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A CLASS OF ARTINIAN LOCAL RINGS OF HOMOGENEOUS TYPE

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ABSTRACT. Let I be an ideal in a regular local ring (R, \mathfrak{n}) , we will find bounds on the first and the last Betti numbers of $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$. if A is an Artinian ring of the embedding codimension h, I has the initial degree t and $\mu(\mathfrak{m}^t) = 1$, we call A a t-extended stretched local ring. This class of local rings is a natural generalization of the class of stretched local rings studied by Sally, Elias and Valla. For a t-extended stretched local ring, we show that $\binom{h+t-2}{t-1} - h + 1 \leq \tau(A) \leq \binom{h+t-2}{t-1}$ and $\binom{h+t-1}{t} - 1 \leq \mu(I) \leq \binom{h+t-1}{t}$. Moreover $\tau(A)$ reaches the upper bound if and only if $\mu(I)$ is the maximum value. Using these results, we show when $\beta_i(A) = \beta_i(\operatorname{gr}_{\mathfrak{m}}(A))$ for each $i \geq 0$. Beside, we will investigate the rigid behavior of the Betti numbers of A in the case that I has initial degree t and $\mu(\mathfrak{m}^t) = 2$. This class is a natural generalization of almost stretched local rings again studied by Elias and Valla. Our research extends several results of two papers by Rossi, Elias and Valla.

Keywords: Artinian rings, Hilbert function, number of generators, Cohen-Macaulay type.

MSC(2010): Primary: 13E10; Secondary: 13H10, 13D40, 13D02.

1. Introduction

Suppose I is an ideal of the regular local ring (R, \mathfrak{n}) and R/\mathfrak{n} is an algebraically closed field of characteristic zero. Let $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$, it is a classical problem to study the Hilbert function of A and the numerical invariants of its minimal R-free resolution.

157

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Denote by $\mu()$ the minimal number of generators of an ideal of A (or R), the Hilbert function of A is, by definition,

$$\mathrm{HF}_{\mathrm{A}}(j) := \dim_{\mathbf{k}} \mathfrak{m}^{j}/\mathfrak{m}^{j+1} = \mu(\mathfrak{m}^{j}) \ \, \mathrm{where} \ \, \mathbf{k} = \mathrm{R}/\mathfrak{n} = \mathrm{A}/\mathfrak{m}.$$

Hence HF_A is the Hilbert function of the homogeneous k-standard algebra

$$\operatorname{gr}_{\mathfrak{m}}(A) = \bigoplus_{j \ge 0} \mathfrak{m}^j / \mathfrak{m}^{j+1}$$

which is called the associated graded ring of A. If $\dim(R) = n$ then the associated graded ring of R, $\operatorname{gr}_{\mathfrak{n}}(\mathbf{R})$, is the polynomial ring over the n variables, say $P = \mathbf{k}[x_1, \ldots, x_n]$.

By Macaulay's Theorem, there exists a lexsegment ideal $L = \text{Lex}(I) \subset P = \text{gr}_n(R)$ with $\text{HF}_A = \text{HF}_{P/L}$ and it is known that

$$\beta_i(A) \leq \beta_i(\operatorname{gr}_{\mathfrak{m}}(A)) \leq \beta_i(P/L) \text{ for each } i \geq 0.$$

The local ring A is called of homogeneous type if for each $i \geq 0$, $\beta_i(A) = \beta_i(\operatorname{gr}_{\mathfrak{m}}(A))$. By a result of Rossi and Sharifan the Betti numbers $\beta_i(A)$ can be obtained from the Betti numbers $\beta_i(P/L)$ by a sequence of zero and negative consecutive cancellations (see [10, Theorem 4.1]). In particular, $\beta_i(A) = \beta_i(P/L)$ for each $i \geq 0$ if and only if $\mu(I) = \mu(L)$ (see [9, Corollary 3.4]. In this paper, we apply this approach to study the numerical invariants of some classes of Artinian local rings.

Assume that A is Cohen-Macaulay and $d = \dim(A)$, e is the multiplicity of A and $h = \mu(\mathfrak{m}) - d$ is the embedding codimension of A. By a theorem of Abhyankar, we know that $e \ge h + 1$, and if the equality e = h + 1 holds we say that A has minimal multiplicity and in this case $\beta_i(A) = i \binom{h+1}{i+1}$ for each $i \ge 0$.

More generally, let A = R/I be a Cohen-Macaulay local ring and $t \ge 2$ be the largest integer such that $I \subseteq \mathfrak{n}^t$, i.e., t is the initial degree of I. Then it is easy to see that $e \ge \binom{h+t-1}{h}$ and if the equality $e = \binom{h+t-1}{h}$ holds then

$$\beta_i(A) = \sum_{k=1}^h \binom{t+k-2}{k-1} \binom{k-1}{i-1} \text{ for each } i \ge 0.$$

In this paper, we study ideal I of initial degree t and show that if A = R/I is Artinian then

(1.1)
$$\binom{h+t}{t} - e \le \binom{h+t-1}{t} - \mu(m^t) \le \mu(I).$$

These bounds generalize the bounds given in [4, Lemma 2.1]. As a consequence of (1.1) and by considering the lexsegment ideal corresponding to an Artinian reduction of A, we prove that if $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ is a Cohen-Macaulay local ring, t is the initial degree of I and $e \leq \binom{h+t-1}{h}+2$, then

$$\binom{h+t-1}{t} - 2 \le \mu(I) \le \binom{h+t-1}{t}.$$

An Artinian local ring (A, \mathfrak{m}) , not necessary Gorenstein, is called *stretched* if $\mu(\mathfrak{m}^2) = 1$. Stretched Artinian local rings have been studied by J. Sally in [11] and Elias and Valla in [4]. They found a very nice structure theorem for the defining ideal I of the stretched Artinian local ring A = R/I of Cohen-Macaulay type type $1 \leq \tau(A) \leq h$. In particular, they showed that:

(1.2)
$$\mu(I) = \begin{cases} \binom{h+1}{2} - 1 & \text{if } A \text{ is stretched and } \tau(A) < h; \\ \binom{h+1}{2} & \text{if } A \text{ is stretched and } \tau(A) = h. \end{cases}$$

If (A, \mathfrak{m}) is an Artinian local ring of initial degree t and $\mu(\mathfrak{m}^t) = 1$, we say that A is a *t*-extended stretched Artinian local ring. Our goal is to extend Elias and Valla's results to this larger class of local rings.

By Macaulay's theorem, the Hilbert function of a t-extended stretched Artinian local ring A is given by:

$$\frac{j \quad 0 \quad 1 \quad \cdots \quad t-1 \quad t \quad \cdots \quad s \quad s+1}{\mathrm{HF}_{\mathrm{A}}(\mathbf{j}) \quad 1 \quad h \quad \cdots \quad \binom{h+t-2}{t-1} \quad 1 \quad \cdots \quad 1 \quad 0},$$

for some $s \geq t$. It is clear that $\tau(A) \leq \tau(P/L)$ where L is the lexsegment ideal corresponding to A. To find a lower bound for the Cohen-Macaulay type of A, we apply the cancellation method for a larger class of local rings. Let

be the Hilbert function of an Artinian local ring (A, \mathfrak{m}) . If v is the least integer such that $h_v \leq 2$, then $\tau(A) \geq \max\{1, h_{v-1} - h_v(h-1)\}$ (see Theorems 3.4, 3.5). So, if (A, \mathfrak{m}) is a *t*-extended stretched local ring, then $\binom{h+t-2}{t-1} - h + 1 \leq \tau(A) \leq \binom{h+t-2}{t-1}$.

Let $R = \mathbf{k}[[x_1, \ldots, x_h]]$ be the power series ring in the variables x_1, \ldots, x_h and A = R/I a *t*-extended stretched Artinian local ring. In Corollary 4.7 we show that

(1.3)

$$\mu(I) = \begin{cases} \binom{h+t-1}{t} - 1 & \text{if } A \text{ is } t \text{- extended stretched and } \tau(A) < \binom{h+t-2}{t-1}; \\ \binom{h+t-1}{t} & \text{if } A \text{ is } t \text{- extended stretched and } \tau(A) = \binom{h+t-2}{t-1}. \end{cases}$$

Note that (1.3) extends (1.2) to the case t > 2. In order to prove Corollary 4.7 we use [9, Corollary 3.4], and standard basis theory to find a minimal system of generators for I. Our method leads to a structure theorem for the extended stretched Artinian local rings with the maximal Cohen-Macaulay type (see Theorem 4.6).

Next, using Corollary 4.7, we characterize extended stretched Artinian local rings of homogeneous type (see Corollary 4.8).

It is worth remarking that Rossi and Valla [8] studied extended stretched local rings in the case s = t. They called the defining ideal of this kind of rings, *almost* t-*extremal*. Our research extends and reprove some of their results (see [8, Lemma 3.7 and Theorem 3.10]).

The last section is devoted to study an Artinian local ring $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ of initial degree t, embedding codimension h and $\mu(\mathfrak{m}^t) = 2$. In this case, by Proposition 5.1,

$$\binom{h+t-1}{t} - 2 \le \mu(I) \le \binom{h+t-1}{t},$$

and

$$\binom{h+t-2}{t-1} - 2(h-1) \le \tau(A) \le \binom{h+t-2}{t-1}.$$

In Theorem 5.5 we find a minimal system of generators for I when $\tau(A)$ is the maximal one. Finally, we find some classes of Artinian local rings of homogeneous type when $\mu(\mathfrak{m}^t) = 2$ (see Corollaries 5.6,5.7 and 5.8). In particular, we prove that the following statements are equivalents:

(1)
$$\beta_i(A) = \beta_i(P/I^*) = \beta_i(P/\text{Lex}(I))$$
 for each $i \ge 0$
(2) $\mu(I) = {h+t-1 \choose t}$.
(3) $\tau(A) = {h+t-2 \choose t-1}$.

2. Preliminaries

Let (R, \mathfrak{n}) be a regular local ring of dimension n. Assume that $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ has dimension d, embedding codimension h and multiplicity e. The Hilbert function of A is defined as:

$$\operatorname{HF}_{A}(j) := \dim_{k}(\mathfrak{m}^{j}/\mathfrak{m}^{j+1}) \text{ for } j \geq 0.$$

. . .

Let $P = \mathbf{k}[x_1, \dots, x_n]$ then that the associated graded ring of the local ring A is

$$\operatorname{gr}_{\mathfrak{m}}(A) = \oplus_{j \ge 0} \mathfrak{m}^j / \mathfrak{m}^{j+1} = P/I^*,$$

where I^* is a homogeneous ideal of P generated by the initial forms of the elements of I. Here, if $f \in R$ is a nonzero element and and m is the largest integer such that $f \in \mathfrak{n}^m$, we let $f^* := \bar{f} \in \mathfrak{n}^m/\mathfrak{n}^{m+1}$ and we say that f^* is the initial form of f. So

$$I^* = \langle f^* : f \in I \rangle.$$

Therefore the Hilbert function of A = R/I is the same as the Hilbert function of the standard graded algebra P/I^* .

Numerical invariants of the minimal R-free resolution of A can be studied by taking advantage of the rich literature in the graded case. Namely, by a result of Robbiano (see [6] and [9, Theorem 1.8]) from a minimal P-free resolution of $\operatorname{gr}_{\mathfrak{m}}(A)$ we can build up an R-free resolution of A which is not necessarily minimal. Hence, for the Betti numbers of A and $\operatorname{gr}_{\mathfrak{m}}(A)$ one has

$$\beta_i(A) \leq \beta_i(\operatorname{gr}_{\mathfrak{m}}(A))$$
 for every $i \geq 0$.

According to [6], A is called of homogeneous type, if $\beta_i(A) = \beta_i(\operatorname{gr}_{\mathfrak{m}}(A))$ for each $i \geq 0$.

Let A = R/I be a local ring and $L = \text{Lex}(I) = \text{Lex}(I^*)$ be the corresponding lexsegment ideal then $\beta_i(A) = \beta_i(P/I^*) = \beta_i(P/L)$ for each $i \ge 0$ if and only if $\mu(I) = \mu(L)$ (see[9, Corollary 3.4]). Moreover, by [10, Theorem 4.1], for each local ring A = R/I the Betti numbers, $\beta_i(A)$, can be obtained from the Betti numbers of P/Lex(I) by a sequence of negative and zero consecutive cancellations where negative and zero cancellation is defined as follows.

Given a sequence of integer numbers $\{c_i\}$ such that $c_i = \sum_{j \in \mathbb{N}} c_{ij}$, we obtain a new sequence by a consecutive cancellation as follows: fix an index *i*, and choose *j* and *j'* such that $j \leq j'$ and $c_{ij}, c_{i-1,j'} > 0$. Then replace c_{ij} by $c_{ij} - 1$ and $c_{i-1,j'}$ by $c_{i-1,j'} - 1$ and, accordingly, replace in the sequence c_i by $c_i - 1$ and c_{i-1} by $c_{i-1} - 1$. If j < j' we call it an *i*-negative consecutive cancellation, and if j = j', an *i*-zero consecutive cancellation. A sequence of consecutive cancellations will mean a finite number of consecutive cancellations performed on a given sequence.

Let N be a homogeneous P-module with P- free graded resolution given by:

$$\mathbf{G}: \ 0 \to \bigoplus_{i \ge 0} P^{\beta_{lj}}(-j) \xrightarrow{d_l} \bigoplus_{i \ge 0} P^{\beta_{l-1,j}}(-j) \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_1} \bigoplus_{i \ge 0} P^{\beta_{0j}}(-j).$$

According to the above definition, we will say that the sequence of the Betti numbers $\{\beta_i = \sum_{j \in \mathbb{N}} \beta_{ij}\}$ of N admits an *i negative consecutive cancellation* (resp. *i zero consecutive cancellation*) if there exist integers j < j' (resp. j = j') such that $\beta_{ij}, \beta_{i-1,j'} > 0$.

Note that [10, Theorem 4.1] is an extension of Peeva's result to the local case. Peeva [7] has proved that the graded Betti numbers of P/I ($I \subseteq P$ is a homogeneous ideal) can be obtained from the graded Betti numbers of P/L (L = Lex(I)) by a sequence of zero consecutive cancellations.

For more information about the cancellation method in finding the Betti numbers of local rings we refer to [10].

We recall that if A is an Artinian local ring, then the *socle degree* of A is the last integer s = s(A) such that $HF_A(s) \neq 0$ and the Cohen-Macaulay type of A is

$$\tau(A) := \dim_k(0:\mathfrak{m}).$$

Moreover, A is called Gorenstein if $\tau(A) = 1$. In this paper we apply the cancellation method to find some bounds on $\tau(A)$ and $\mu(I)$.

Another tool that we will use in this research is the standard basis theory. We refer the reader to [5, Section 6.4] for the details and proofs of what we recall in the following.

Assume that $R = \mathbf{k}[[x_1, \ldots, x_n]]$ be the power series ring in the variables x_1, \ldots, x_n and as before $P = \mathbf{k}[x_1, \ldots, x_n] = \operatorname{gr}_{\mathbf{n}}(\mathbf{R})$.

We denote by \mathbb{T}^n the set of terms or monomials of P; let σ be a term ordering on \mathbb{T}^n and assume that $x_1 > \cdots > x_n$. We define a new total order $\bar{\sigma}$ on \mathbb{T}^n in the following way: given $m_1, m_2 \in \mathbb{T}^n$ we let $m_1 >_{\bar{\sigma}} m_2$ if and only if deg $(m_1) < \deg(m_2)$, or deg $(m_1) = \deg(m_2)$ and $m_1 >_{\sigma} m_2$. $\bar{\sigma}$ is called *local degree ordering* induced by σ . Given $f \in R$ we denote by supp(f) the support of f, i.e., if $f = \sum_{(i_1,\ldots,i_n) \in \mathbb{N}^n} a_{(i_1,\ldots,i_n)} x_1^{i_1} \cdots x_n^{i_n}$ then supp(f) is the set of terms $x_1^{i_1} \cdots x_n^{i_n}$ such that $a_{(i_1,\ldots,i_n)} \neq 0$. So for a given $f \in R$, there is a monomial which is the biggest of the monomials in supp(f) with respect to $\bar{\sigma}$. This monomial is called the

leading monomial of f with respect to $\bar{\sigma}$ and is denoted by $L_{\bar{\sigma}}(f)$. It is clear that

$$L_{\bar{\sigma}}(f) = L_{\sigma}(f^*).$$

If I is an ideal of R we denote by $L_{\bar{\sigma}}(I)$ the monomial ideal of $P = \mathbf{k}[x_1, \ldots, x_n]$ generated by the leading monomials of the elements of I and call it the *leading ideal* of I.

$$L_{\bar{\sigma}}(I) = \langle L_{\bar{\sigma}}(f) : f \in I \rangle.$$

There is a unique reduced standard basis for I with respect to the local ordering $\bar{\sigma}$ that is a finite set $G = \{f_1, \ldots, f_m\} \subset I$ where 1) $L_{\bar{\sigma}}(I) = \langle L_{\bar{\sigma}}(f_1), \ldots, L_{\bar{\sigma}}(f_m) \rangle$,

2) for each $1 \leq i \leq m$, $LC(f_i) = 1$,

3) if $u \in \text{supp}(f_i) \setminus \{L_{\bar{\sigma}}(f_i)\}$ for some *i* then $u \notin \langle L_{\bar{\sigma}}(f_1), \dots, L_{\bar{\sigma}}(f_m) \rangle$ and

4) for each $1 \leq i \leq m$, $L_{\bar{\sigma}}(f_i) \notin \langle L_{\bar{\sigma}}(f_1), \dots, L_{\bar{\sigma}}(f_i), \dots, L_{\bar{\sigma}}(f_m) \rangle$.

If $\{f_1, \ldots, f_m\}$ is a reduced standard basis of I then one can easily show that $I = \langle f_1, \ldots, f_m \rangle$ and $I^* = \langle f_1^*, \ldots, f_m^* \rangle$.

3. Ideals of initial degree t

Lemma 3.1. Let A = R/I be an Artinian local ring with the maximal ideal $\mathfrak{m} = \mathfrak{n}/I$, multiplicity e and embedding codimension h. If I has initial degree t then

$$\binom{h+t}{t} - e \le \binom{h+t-1}{t} - \mu(\mathfrak{m}^t) \le \mu(I) \le \mu(\operatorname{Lex}(I)).$$

Proof. Let $L = \text{Lex}(I) \subset P = \mathbf{k}[x_1, \ldots, x_h]$. By [10, Theorem 4.1], it is clear that $\mu(L_{\langle t \rangle}) \leq \mu(I) \leq \mu(\text{Lex}(I))$ where $L_{\langle t \rangle}$ is the ideal generated by homogenous elements of L of degree t. Thus,

$$\mu(L_{\langle t \rangle}) = \mu(\mathfrak{n}^t) - \mu(\mathfrak{m}^t) = \binom{h+t-1}{t} - \mu(\mathfrak{m}^t) \le \mu(I).$$

By definition $e = \sum_{i=0}^{s} \mu(\mathfrak{m}^i)$ where s = s(A). Thus $e \ge 1 + h + \dots + {\binom{h+t-2}{t-1}} + \mu(\mathfrak{m}^t)$ and

$$\begin{pmatrix} t \\ t \end{pmatrix} - e \leq \begin{pmatrix} h+t \\ t \end{pmatrix} - (1+h+\dots+\begin{pmatrix} h+t-2 \\ t-1 \end{pmatrix} + \mu(\mathfrak{m}^t))$$
$$= \begin{pmatrix} h+t-1 \\ t \end{pmatrix} - \mu(\mathfrak{m}^t).$$

We remark that if $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ is a Cohen-Macaulay local ring of dimension d and embedding codimension h, then there exists an Artinian reduction $(B, \overline{\mathfrak{m}})$ such that the embedding codimension of Bis h, e(A) = e(B) and for each $i \geq 0, \beta_i(A) = \beta_i(B)$. For example if $J = \langle a_1, \ldots, a_d \rangle$ is an ideal generated by a maximal \mathfrak{n} -superficial sequence for A, then a_1, \ldots, a_d is a regular sequence and $I \cap J = IJ$. Let

$$\overline{I} = (I+J)/J, \quad \overline{R} = R/J, \quad B = A/(a_1, \dots, a_d)A = \overline{R}/\overline{I}, \quad \overline{\mathfrak{m}} = m/J.$$

Then one can see that e(A) = e(B) and $\beta_i(A) = \beta_i(B)$ for each $i \ge 0$ if we regard A as an R-module and B as an \overline{R} -module (see ([4, Proposition 2.2] and [1, Lemma 1.3.5])).

Corollary 3.2. Let A = R/I be a Cohen-Macaulay local ring with dimension d, embedding codimension h and multiplicity $e \leq {\binom{h+t-1}{t-1}} + 2$ where t is the initial degree of I, then

$$\binom{h+t}{t} - e \le \binom{h+t-1}{t} - 2 \le \mu(I) \le \binom{h+t-1}{t}.$$

Proof. By the discussion just before the corollary, we can assume that A is an Artinian ring. One can easily see that $\mu(\mathfrak{m}^t) \leq 2$ and $\mu(\text{Lex}(I)) = \binom{h+t-1}{t}$. So the bounds follows from Lemma 3.1. \Box

Note that Corollary 3.2 extends the bounds given in ([4], equation(1)) to the case that initial degree is t > 2.

In order to find some bounds on the Cohen-Macaulay type of Artinian rings we first remark that it is possible to compute the Betti numbers of lexsegment ideals from the corresponding Hilbert function. In particular, for the last Betti numbers we have an easy formula

Given the positive integers a and d, the d-binomial expansion of a is:

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \dots + \binom{k(j)}{j},$$

where $k(d) > k(d-1) > \dots > k(j) > j \ge 1$. Let

$$a_{[d]} = \binom{k(d) - 1}{d - 1} + \binom{k(d - 1) - 1}{d - 2} + \dots + \binom{k(j) - 1}{j - 1}.$$

We have:

Remark 3.3. Let $L \subset P = \mathbf{k}[x_1, \ldots, x_h]$ be a lexsegment Ideal with the Hilbert function $HF_{P/L}$. Then

$$\beta_{h,h+i}(P/L) = \mathrm{HF}_{P/L}(i) - (\mathrm{HF}_{P/L}(i+1))_{[i+1]} \quad for \ each \ i \ge 1.$$

Proof. See [12, Proof of Theorem 2.1].

So by Remark 3.3, the Cohen-Macaulay type of P/L can immediately be computed from its Hilbert function. We apply this fact in the next theorem in order to find a lower bound on the Cohen-Macaulay type of a class of Artinian local rings.

Theorem 3.4. Let $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ be an Artinian local ring and

j	0	1		v-1	v		s	s+1
$HF_A(j)$	1	h	• • •	h_{v-1}	1	• • •	1	0

be the Hilbert function of A where v is the least integer such that $h_v = 1$, then

$$\tau(A) \ge \max\{1, h_{v-1} - h + 1\}.$$

Proof. It is enough to apply the cancellation method. Let $L \subset \mathbf{k}[x_1, \ldots, x_h]$ be the lexsegment ideal corresponding to the Hilbert function of A. By the shape of the Hilbert function, it is easy to see that G(L), the minimal set of monomial generators of L, has no element in degree i where $v < i \leq s$ and it has just one element of degree s + 1 that is x_h^{s+1} . Let $\{\beta_{ij}\}$ be the sequence of the Betti numbers of P/L, since L is a stable ideal, the Eliahou-Kervaire formula (see [3]) and Remark 3.3 show that

$$\beta_{h,h+v-1} = h_{v-1} - 1, \quad \beta_{h,h+j} = 0 \text{ for each } v-1 < j < s \text{ and } \beta_{h,h+s} = 1,$$

$$\beta_{h-1,h+j-1} = 0 \text{ for each } v-1 < j < s \text{ and } \beta_{h-1,h+s-1} = h-1.$$

If we consider any sequence of consecutive cancellations in the h position, $\beta_{h,h+v-1}$ could decrease just by $\beta_{h-1,h+s-1}$ and $\beta_{h,h+s}$ can not be cancelled. So, the conclusion follows.

Note that Theorem 3.4 is an extension of [10, Corollary 4.5]. Let us give another application of cancellation method in finding a lower bound on the Cohen-Macaulay type of another class of Artinian local rings.

Theorem 3.5. Let $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ be an Artinian local ring of embedding codimension h and v is the least integer such that $HF_A(v) = 2$, then

$$\tau(A) \ge \max\{1, \operatorname{HF}_{A}(v-1) - 2(h-1)\}.$$

Proof. Let s be the socle degree of A. By Macaulay's theorem, for the Hilbert function of A there are only two possibilities:

1) For each $v \leq j \leq s$, $HF_A(j) = 2$.

2) There exists $v \leq r < s$ such that $HF_A(j) = 2$ for each $v \leq j \leq r$ and $HF_A(j) = 1$ for each $r < j \leq s$.

We prove the theorem in the case 2, because the other case is easier. So we assume that HF_A is given by the following table.

Let $L \subset \mathbf{k}[x_1, \ldots, x_h]$ be the lexsegment ideal that $\mathrm{HF}_{\mathrm{P/L}} = \mathrm{HF}_{\mathrm{R/I}}$ and G(L) be the minimal set of monomial generators of L. Then G(L)has no element in degree i where $v < i \leq r$ or $r + 1 < i \leq s$, it has one element of degree r + 1 that is $x_{h-1}x_h^r$ and one element of degree s + 1that is x_h^{s+1} .

Let $\{\beta_{ij}\}\$ be the sequence of the Betti numbers of P/L, since L is a stable ideal, the Eliahou-Kervaire formula (see [3]) and Remark 3.3 show that

$$\begin{split} \beta_{h,h+v-1} &= h_{v-1} - 2, \\ \beta_{h,h+j} &= 0 \text{ for each } j \text{ where } v-1 < j < r \text{ or } r < j < s, \\ \beta_{h,h+r} &= \beta_{h,h+s} = 1, \\ \beta_{h-1,h+j-1} &= 0 \text{ for each } j \text{ where } v-1 < j < r \text{ or } t < j < s \end{cases}$$

and $\beta_{h-1,h+r-1} = \beta_{h-1,h+s-1} = h - 1.$

If we consider any sequence of consecutive cancellations in the h position, $\beta_{h,h+v-1}$ could decrease just by $\beta_{h-1,h+s-1}$ or $\beta_{h-1,h+r-1}$, $\beta_{h,h+r}$ can be cancelled by $\beta_{h-1,h+s-1}$ and $\beta_{h,h+s}$ can not be cancelled. So the conclusion follows.

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4. Extended stretched Artinian local rings

In this section, we apply the results of section 3 to a special class of Artinian local rings. We generalize some results of [4] where the authors showed that if A = R/I is an Artinian stretched local ring with the Hilbert function

and the Cohen-Macaulay type $1 \leq \tau \leq h$ then we can find a basis $\{x_1, \ldots, x_h\}$ of \mathfrak{n} such that the ideal I is minimally generated by:

- $\{x_i x_j\}_{1 \le i < j \le h}, \{x_j^2\}_{2 \le j \le \tau}, \{x_i^2 x_1^s\}_{\tau + 1 \le i \le h}, \text{ if } \tau < h.$
- $\{x_1x_j\}_{2 \le j \le h}, \{x_ix_j\}_{2 \le i \le j \le h}, x_1^{s+1}, \text{ if } \tau = h.$

So, the minimal number of generators of the defining ideal I is:

$$\mu(I) = \begin{cases} \binom{h+1}{2} - 1 & \text{if } A \text{ is stretched and } \tau(A) < h;\\ \binom{h+1}{2} & \text{if } A \text{ is stretched and } \tau(A) = h. \end{cases}$$

Now let $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ be an Artinian local ring of embedding codimension h and assume that there exist integers $1 < t \leq s$ such that I has initial degree t, socle degree s and $\mu(\mathfrak{m}^t) = 1$. In other words A has the following Hilbert function:

(4.1)
$$\frac{j \quad 0 \quad 1 \quad \dots \quad t-1 \quad t \quad \dots \quad s \quad s+1}{\operatorname{HF}_{A}(j) \quad 1 \quad h \quad \dots \quad \binom{h+t-2}{t-1} \quad 1 \quad \dots \quad 1 \quad 0}.$$

In this case, we say that A is a t-extended stretched Artinian local ring. Because if t = 2 then A is a stretched Artinian local ring. An arbitrary Cohen-Macaulay local ring (A, \mathfrak{m}) is called t-extended stretched local ring if it has an Artinian reduction B such that B is a t-extended stretched artinian ring.

Proposition 4.1. Let (A, \mathfrak{m}) be a *t*-extended stretched Artinian local ring. Then

$$\binom{h+t-2}{t-1} - h + 1 \le \tau(A) \le \binom{h+t-2}{t-1}.$$

Proof. For the lower bound it is enough to apply Theorem 3.4 and the upper bound follows from the inequality $\tau(A) \leq \tau(P/L)$ and Remark 3.3 (*L* is the lexsegment ideal of $P = \mathbf{k}[x_1, \ldots, x_h]$ corresponding to the Hilbert function of *A*).

To continue, we need one more definition.

Definition 4.2. Let $Y = \{y_1, \ldots, y_h\} \subset R$ be a minimal system of generators of \mathfrak{n} and t a positive integer. We define

$$\mathrm{Mon}(Y,t):=\{y_1^{\alpha_1}\cdots y_h^{\alpha_h}\ :\ \alpha_i\in\mathbb{N},\ \sum_{i=1}^h\alpha_i=t\}.$$

Remark 4.3. For each

$$\binom{h+t-2}{t-1} - h + 1 \le \tau \le \binom{h+t-2}{t-1},$$

we can find an ideal $I \subset R$ such that R/I is a t-extended stretched Artinian local ring with $\tau(R/I) = \tau$. For example, let $X = \{x_1, \ldots, x_h\}$ be a minimal basis of \mathfrak{n} . If $\tau \neq \binom{h+t-2}{t-1}$ and $\ell = \binom{h+t-2}{t-1} - \tau$, it is enough to take I the ideal generated by $Mon(X,t) \setminus \{x_{h-\ell}^t, \ldots, x_h^t\}$ and $\{x_j^t - x_h^s\}_{h-\ell \le j < h}$. If $\tau = \binom{h+t-2}{t-1}$, we consider the ideal generated by $\{x_h^{s+1}\} \cup \operatorname{Mon}(\mathbf{X}, \mathbf{t}) \setminus \{\mathbf{x}_h^{\mathbf{t}}\}.$

Proposition 4.4. Let (A, \mathfrak{m}) be a *t*-extended stretched Artinian local ring. Then

$$\binom{h+t-1}{t} - 1 \le \mu(I) \le \binom{h+t-1}{t}$$

and if the Cohen-Macaulay type is not the maximal one, $\tau \neq {h+t-2 \choose t-1}$, then $\mu(I) = {h+t-1 \choose t} - 1.$

Proof. By Lemma 3.1 we get the lower bound. Let $P = \mathbf{k}[x_1, \ldots, x_h]$. If the socle degree of A is s, then by the shape of the Hilbert function, the lexsegment ideal Lex(I) is: (4.2)

$$\mathrm{Lex}(I) = \langle x_h^{s+1} \rangle + \langle x_1^{a_1} \dots x_h^{a_h} \ : \ (a_1, \dots, a_h) \in \mathbb{N}^h, \sum_{i=1}^h a_i = t \ \mathrm{and} \ a_h \neq t \rangle.$$

So $\mu(I) \leq \mu(\text{Lex}(I)) = {\binom{h+t-1}{t}}$. Now if $\mu(I) = {\binom{h+t-1}{t}}$ then by ([9, Corollary 3.4]) all the Betti numbers of I and Lex(I) are the same. In particular, $\tau(A) = \binom{h+t-2}{t-1}$. Therefore, in the case that the Cohen-Macaulay type is not maximal, the minimal number of generators is exactly $\binom{h+t-1}{t} - 1$.

We remark that if t < s then $\mu(I^*) = \binom{h+t-1}{t}$ because there is no zero cancellation in the first Betti number of P/Lex(I).

Next Lemma shows that if R/I is a *t*-extended stretched Artinian local ring then the Hilbert function of R/I: \mathfrak{n} is uniquely characterized by the Cohen-Macaulay type of R/I.

Lemma 4.5. Let I be an ideal of R such that A = R/I is a t-extended stretched Artinian local ring with the Cohen-Macaulay type τ and the socle degree s. Then the Hilbert function of R/I: \mathfrak{n} is given by

j	0	1	•••	t-2	t-1	t	 s-1	s	
$\mathrm{HF}_{\mathrm{A}}(j)$	1	h	• • •	$\binom{h+t-3}{t-2}$	x	1	 1	0) '

where $x = \binom{h+t-2}{t-1} + 1 - \tau$. In particular, $R/I : \mathfrak{n}$ is a t-1-extended stretched Artinian local ring if and only if τ is the maximum one.

Proof. First, note that the initial degree of $I : \mathfrak{n}$ is at least t-1 because I has initial degree t. Moreover, $I \subseteq I : \mathfrak{n}$. So $I^* \subseteq (I : \mathfrak{n})^*$ and

$$\mathrm{HF}_{R/I:\mathfrak{n}}(j) = \mathrm{HF}_{P/(I:\mathfrak{n})^*}(j) \leq \mathrm{HF}_{P/I^*}(j) = \mathrm{HF}_{R/I}(j), \ \text{ for each } j \geq 0.$$

Since $\mathfrak{n}^{s+1} \subset I$ it is clear that $\mathfrak{n}^s \subset I : \mathfrak{n}$. So $\operatorname{HF}_{R/I:\mathfrak{n}}(s) = 0$. If $\operatorname{HF}_{R/I:\mathfrak{n}}(s-1) = 0$ then $\mathfrak{n}^{s-1} \subset I : \mathfrak{n}$ and so $\mathfrak{n}^s \subset I$ which is a contradiction. Thus, $\operatorname{HF}_{R/I:\mathfrak{n}}$ has the desired shape. It remains to compute x. Consider the following short exact sequence:

$$0 \to I: \mathfrak{n}/I \to R/I \to R/I: \mathfrak{n} \to 0.$$

Denote by $\ell(M)$, the length of a module over R or P, $\ell(R/I : \mathfrak{n}) = \ell(R/I) - \ell(I : \mathfrak{n}/I)$. Thus $e(R/I : \mathfrak{n}) = e(R/I) - \tau$. So it follows that $x = \binom{h+t-2}{t-1} + 1 - \tau$.

In the following of this section, we assume that $R = \mathbf{k}[[x_1, \ldots, x_h]]$. Our goal is to find a structure theorem for t-extended stretched Artinian local ring R/I when $\tau(R/I)$ is the maximal one. Given a set of minimal generators $Y = \{y_1, \ldots, y_h\}$ of \mathfrak{n} , let ϕ_Y be the automorphism of Rinduced by substituting y_i for x_i in each power series $f(x_1, \ldots, x_h) \in R$. Given two ideals I and J in R, there exists a \mathbf{k} -algebra isomorphism $\alpha : R/I \to R/J$ if and only if for a set of generators $Y = \{y_1, \ldots, y_h\}$ of \mathfrak{n} we have $I = \phi_Y(J)$.

Theorem 4.6. Let A = R/I be a t-extended stretched Artinian local ring with the Cohen-Macaulay type τ and the socle degree s then

1) If $\tau < \binom{h+t-2}{t-1}$, then we can find a minimal system of generators $Y = \{y_1, \ldots, y_h\}$ of \mathfrak{n} such that I is minimally generated by the following

 $\{y_iy_h^{t-1}: 1 \le i \le h-1\} \cup \{u-\lambda_u y_h^s : u \in \mathrm{Mon}(\mathbf{Y}, \mathbf{t}) \setminus \{\mathbf{y}_i\mathbf{y}_h^{\mathbf{t}-1}: 1 \le \mathbf{i} \le \mathbf{h}\}\},\$

where for each $u, \lambda_u \in \mathbf{k}$ and at least one of the λ_u 's is nonzero.

2) If $\tau = \binom{h+t-2}{t-1}$, then we can find a minimal system of generators $Y = \{y_1, \ldots, y_h\}$ of \mathfrak{n} such that I is minimally generated by Mon(Y,t) $\{y_h^t\} \cup \{y_h^{s+1}\}.$

Proof. Assume that σ be the lexicographic order induced by $x_1 > \cdots > x_h$.

Let A = R/I be an arbitrary *t*-extended stretched Artinian local ring and set $X = \{x_1, \ldots, x_h\}$. By the shape of the Hilbert function of A we can assume that, after a generic change of variable,

$$L_{\bar{\sigma}}(I) = \langle \{x_h^{s+1}\} \cup \operatorname{Mon}(\mathbf{X}, \mathbf{t}) \setminus \{\mathbf{x}_h^{t}\} \rangle.$$

Thus the reduced standard basis of I has the following shape:

 $G(I) = \{x_h^{s+1}\} \cup \{u + f_u(x_h) : u \in Mon(X, t) \setminus \{x_h^t\}\}, \text{ where } f_u(x_h) = \lambda_{u,t} x_h^t + \dots + \lambda_{u,s} x_h^s, \ \lambda_{u,j} \in \mathbf{k}.$

For each i = 1, ..., h-1, if we set $u_i = x_i x_h^{t-1}$, then $u_i + \sum_{j=t}^s \lambda_{u_i,j} x_h^j \in G(I)$. Suppose $S = \mathbf{k}[[y_1, ..., y_h]]$ is a power series ring on the variables $y_1, ..., y_h$ and $Y = \{y_1, ..., y_h\}$. Define the \mathbf{k} -algebra isomorphism $\phi : R \to S$ induced by the map which sends each x_i to $y_i - \sum_{j=t}^s \lambda_{u_i,j} y_h^{j-t+1}$ for i = 1, ..., h-1 and $\phi(x_h) = y_h$. This isomorphism will send each $u \in G(I)$ to $\phi(u) \in J := \phi(I)$.

One can easily see that the set $\mathcal{A} = \{\phi(u) : u \in G(I)\}$ is a standard basis of J for the local lexicographic ordering on S. Starting from \mathcal{A} , it is straight forward to check that the reduced standard basis of J has the following shape:

 $G(J) = \{y_h^{s+1}\} \cup \{u + g_u(y_h) : u \in \operatorname{Mon}(Y, t) \setminus \{y_h^t\}\}, \text{ where } g_u(y_h) = \gamma_{u,t}y_h^t + \dots + \gamma_{u,s}y_h^s, \ \gamma_{u,j} \in \mathbf{k} \text{ and if } u = y_iy_h^{t-1} \text{ for some } i = 1, \dots, h-1, \text{ then } g_u(y_h) = 0.$

Next, we show that $\gamma_{u,j} = 0$ for each monomial $u \in \text{Mon}(Y, t) \setminus \{y_h^t\}$ and each $j \neq s$. By contrary assume that $t \leq r < s$ is the least integer that there exists $u = y_1^{\alpha_1} \cdots y_h^{\alpha_h}$ with $\alpha_h < t - 1$ with $\gamma_{u,r} \neq 0$. So $h_u = y_1^{\alpha_1} \cdots y_h^{\alpha_h} + \gamma_{u,r} y_h^r + \cdots + \gamma_{u,s} y_h^s \in G(J)$. Clearly there exists j < hwith $\alpha_j > 0$. Take $v = y_h u/y_j$. So, $h_v = v + \gamma_{v,r'} y_h^{r'} + \cdots + \gamma_{v,s} y_h^s \in G(J)$, where $r' \geq r$. Now it is easy to show that there exists $f \in \mathbf{k}[[y_1, \ldots, y_h]]$

set.

such that $y_h h_u - y_j h_v - f y_j y_h^{t-1} = \gamma_{u,r} y_h^{r+1} + \dots + \gamma_{u,s} y_h^{s+1} \in J$. This shows that $y_h^{r+1} \in L_{\bar{\sigma}}(J)$ which is a contradiction because r < s.

To summarize, with respect to the minimal system of generators $\{y_1, \ldots, y_h\}$ where $y_h = x_h$ and for $1 \le i < h$, $y_i = x_i + \sum_{j=t}^s \lambda_{u_i,j} x_h^{j-t+1}$, I has the following system of generators:

$$M = \{y_h^{s+1}\} \cup \{y_i y_h^{t-1} : 1 \le i \le h-1\} \cup \{u - \lambda_u y_h^s : u \in \operatorname{Mon}(Y, t) \setminus \{y_i y_h^{t-1} : 1 \le i \le h\}\},\$$
where for each $u, \lambda_u \in \mathbf{k}$

Now if $\tau < \binom{h+t-2}{t-1}$, we remark that by Proposition 4.4, $\mu(I) = \binom{h+t-1}{t} - 1$. To find a minimal system of generators for I which is contained in M, the only generator that we can remove is y_h^{s+1} . So

$$\{y_i y_h^{t-1} : 1 \le i \le h-1\} \cup \{u - \lambda_u y_h^s : u \in \operatorname{Mon}(\mathbf{Y}, \mathbf{t}) \setminus \{y_i y_h^{t-1} : 1 \le i \le h\}\}$$

is a minimal system of generators for I and at least one of the λ_u 's is nonzero because $\mathfrak{n}^{s+1} \subset I$.

If $\tau = \binom{h+t-2}{t-1}$, then by Lemma 4.5, $I : \mathfrak{n}$ is a t-1 extended stretched Artinian local ring. So as we showed before, there exists a minimal system of generators y_1, \ldots, y_h for n such that $I : \mathfrak{n}$ is generated by

$$\{y_h^s\} \cup \{y_i y_h^{t-2} : 1 \le i \le h-1\} \cup \{u - \lambda_u y_h^{s-1} : u \in Mon(\mathbf{Y}, t-1) \setminus \{y_i y_h^{t-2} : 1 \le i \le h\} \}.$$

This shows that $Mon(Y, t) \setminus \{y_h^t\} \cup \{y_h^{s+1}\} \subset I$. Comparing the Hilbert functions we see that $I = \langle Mon(Y, t) \setminus \{y_h^t\} \cup \{y_h^{s+1}\} \rangle$.

We remark that Theorem 4.6 is a generalization of [4, Theorem 3.1]. Actually, it gives a a structure theorem for extended stretched Artinian local rings with the maximal Cohen-Macaulay type. In the following we give some application of Theorem 4.6.

Corollary 4.7. Let A = R/I be a t- extended stretched Cohen-Macaulay local ring then

$$\mu(I) = \begin{cases} \binom{h+t-1}{t} - 1 & \text{if } \tau(A) < \binom{h+t-2}{t-1}; \\ \binom{h+t-1}{t} & \text{if } \tau(A) = \binom{h+t-2}{t-1}. \end{cases}$$

Proof. It is enough to recall that A has a t-extended stretched Artinian reduction with the same Betti numbers as A. So, by Theorem 4.6, the conclusion follows.

If (A, \mathfrak{m}) is a *t*-extended stretched Cohen-Macaulay local ring of the embedding codimension *h* and the socle degree *s*, then the bounds in Propositions 4.1 and 4.4 hold for *A*. Moreover, the Betti numbers of the corresponding lexsegment ideal (see (4.2)) can be computed by the Eliahou-Kervaire resolution and we have: (4.3)

$$\beta_i(A) \le \beta_i(P/\text{Lex}(\mathbf{I})) = \binom{\mathbf{t} + \mathbf{i} - 2}{\mathbf{i} - 1} \binom{\mathbf{h} + \mathbf{t} - 1}{\mathbf{t} + \mathbf{i} - 1} \text{ for each } 1 \le \mathbf{i} \le \mathbf{h}.$$

In particular we can present the following corollary:

Corollary 4.8. Let A be an Artinian t-extended stretched local ring of the socle degree s. Then A is of homogeneous type if and only if one of the following condition holds:

(1) s = t. (2) $\tau(A) = \binom{h+t-2}{t-1}$.

Proof. If A is an Artinian t-extended stretched local ring and s = t then the the corresponding lexsegment ideal is generated in two successive degrees so the conclusion follows by [10, Corollary 4.3].

If s > t then it is easy to see that $\mu(I^*) = \mu(\text{Lex}(I)) = \binom{h+t-1}{t}$. So in this case the result follows by Corollary 4.7, and [9, Corollary 3.4]. \Box

Let (A, \mathfrak{m}) be a t- extended stretched Artinian local ring with the socle degree t, following Rossi and Valla, we say that the ideal I is almost t-extremal. They proved that in this case $\tau(A) = \tau(\operatorname{gr}_{\mathfrak{m}}(A)) \leq \binom{h+t-2}{t-1}$ (see [8, Lemma 3.7]) and the minimal number of generators of I is $\binom{h+t-1}{t} - 1$ if and only if $\tau < \binom{h+t-2}{t-1}$ (see [8, Theorem 3.10]). So Corollaries 4.7 and 4.8 reprove and extend the mentioned results of [8].

Also, by Corollary 4.8, the given bounds for the Betti numbers of a t- extended stretched Artinian local ring A (see equation (4.3)) are achieved when $\tau(A)$ is the maximal one.

In the last corollary of this section we find the Betti numbers of an arbitrary t- extended stretched Cohen-Macaulay local ring of embedding codimension 3.

Corollary 4.9. Let (A, \mathfrak{m}) be a *t*-extended stretched Cohen-Macaulay local ring of the embedding codimension 3 and the Cohen-Macaulay type $\tau < \binom{t+1}{2}$ then

$$\beta_1(A) = {\binom{t+2}{2}} - 1, \quad \beta_2(A) = t(t+2) - {\binom{t+1}{2}} - \tau - 1 \text{ and } \beta_3(A) = \tau.$$

Proof. It is enough to notice that, in this case, Corollary 4.7 gives all the cancellations in the Betti numbers of the corresponding lexsegment ideal. \Box

5. The case
$$\mu(\mathfrak{m}^t) = 2$$

In this section, we study Artinian local ring A = R/I of the embedding codimension h when the initial degree is t and $HF_A(t) = 2$. In this case, HF_A is given by one of the following tables:

(5.2)
$$\frac{j \quad 0 \quad 1 \quad \cdots \quad t-1 \quad t \quad \cdots \quad s \quad s+1}{\operatorname{HF}_{A}(j) \quad 1 \quad h \quad \cdots \quad \binom{h+t-2}{t-1} \quad 2 \quad \cdots \quad 2 \quad 0}.$$

If the Hilbert function is as (5.1), we say that A is of type (t, r, s)where $t \leq r < s$. In this case

(5.3) Lex(I) =
$$\langle \mathbf{x}_{\mathbf{h}}^{\mathbf{s}+1}, \mathbf{x}_{\mathbf{h}-1}\mathbf{x}_{\mathbf{h}}^{\mathbf{r}} \rangle + \langle x_{1}^{a_{1}} \dots x_{h}^{a_{h}} : \sum_{i=1}^{h} a_{i} = t, a_{i} \in \mathbb{N} \text{ and } (a_{h-1}, a_{h}) \neq (0, t), (1, t-1) \rangle.$$

If the Hilbert function is as (5.2), we say that A is of type (t, s, s) where $t \leq s$. For this type the corresponding lexsegment ideal is:

(5.4) Lex(I) =
$$\langle \mathbf{x}_{\mathbf{h}}^{\mathbf{s}+1}, \mathbf{x}_{\mathbf{h}-1}\mathbf{x}_{\mathbf{h}}^{\mathbf{s}} \rangle$$
 +
 $\langle x_{1}^{a_{1}} \dots x_{h}^{a_{h}} : \sum_{i=1}^{h} a_{i} = t, a_{i} \in \mathbb{N} \text{ and } (a_{h-1}, a_{h}) \neq (0, t), (1, t-1) \rangle.$

By Lemma 3.1 and Theorem 3.5 we have:

Proposition 5.1. Let (A, \mathfrak{m}) be an Artinian local ring of initial degree t and assume that $\mu(\mathfrak{m}^t) = 2$ then

$$\binom{h+t-1}{t} - 2 \le \mu(I) \le \binom{h+t-1}{t},$$

and

$$\binom{h+t-2}{t-1} - 2(h-1) \le \tau(A) \le \binom{h+t-2}{t-1}.$$

In the previous section, we showed that if $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ is a *t*-extended stretched Artinian local ring then the Hilbert function of R/I: \mathfrak{n} is characterized by the Cohen-Macaulay type of R/I. Next example shows that for an Artinian local ring of type (t, r, s) a similar result does not hold.

Example 5.2. Let $R = \mathbf{k}[[x, y]]$, $I = \langle x^2y^2 - y^5, -x^3y, x^4 \rangle$ and $J = \langle -x^4 + x^2y^3 + y^7, x^3y - xy^4, -x^2y^2 \rangle$. Then the Hilbert function of both R/I and R/J is given by the following table:

Moreover $\tau(R/I) = \tau(R/J) = 2$. But the Hilbert functions of $R/I : \mathfrak{n}$ is

j	0	1	2	3	4	5	6	7
$\operatorname{HF}_{R/I:\mathfrak{n}}(j)$	1	2	3	3	2	2	1	0

and the Hilbert function of $R/J: \mathfrak{n}$ is

j	0	1	2	3	4	5	6	7	
$\operatorname{HF}_{R/J:\mathfrak{n}}(j)$	1	2	3	4	2	1	1	0	•

If the Artinian local ring R/I is of type (t, r, s) then the Hilbert function of R/I: \mathfrak{n} is uniquely determined by $\tau(R/I)$ provided that one of the extra conditions of the next lemma holds.

Lemma 5.3. Let $(A, \mathfrak{m}) = (R/I, \mathfrak{n}/I)$ be an Artinian local ring then

1) If A is of type (t, r, s) where $t \le r < s$ and $\tau(A) = \binom{h+t-2}{t-1}$ then $R/I : \mathfrak{n}$ is of type (t-1, r-1, s-1). So: (5.5) $\frac{j \quad 0 \quad 1 \quad \cdots \quad t-2 \quad t-1 \quad \cdots \quad r-1 \quad r \quad \cdots \quad s-1 \quad s}{\operatorname{HF}_{R/I:\mathfrak{n}}(j) \quad 1 \quad h \quad \cdots \quad \binom{h+t-3}{t-2} \quad 2 \quad \cdots \quad 2 \quad 1 \quad \cdots \quad 1 \quad 0}.$

2) If A is of type (t, s, s) where $t \leq s$, then $HF_{R/I:n}$ is:

(5.6)
$$\frac{j \quad 0 \quad 1 \quad \cdots \quad t-2 \quad t-1 \quad t \quad \cdots \quad s-1 \quad s}{\operatorname{HF}_{\mathrm{R/I:n}}(j) \quad 1 \quad h \quad \cdots \quad \binom{h+t-3}{t-2} \quad x \quad 2 \quad \cdots \quad 2 \quad 0}$$

where $x = \binom{h+t-2}{t-1} + 2 - \tau$. In particular, $R/I : \mathfrak{n}$ is of type (t-1, s-1, s-1) if and only if $\tau(A) = \binom{h+t-2}{t-1}$.

Proof. By assumption, the initial degree of I is t and therefore the initial degree of $I : \mathfrak{n}$ is at least t - 1. Also, by the definition of $I : \mathfrak{n}, R/I : \mathfrak{n}$ has the socle degree s - 1. since $I^* \subseteq (I : \mathfrak{n})^*$,

$$\mathrm{HF}_{\mathrm{R}/\mathrm{I}:\mathfrak{n}}(j) = \mathrm{HF}_{\mathrm{P}/(\mathrm{I}:\mathfrak{n})^*}(j) \leq \mathrm{HF}_{\mathrm{P}/\mathrm{I}^*}(j) = \mathrm{HF}_{\mathrm{R}/\mathrm{I}}(j), \ \text{ for each } j \geq 0.$$

We show that if $\operatorname{HF}_{R/I:n}(\ell) = 1$ for some $t-1 \leq \ell \leq s-1$ then $\operatorname{HF}_{R/I}(\ell+1) = 1$.

Assume that σ be the lexicographic ordering induced by $x_1 > \cdots > x_h$. Let $b = \binom{\ell+h-1}{\ell} - 1$. If $\operatorname{HF}_{R/I:n}(\ell) = 1$ then after a suitable change of basis we can assume that there exist $f_1, \ldots, f_b \in (I:\mathfrak{n}) \cap \mathfrak{n}^{\ell}$ that $\{L_{\bar{\sigma}}(f_1), \ldots, L_{\bar{\sigma}}(f_b)\} = \operatorname{Mon}(X, \ell) \setminus \{x_h^{\ell}\} \subset L_{\bar{\sigma}}(I:\mathfrak{n})$. Since $\mathfrak{n}\{f_1, \ldots, f_b\} \subset I$ one can easily see that $\operatorname{Mon}(X, \ell+1) \setminus \{x_h^{\ell+1}\} \subset L_{\bar{\sigma}}(I)$. So $\operatorname{HF}_{R/I}(\ell+1) = 1$.

Finally, we remark that if $HF_{R/I}(\ell) = 2$ for some ℓ then by Macaulay's theorem we conclude that $HF_{R/I}(\ell+1) \leq 2$.

Now if $\operatorname{HF}_{\mathrm{R/I}}$ is as equation (5.1) then by the above facts, $\operatorname{HF}_{\mathrm{R/I:n}}(t-1) \geq 2$, for each $t \leq \ell \leq r-1$ we have $\operatorname{HF}_{\mathrm{R/I:n}}(\ell) = 2$, $\operatorname{HF}_{\mathrm{R/I:n}}(\mathbf{r}) = 2$ or 1, and $\operatorname{HF}_{\mathrm{R/I:n}}(\ell) = 1$ for each $r < \ell \leq s-1$.

As the proof of Lemma 4.5, we can see that

$$e(R/I:\mathfrak{n}) = e(R/I) - \tau(R/I).$$

So if $\tau(R/I) = \binom{h+t-2}{t-1}$ then $\operatorname{HF}_{R/I}(t-1) - \operatorname{HF}_{R/I:\mathfrak{n}}(t-1) + \operatorname{HF}_{R/I}(r) - \operatorname{HF}_{R/I:\mathfrak{n}}(r) + 1 = \binom{h+t-2}{t-1}$. This shows that $\operatorname{HF}_{R/I:\mathfrak{n}}(t-1) + \operatorname{HF}_{R/I:\mathfrak{n}}(r) = 3$ and so $\operatorname{HF}_{R/I:\mathfrak{n}}(t-1) = 2$ and $\operatorname{HF}_{R/I:\mathfrak{n}}(r) = 1$. So, the Hilbert function of $R/I:\mathfrak{n}$ is given by table (5.5).

If $HF_{R/I}$ is as equation (5.2) then by a similar argument we can see that $HF_{R/I:n}$ is as (5.6).

In the following of this section we assume that $R = \mathbf{k}[[x_1, \ldots, x_h]]$. Our goal is to characterize Artinan local ring A of homogeneous type when A = R/I is of type (t, r, s).

Lemma 5.4. Let A = R/I be an Artinian local ring of type (t, r, s)or (t, s, s), then one can find a minimal system of generators $Y = \{y_1, \ldots, y_h\}$ of \mathfrak{n} such that I is generated by y_h^{s+1} , $y_{h-1}y_h^r + \sum_{i=r+1}^s \lambda_i y_h^i (\lambda_i \in \mathbf{k}),$ $y_j y_h^{t-1}$ $(1 \le j \le h-2),$ $u + \sum_{i=t-1}^{r-1} \lambda_{u,i} y_{h-1} y_h^i + \sum_{i=t}^s \gamma_{u,i} y_h^i$ $(u \in \operatorname{Mon}(Y, t) \setminus \{y_j y_h^{t-1} \mid 1 \le j \le h\}, \ \lambda_{u,i}, \gamma_{u,i} \in \mathbf{k}),$ where if A is of type (t, s, s) then r = s in the above generators.

Proof. Assume that σ be the lexicographic ordering induced by $x_1 > \cdots > x_h$.

Let A = R/I be an Artinian local ring with the Hilbert function (5.1) and set $X = \{x_1, \ldots, x_h\}$. By the shape of the Hilbert function of A we can assume that, after a generic change of variable,

$$L_{\bar{\sigma}}(I) = \langle \{x_{h-1}x_h^r, x_h^{s+1}\} \cup \operatorname{Mon}(\mathbf{X}, \mathbf{t}) \setminus \{\mathbf{x}_{h-1}\mathbf{x}_h^{t-1}, \mathbf{x}_h^t\} \rangle.$$

Thus the reduced standard basis of I has the following shape:

$$G(I) = \{x_h^{s+1}, x_{h-1}x_h^r + \sum_{i=r+1}^s \lambda_i x_h^i\} \cup \{u + f_u(x_{h-1}, x_h) : u \in \text{Mon}(\mathbf{X}, \mathbf{t}) \setminus \{\mathbf{x}_{h-1}\mathbf{x}_h^{t-1}, \mathbf{x}_h^t\}\},\$$
where $f_u(x_{h-1}, x_h) = \sum_{i=t}^s \lambda_{u,i} x_h^i + \sum_{i=t-1}^{r-1} \gamma_{u,i} x_{h-1} x_h^i$ for some $\lambda_i, \lambda_{u,i}, \gamma_{u,i} \in \mathbf{k}.$

For each $j = 1, \ldots, h-2$, let $u_j = x_j x_h^{t-1}$. Suppose $S = \mathbf{k}[[y_1, \ldots, y_h]]$ is a power series ring on the variables y_1, \ldots, y_h and $Y = \{y_1, \ldots, y_h\}$. Define the \mathbf{k} -algebra isomorphism $\phi : R \to S$ induced by the map which sends each x_j to $y_j - \sum_{i=t}^s \lambda_{u_j,i} y_h^{i-t+1} - \sum_{i=t-1}^{r-1} \gamma_{u_j,i} y_{h-1} y_h^{i-t+1}$ for $j = 1, \ldots, h-2$, $\phi(x_{h-1}) = y_{h-1}$ and $\phi(x_h) = y_h$. This isomorphism will send each $u \in G(I)$ to $\phi(u) \in J := \phi(I)$.

One can easily see that the set $\mathcal{A} = \{\phi(u) : u \in G(I)\}$ is a standard basis of J for the local lexicographic ordering on S. Starting from \mathcal{A} , it is straight forward to check that the reduced standard basis of J has

the following shape:

$$G(J) = \{y_{h-1}y_h^r + \sum_{i=r+1}^s \lambda_i y_h^i, y_h^{s+1}\} \cup \{u - g_u(y_{h-1}, y_h) : u \in Mon(Y, t) \setminus \{y_{h-1}y_h^{t-1}, y_h^t\}\} \text{ where}$$
$$g_u(y_{h-1}, y_h) = \sum_{i=t}^s \lambda_{u,i} y_h^i + \sum_{i=t-1}^{r-1} \gamma_{u,i} y_{h-1} y_h^i \text{ for some } \gamma_{u,j}, \lambda_{u,j} \in \mathbf{k}$$
and if $u = y_j y_h^{t-1}$ for some $j = 1, \dots, h-2$ then $g_u(y_{h-1}, y_h) = 0.$

So, if we let $y_j = x_j + \sum_{i=t}^s \lambda_{u_j,i} x_h^{i-t+1} + \sum_{i=t-1}^{r-1} \gamma_{u_j,i} x_{h-1} x_h^{i-t+1}$ for $1 \leq j \leq h-2, y_{h-1} = x_{h-1}$ and $y_h = x_h$, with respect to the basis $\{y_1, \ldots, y_h\}$ the ideal I has the desired generating set.

If R/I is of type (t, s, s), an easy modification of the above proof shows the result.

Theorem 5.5. Let A = R/I be an Artinian local ring of type (t, r, s) or (t, s, s) and maximal Cohen-Macaulay type. Then one can find a minimal system of generators $Y = \{y_1, \ldots, y_h\}$ of \mathfrak{n} such that I is minimally generated by y_h^{s+1} , $y_{h-1}y_h^r + \sum_{i=r+1}^s \lambda_i y_h^i$, $u \ (u \in \operatorname{Mon}(Y, t) \cap \langle y_1, \ldots, y_{h-2} \rangle)$,

 $\begin{array}{l} y_{h-1}y_{h}^{r} + \sum_{i=r+1}^{s} \lambda_{i}y_{h}^{i}, \\ u \; (u \in \operatorname{Mon}(\mathbf{Y}, \mathbf{t}) \cap \langle \mathbf{y}_{1}, \dots, \mathbf{y}_{\mathbf{h}-2} \rangle), \\ y_{h-1}^{\ell}y_{h}^{t-\ell} + \sum_{i=t-1}^{r-1} \lambda_{i,\ell}y_{h-1}y_{h}^{i} + \sum_{i=t}^{s} \gamma_{i,\ell}y_{h}^{i} \\ (2 \leq \ell \leq t, \; \text{for some } \lambda_{i,\ell}, \gamma_{i,\ell} \in \mathbf{k}), \\ where \; if \; \operatorname{HF}_{\mathbf{A}} \; is \; given \; by \; (5.2) \; then \; r = s \; in \; the \; above \; generators. \end{array}$

Proof. We prove theorem in the case that $HF_{R/I}$ is given by (5.1) because the other case can be proved by the same method.

Since $\tau(A)$ is the maximal one, Lemma 5.3 shows that $R/I : \mathfrak{n}$ is an Artinian local ring of type (t-1, r-1, s-1). By Lemma 5.4, one can find a minimal system of generators $Y = \{y_1, \ldots, y_h\}$ of \mathfrak{n} such that $I : \mathfrak{n}$ is generated by the following elements.

 $\begin{array}{l} y_{h}^{s}, \\ y_{h-1}y_{h}^{r-1} + \sum_{i=r}^{s-1} \lambda_{i}y_{h}^{i}, \\ y_{j}y_{h}^{t-2} \ (1 \leq j \leq h-2), \\ u + \sum_{i=t-2}^{r-2} \lambda_{u,i}y_{h-1}y_{h}^{i} + \sum_{i=t-1}^{s-1} \gamma_{u,i}y_{h}^{i} \\ (u \in \operatorname{Mon}(\mathbf{Y}, \mathbf{t}-1) \setminus \{\mathbf{y}_{j}\mathbf{y}_{h}^{\mathbf{t}-2} \mid 1 \leq j \leq h\}, \ \lambda_{\mathbf{u},i}, \gamma_{\mathbf{u},i} \in \mathbf{k}), \end{array}$

Since $\mathfrak{n}(I:\mathfrak{n}) \subseteq I$ we have $J_1 + J_2 \subseteq I$ where

$$J_1 = \langle u : u \in Mon(Y, t) \cap \langle y_1, \dots, y_{h-2} \rangle \rangle$$

and

$$J_{2} = \langle y_{h-1}, y_{h} \rangle \langle y_{h}^{s}, y_{h-1}y_{h}^{r-1} + \sum_{i=r}^{s-1} \lambda_{i}y_{h}^{i}, u + \sum_{i=t-2}^{r-2} \lambda_{u,i}y_{h-1}y_{h}^{i} + \sum_{i=t-1}^{s-1} \gamma_{u,i}y_{h}^{i} : u = y_{h-1}^{\ell}y_{h}^{t-1-\ell}, \ 2 \le \ell \le t-1 \rangle$$
$$= \langle y_{h}^{s+1}, y_{h-1}y_{h}^{r} + \sum_{i=r+1}^{s} \lambda_{i}y_{h}^{i}, y_{h-1}^{\ell}y_{h}^{t-\ell} + \sum_{i=t-1}^{r-1} \lambda_{i,\ell}'y_{h-1}y_{h}^{i} + \sum_{i=t-1}^{s} \gamma_{i,\ell}'y_{h}^{i} : 2 \le \ell \le t, \text{ for some } \lambda_{i}, \lambda_{i,\ell}', \gamma_{i,\ell}' \in \mathbf{k} \rangle.$$

But $L_{\bar{\sigma}}(I) = L_{\bar{\sigma}}(J_1) + L_{\bar{\sigma}}(J_2) \subseteq L_{\bar{\sigma}}(J_1 + J_2) \subseteq L_{\bar{\sigma}}(I)$. So $I = J_1 + J_2$. Since $\{u : u \in Mon(Y, t) \cap \langle y_1, \dots, y_{h-2} \rangle\}$ is a minimal system of generators for J_1 the proof will be completed if we show that

$$\{y_{h}^{s+1}, y_{h-1}y_{h}^{r} + \sum_{i=r+1}^{s} \lambda_{i}y_{h}^{i}, y_{h-1}^{\ell}y_{h}^{t-\ell} + \sum_{i=t-1}^{r-1} \lambda_{i,\ell}' y_{h-1}y_{h}^{i} + \sum_{i=t}^{s} \gamma_{i,\ell}' y_{h}^{i} : 2 \le \ell \le t, \ \lambda_{i}, \lambda_{i,\ell}', \gamma_{i,\ell}' \in \mathbf{k}\}$$

is a minimal system of generators for J_2 .

By $J_1 \subset I$, it is clear that $\langle Mon(Y, t-1) \cap \langle y_1, \dots, y_{h-2} \rangle \rangle \subseteq I : \mathfrak{n}$. So $I : \mathfrak{n} = J'_1 + J'_2$ where

$$J'_1 = \langle u : u \in Mon(Y, t-1) \cap \langle y_1, \dots, y_{h-2} \rangle \rangle$$

and

$$J_{2}' = \langle y_{h}^{s}, y_{h-1}y_{h}^{r-1} + \sum_{i=r}^{s-1} \lambda_{i}y_{h}^{i}, u + \sum_{i=t-2}^{r-2} \lambda_{u,i}y_{h-1}y_{h}^{i} + \sum_{i=t-1}^{s-1} \gamma_{u,i}y_{h}^{i} :$$
$$u = y_{h-1}^{\ell}y_{h}^{t-1-\ell}, \ 2 \le \ell \le t-1 \rangle.$$

So, as a \mathbf{k} -vector space we have:

$$I:\mathfrak{n}/I=(J_1'+I)/I+(J_2'+I)/I.$$

Comparing the **k**-vector space dimension of two sided of the above equation we see that $\dim_k(J'_2 + I/I) = t$.

Regarding J_2 and J'_2 as ideals of $\mathbf{k}[[y_{h-1}, y_h]]$, we have

$$J_2' \subseteq J_2 :_{\mathbf{k}[[y_{h-1}, y_h]]} \langle y_{h-1}, y_h \rangle.$$

Therefore, one can find a natural one to one linear map

$$g: (J_2' + I/I) \to (J_2:_{\mathbf{k}[[y_{h-1}, y_h]]} \langle y_{h-1}, y_h \rangle)/J_2.$$

So

$$\dim_k(J'_2 + I/I) \leq \dim_k((J_2 :_{\mathbf{k}[[y_{h-1}, y_h]]} \langle y_{h-1}, y_h \rangle)/J_2) \\ = \tau(\mathbf{k}[[y_{h-1}, y_h]]/J_2).$$

This means that $t \leq \mu(J_2) - 1$. Therefore

$$\{y_{h}^{s+1}, y_{h-1}y_{h}^{r} + \sum_{i=r+1}^{s} \lambda_{i}y_{h}^{i}, y_{h-1}^{\ell}y_{h}^{t-\ell} + \sum_{i=t-1}^{r-1} \lambda_{i,\ell}' y_{h-1}y_{h}^{i} + \sum_{i=t}^{s} \gamma_{i,\ell}' y_{h}^{i} : 2 \le \ell \le t, \ \lambda_{i}, \lambda_{i,\ell}', \gamma_{i,\ell}' \in \mathbf{k}\}$$

is a minimal system of generators for J_2 and the proof is completed.

Next we give some applications of Theorem 5.5 in finding Artinian local rings of homogeneous type.

Corollary 5.6. Let A = R/I be an Artinian local ring of type (t, r, s) where $t \le r \le s$ then the following statements are equivalent.

(1) $\beta_i(A) = \beta_i(P/I^*) = \beta_i(P/\text{Lex}(I))$ for each $i \ge 0$. (2) $\mu(I) = \binom{h+t-1}{t}$. (3) $\tau(A) = \binom{h+t-2}{t-1}$.

In particular, if one of the above equivalent conditions holds then A is of homogeneous type.

Proof. $(2) \rightarrow (1)$ is followed by [9, Corollary 3.4]. $(1) \rightarrow (3)$ is clear and $(3) \rightarrow (1)$ is an immediate corollary of Theorem 5.5.

Corollary 5.7. Let A = R/I be an Artinian local ring of type (t, s, s) where $t \leq s$ then A is of homogeneous type if and only if one of the following conditions holds.

(1)
$$t = s$$
.

(2)
$$\tau(A) = \binom{h+t-2}{t-1}$$
.

Proof. If s = t then the conclusion follows by [10, Corollary 4.3]. If t < s then Lex(I) is generated in two non-successive degrees. So there is no zero cancellation in the first Betti number of P/Lex(I). So $\mu(I^*) = \binom{h+t-1}{t}$ and the result follows by Corollary 5.6.

Corollary 5.8. Let A = R/I be an Artinian local ring of type (t, r, s) where t < r < s - 1 then A is of homogeneous type if and only if $\tau(A) = \binom{h+t-2}{t-1}$.

Proof. It is enough to remark that Lex(I) is generated in degrees t, r + 1, s + 1. So there is no zero cancellation in the first Betti number of P/Lex(I) and $\mu(I^*) = \binom{h+t-1}{t}$. Now the result follows by Corollary 5.6.

Finally we remark that if A = R/I is an Artinian local ring of type (t,r,s) where $t \leq r \leq s$ and $\tau(A) < \binom{h+t-2}{t-1}$ then $\binom{h+t-1}{t} - 2 \leq \mu(I) \leq \binom{h+t-1}{t} - 1$. In this case, $\mu(I)$ is not uniquely determined by $\tau(A)$. For example let $R = \mathbf{k}[[x,y,z]]$. Then for both ideals $I = \langle x^2 + z^4, xy, xz, y^2, yz^3 \rangle$ and $J = \langle x^2 + z^4, xy + yz^2, xz, y^2 \rangle$, the rings A = R/I and B = R/J are of type (2,3,4) and $\tau(A) = \tau(B) = 2$. But $\mu(I) = 5$ and $\mu(J) = 4$.

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