$

Also if A is a bounded linear operator from X onto a real separable Hilbert space K, then AX will be an α-stable random vector in K, and it follows from the uniqueness of the spectral measure that

$$\Gamma_{AX}(E) = \int_{T_{H,K}^{-1}(E)} \|As\|_K^\alpha \Gamma_X(ds), \quad E \in \mathcal{B}(S_K),$$

where S_H and S_K are the surfaces of the unit balls in H and K respectively, $T_{H,K} : S_H \to S_K,\ T(s) = \|As\|_K^{-1} As$; moreover $\gamma_{AX} = A\gamma_X$.

References

[1] A. De Acosta, Stable measures and seminorms, *Ann. Prob.* **3** (1975), no. 5, 865–875.

[2] P. Billingsley, Probability and Measure, John Wiley & Sons, Inc., New York, 1995.

[3] E. Dettweiler, Stabile Maβe auf Badrikianachen Rüumen, *Math. Z.* **146** (1976), no. 2, 149–166.

[4] W. Feller, Probability Theorey And its Applications, Second edition, John Wiley & Sons, Inc., New York, 1971.

[5] M. Kanter, The L^p norm of sums of translates of a function, *Trans. Amer. Math. Soc.* **179** (1973) 35–47.

[6] J. Kuelbs, A representation theorem for symmetric stable processes and stable measures on H, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **26** (1973) 269–271.

[7] P. Levy, Theorie de l'addition des variables aleatoires, 3rd ed. Gauthier-Villais, Paris, 1954.

[8] W. Linde, Infinitely Divisible and Stable Measures on Banach Spaces, Teubner-Texte zur Mathematik, Band 58, DDR, 1983.

[9] A. Mohammadpour and A. R. Soltani, Exchangeable stable random vectors and their simulations, *Comput. Statist.* **15** (2000) 205–217.

[10] E. L. Rvaceva, On domains of attraction of multi-dimensional distributions, Select. Transl. Math. Statist. and Probability, 2, 183–205, Amer. Math. Soc., Providence, 1962.

[11] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance.* Chapman & Hall, New York, 1994.

[12] A. Tortrate, Lois e(λ) dans les espaces vectoriels et lois stables, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **37** (1976), no. 2, 179–182.

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