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REALIZATION OF A CERTAIN CLASS OF SEMI-GROUPS AS VALUE SEMI-GROUPS OF VALUATIONS

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Abstract. Given a well-ordered semi-group Γ with a minimal system of generators of ordinal type at most ωn^1 and of rational rank r, which satisfies a positivity and increasing condition, we construct a zero-dimensional valuation centered on the ring of polynomials with r variables such that the semi-group of the values of the polynomial ring is equal to Γ. The construction uses a generalization of Favre and Jonsson's version of MacLane's sequence of key-polynomials [3].

1. Introduction

Recently, the interest for studying the structure of the value semi-groups of the valuations centered on a noetherian local-ring has increased (see, for example, [2]). Several examples (e.g., plane branches, irreducible quasi-ordinary hypersurface singularities) suggest that the structure of these semi-groups contains important information on the local uniformization process of the valuation. What type of semi-groups can be realized as the semi-group of values of a noetherian local ring dominated by a valuation ring? Little is known in this respect. We

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know they are well-ordered of ordinal type $< \omega^h$, for some natural number h ([15], Appendix 3, Proposition 2). Abhyankar's inequality holds between numerical invariants of these valuations (see below). And, such semi-groups have no accumulation point when they are considered as semi-groups of $(\mathbb{R}^n, <_{lex})$ [2].

Here, we show that given a semi-group Γ of rational rank r, with a given minimal system of generators which is well-ordered of ordinal type at most ωn, $n \in \mathbb{N}$, which satisfies a positivity and increasing condition (Definition 2.2 and Theorem 7.1), there is a polynomial ring $R = k[X_1, \ldots, X_r]$, where k is an arbitrary field, and a valuation ν, which is positive on R, such that the value semi-group $\nu(R \setminus \{0\})$ is equal to Γ.

Our basic tool is a generalization of Favre and Jonsson's version of MacLane's sequence of key-polynomials ([3], [7]) for polynomial rings with arbitrary number of variables. The technique of sequences of key-polynomials was first invented by MacLane [7], following ideas of Ostrowski, to produce and describe all the extensions of a discrete rank one valuation ν of a field K to the extension field $L = K(x)$. He attached to any extension, say μ, of the valuation ν, a sequence of polynomials $\phi_i(x)$ of the ring $K[x]$. By induction, one can produce any extension μ to L of the valuation ν using valuations constructed by key-polynomials (augmented valuations). In [13], Vaquié generalized MacLane's method to produce all the extensions of an arbitrary valuation of an arbitrary field K to L. He showed that given such an extension of a valuation, there may be many ways to produce such countable well-ordered sets of key-polynomials and augmented valuations. Later, Favre and Jonsson showed that in the case of $d = 1$ one can consider a rather simple sequence of toroidal key-polynomials (SKP), to produce all the pseudo-valuations centered on the ring $k[[X_0, X_1]]$. Using the arithmetic of the sequence of key-polynomials of the extension μ of the valuation ν, in [14], Vaquié defined a new invariant, called total jump (saut total). In the case where $L = K[x]$ and x are algebraic over K, he gives a formula relating total jump to the classical invariants of the valuation extensions. In [5], the construction of key-polynomials is generalized for the case where L is an arbitrary algebraic extension of K (not necessarily of the form $K[x]$). They give an explicit description of the construction of key-polynomials of the valuation extension (L, μ) of (K, ν). There are several constructions in [5] which are analogous to the present work, for example the notion of *standard monomial* and *standard expansion* corresponds to the *monomial of adic form* and *adic expansion*, respectively, in our terminology.

We give a generalization of the sequence of toroidal key-polynomials of [3] to produce a class of valuations of the field $k((X_0, \ldots, X_d))$, where k is an arbitrary field. Our generalization cannot generate all the valuations centered at $k[[X_0, \ldots, X_d]]$. The construction is explicit enough to describe the value semi-group $\nu(k[[X_0, \ldots, X_d]] \setminus \{0\})$. And, in addition, to realize certain semi-groups as value semi-groups.

Here, we recall the basic definitions associated with valuations.

Definition 1.1. Fix a valuation ν.

- The rank $\mathrm{rk}(\nu)$ of ν, is the Krull dimension of the valuation ring R_ν.
- The rational rank of ν, $\mathrm{r.rk}(\nu)$, is the dimension of the vector space $\nu(\mathrm{Frac}(R_\nu)^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ over the field $\mathbb{Q}$.
- The transcendence degree of ν, $\mathrm{tr.deg}(\nu)$, is the transcendence degree of the extension of k over residue field of ν, $k \subseteq k_\nu := \frac{R_\nu}{\mathfrak{m}_\nu}$.

The principal relation between these numerical invariants is given by Abhyankar's inequalities:

$$\mathrm{rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \mathrm{r.rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \dim R.$$

Moreover, if $\mathrm{r.rk}(\nu) + \mathrm{tr.deg}(\nu) = \dim R$, then value group is isomorphic (as a group) to $\mathbb{Z}^{\mathrm{r.rk}(\nu)}$. When $\mathrm{rk}(\nu) + \mathrm{tr.deg}(\nu) = \dim R$, then the value group is isomorphic as an ordered group to $\mathbb{Z}^{\mathrm{rk}(\nu)}$, endowed with the lex. order.

Let R be an integral domain with field of fractions K and let ν be a valuation of K such that its valuation ring R_ν contains R. In this case, we say the valuation is centered on the ring R. Let us denote by Φ the totally ordered value group of the valuation ν. Denote by Φ_+ the semigroup of positive elements of Φ and set $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \cup \{0\}$; it is the semigroup of (R, ν). Since Γ generates the group Φ, then it is cofinal in the ordered set Φ_+.

For $\phi \in \Phi$, set

$$\mathcal{P}_\phi(R) = \{x \in R \mid \nu(x) \geq \phi\}$$
$$\mathcal{P}^+_\phi(R) = \{x \in R \mid \nu(x) > \phi\},$$

where we agree that $0 \in \mathcal{P}_\phi$, for all ϕ, since its value is larger than any ϕ, so that by the properties of valuations, the $\mathcal{P}_\phi$ are ideals of R. Note that $\bigcap_{\phi\in\Phi_+}\mathcal{P}_\phi=(0)$ and that if ϕ is in the negative part Φ_- of Φ, then $\mathcal{P}_\phi(R)=\mathcal{P}^+_\phi(R)=R$.

For $\phi\notin\Gamma$, $\mathcal{P}_\phi(R)=\mathcal{P}^+_\phi(R)$. For each non zero element $x\in R$, there is a unique $\phi\in\Gamma$ such that $x\in\mathcal{P}_\phi\setminus\mathcal{P}^+_\phi$; the image of x in the quotient $(\mathrm{gr}_\nu R)_\phi=\mathcal{P}_\phi/\mathcal{P}^+_\phi$ is the *initial form* $\mathrm{in}_\nu(x)$ of x.

The graded algebra associated with the valuation ν was introduced in [6, 11] for the very special case of a plane branch (see [4]), and in [10] in full generality. Later it was extensively used in [12] as a tool to solve the local-uniformization problem. It is:

$$\mathrm{gr}_\nu R=\bigoplus_{\phi\in\Gamma}\mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R).$$

2. The inductive definition of the SKP

From now on, by Φ we mean a totally ordered abelian group of rank $d+1$. The total ordering of Φ is denoted by $<$. Let $\Delta_0=(0)\subset\cdots\subset\Delta_{d+1}=\Phi$ be its sequence of isolated subgroups (see [15]). We define the sequence of pre-values and the sequence of values of positive type. Associated with a sequence of values of positive type there exists a sequence of key-polynomials (SKP) which are elements of the power series ring $k^{(d)}=k[[X_0,\ldots,X_d]]$[2]. First, we need a general lemma on abelian groups.

Lemma 2.1. *Let Ψ be an abelian group, α be an ordinal number and $\Gamma=\{\gamma_0,\gamma_1,\ldots,\gamma_\alpha\}$ be a well-ordered sequence of elements of Ψ. For any ordinal $i\le\alpha$, define the subgroups of Ψ, $G_i=(\gamma_j)_{j\le i}$[3], $G_{i^-}=(\gamma_j)_{j<i}$, $n_i=[G_i:G_{i^-}]$, and set $n_0=\infty$. Then, for any $i\le\alpha$ such that $n_i\ne\infty$, we have a unique representation,*

$$n_i\gamma_i=\sum_{j<i}m_j\gamma_j, \tag{2.1}$$

where $0\le m_j<n_j$, when $n_j\ne\infty$, and $m_j\in\mathbb{Z}$, when $n_j=\infty$, and $m_j=0$ except for a finite number of j. More generally, every element of G_{i^-} can be written uniquely in the form (2.1).

[2]For any $i\le d$, we define $k^{(i)}=k[[X_0,\ldots,X_i]]$ and $k_{(i)}=k((X_0,\ldots,X_i))$.

[3]If $a_1,\ldots,a_n$ are elements of a group G, by $(a_1,\ldots,a_n)$, we denote the subgroup generated by these elements and by $\langle a_1,\ldots,a_n\rangle$, the semigroup generated by them.

Proof. Let $i \leq \alpha$ and $n_i \neq \infty$. By definition of n_i, we have $n_i\gamma_i \in G_{i^-}$. Thus, there exists a representation $n_i\gamma_i = \sum_{j<i} p_j\gamma_j$, where $p_j \in \mathbb{Z}$, and $p_j = 0$, except for a finite number of j. We define, inductively, a sequence $A : N' \subset \mathbb{N} \to \{1, \ldots, \alpha\}$ of elements of the index set α, as follows.

Let $j_0 < i$ be the greatest ordinal number such that $n_{j_0} \neq \infty$ and $p_{j_0} \neq 0$. The ordinal j_0 exists, since there is only a finite number of non-zero p_j. Set $A(0) = j_0$. Using Euclidean division, write $p_{j_0} = q_{j_0}n_{j_0}+r_{j_0}$, where $0 \leq r_{j_0} < n_{j_0}$. Substituting this for p_{j_0}, and expanding $n_{j_0}\gamma_{j_0}$ in terms of elements of $G_{j_0^-}$, we get $n_i\gamma_i = \sum_{j<j_0} p'_j\gamma_j + r_{j_0}\gamma_{j_0}$, where $p'_j \neq 0$, except for a finite number of j. Now, as before, let $j_1(< j_0)$ be the first ordinal number such that $n_{j_1} \neq \infty$ and $p'_{j_1} \neq 0$. Set $A(1) = j_1$ and continue as before to obtain $n_i\gamma_i = \sum_{j<j_1} p''_j\gamma_j + r_{j_1}\gamma_{j_1} + r_{j_0}\gamma_{j_0}$, where $0 \leq r_j < n_j$. Continue this construction.

Either this construction stops after a finite number of steps, say j_k, and then we have $n_i\gamma_i = \sum_{j<i} m_j\gamma_j$ such that $m_j = 0$, except for a finite number of j, and $0 \leq m_j < n_j$, when $n_j \neq \infty$. This shows the existence part of the claim in this case. Or, the construction continues forever, in which case we get a strictly decreasing sequence $A : \mathbb{N} \to \alpha$. But this is impossible: It suffices to note that $A(\mathbb{N})$ is a subset of α without least element, which is impossible (as α is well-ordered). Thus, we have proved the existence part of the claim.

For the uniqueness, if we have two such representations, $n_i\gamma_i = \sum_{j<i} m_j\gamma_j = \sum_{j<i} m'_j\gamma_j$, then let j_0 be the greatest index such that $m_{j_0} \neq m'_{j_0}$ (as the number of nonzero m_j and m'_j is finite, this greatest index exists). Suppose $m_{j_0} > m'_{j_0}$. Then, $(m_{j_0} - m'_{j_0})\gamma_{j_0} = \sum_{j<j_0}(m'_j - m_j)\gamma_j \in G_{j_0^-}$, which is a contradiction, because $0 \leq m_{j_0} - m'_{j_0} < n_{j_0}$. □

Definition 2.2. With the notation of Lemma 2.1, we say the sequence Γ *is of positive type in* the group Ψ if for any i, we have all $m_j \in \mathbb{N}$.

This positivity condition implies that for all i, γ_i is in the positive cone generated by the previous γ's. However, the converse of this is not necessarily true. This condition enables us to construct our key-polynomials as binomials in terms of previous key-polynomials (Definition 2.4).

Definition 2.3. A sequence $(\beta_{i,j} \in \Phi)_{i=0..d,j=1..\tilde{\alpha}_i}$ [4], where $\tilde{\alpha}_i$ an ordinal number and $\tilde{\alpha}_0 = 1$, is called a *sequence of pre-values* if for any i and j we have,

- $\beta_{i,j+1} > n_{i,j}\beta_{i,j}$, where $n_{i,j} = \min\{r \in \mathbb{N} \cup \{\infty\} : r\beta_{i,j} \in (\beta_{i',j'})_{(i',j')<_{lex}(i,j)}\}$.
- $n_{i,j} \neq \infty$, for $j < \tilde{\alpha}_i$.
- If j is a limit ordinal, then $\beta_{i,j} > \beta_{i,j'}$, for any $j' < j$.

Consider the index set $\{(i,j)\}_{i=0..d,j=1..\tilde{\alpha}_i}$, ordered by the *lex*. ordering. As, $\tilde{\alpha}_i$ are ordinals, this is a well ordering. According to Lemma 2.1, when $n_{i,j} \neq \infty$, there exists a unique representation,

$$n_{i,j}\beta_{i,j} = \sum_{(i',j') \in S_{i,j} \cup S^c_{i',j'}} m^{(i,j)}_{i',j'} \beta_{i',j'}, \tag{2.2}$$

where $m^{(i,j)}_{i',j'} = 0$, except for a finite number of $(i',j') <_{lex} (i,j)$, $S_{i,j} = \{(i',j') \mid (i',j') <_{lex} (i,j),\ m^{(i,j)}_{i',j'} > 0\}$, and $S^c_{i,j} = \{(i',j') \mid (i',j') <_{lex} (i,j),\ m^{(i,j)}_{i',j'} < 0\}$. By Lemma 2.1, we have $0 \leq m^{(i,j)}_{i',j'} < n_{i',j'}$ if $n_{i',j'} \neq \infty$, and $m^{(i,j)}_{i',j'} \in \mathbb{Z}$ if $n_{i',j'} = \infty$. Thus, if $(i',j') \in S^c_{i,j}$ then $n_{i',j'} = \infty$ and, by definition of pre-values, we have $j' = \tilde{\alpha}_{i'}$.

Let $\Gamma = (\beta_{i,j} \in \Phi)_{i=0..d,j=1..\tilde{\alpha}_i}$, ordered by *lex* ordering, be a sequence of pre-values. Let $\Phi_{d,\tilde{\alpha}_d}$ be the group generated by these elements. We say Γ is a *sequence of values* if it is of positive type in $\Phi_{d,\tilde{\alpha}_d}$. This condition is equivalent to $S^c_{i,j} = \emptyset$, for any i and j.

Definition 2.4. (**SKP**) Given a sequence of values $\Gamma = (\beta_{i,j} \in \Phi)_{i=0..d,j=1..\tilde{\alpha}_i}$, we associate with Γ a sequence of power series $(U_{i,j} \in k^{(d)})_{i=0..d,j=1..\alpha_i}$, $\alpha_i \leq \tilde{\alpha}_i$. It is called the *sequence of key-polynomials* of the sequence of values Γ. It is defined by induction on i. For $i = 0$, we set $\alpha_0 = \tilde{\alpha}_0 = 1$ and $U_{0,1} = X_0$. Suppose $U_{i',j'}$ and $\alpha_{i'}$ are defined for $i' < i$. We set $U_{i,1} = X_i$. Suppose $U_{i,j'}$ are defined for $j' < j$. Then, we define $U_{i,j}$ as follows:

(P1) If j is not a limit ordinal, then

$$U_{i,j} = U^{n_{i,j-1}}_{i,j-1} - \theta_{i,j-1} \prod_{(i',j') \in S_{i,j-1}} U^{m^{(i,j-1)}_{i',j'}}_{i',j'}, \tag{2.3}$$

[4]By $i = 0..d$ and $j = 1..\tilde{\alpha}_i$ we mean $i = 0, \ldots, d$ and $j = 1, \ldots, \tilde{\alpha}_i$.

where $\theta_{i,j} \in k^*$. This can be written as:

$$U_{i,j} = U_{i,j-1}^{n_{i,j-1}} - \theta_{i,j-1} U^{m^{(i,j-1)}}.$$

(P2) If j is a limit ordinal, then

$$U_{i,j} = \lim_{j' \to j} U_{i,j'} \in k^{(i-1)}[[X_i]].$$

In Proposition 2.11, we prove that this limit exists in the ring $k^{(i-1)}[X_i]$. If this limit is equal to zero, then we set $\alpha_i = j$, $\beta_{i,j} = \infty$, and we stop the construction of the key-polynomials at this step, for i. Otherwise, we continue to construct $U_{i,j'}$ for $j' > j$.

If the construction of the $U_{i,j}$ continues for every $j \le \tilde{\alpha}_i$, then we set $\alpha_i = \tilde{\alpha}_i$.

We denote an SKP by $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$.

Remark 2.5. The following remarks are in order.

(i) Given any sequence of key-polynomials as above, if we consider the data $[U_{i,j}, \beta_{i,j}]_{i=0,1,j=1..\alpha_i}$, then it is a $\Gamma-$SKP for the ring $k[[X_0, X_1]]$ in the sense of [3] for the group $\Gamma = \Phi$.

(ii) The formula of (P1) can be rewritten in the following way:

$$U_{i,j+1} = U_{i,j}^{n_{i,j}} - \theta_{i,j} U_0^{m_0^{(i,j)}} U_1^{m_1^{(i,j)}} \cdots U_{i-1}^{m_{i-1}^{(i,j)}} (U_{i,1}^{m_{i,1}^{(i,j)}} \cdots U_{i,j-1}^{m_{i,j-1}^{(i,j)}}),$$

where, $U_{i'}^{m_{i'}^{(i,j)}} = \prod_{j' \le \alpha_{i'}} U_{i',j'}^{m_{i',j'}^{(i,j)}}$, for $i' = 0..i-1$.

(iii) For a fixed i, when α_i is a limit ordinal:

- If there exists an infinite number of j such that $n_{i,j} > 1$, then we have,
  - $\deg_{X_i}(U_{i,j}) \to \infty$ $(j \to \alpha_i)$.
  - We have $U_{i,\alpha_i} = \lim_{j \to \alpha_i} U_{i,j} = 0$ (Lemma 2.10.(ii)).
- Otherwise, (we denote this case by writing $U_{i,\alpha_i} \neq 0$), we have,
  - $n_{i,j} = 1$, except for a finite number of ordinals j.
  - There is some ordinal j_0 such that $\deg_{X_i}(U_{i,\alpha_i}) = \deg_{X_i}(U_{i,j})$ and $n_{i,j} = 1$, for all $j > j_0$.

(iv) For any limit ordinal $j < \alpha_i$, there are only finitely many $j' < j$ such that $n_{i,j'} > 1$: Suppose the contrary, and let $j < \alpha$ be an ordinal such that there is an infinitely many $j' < j$ such that $n_{i,j'} > 1$. The argument of the proof of Lemma 2.10(ii) shows that $U_{i,j} = 0$. Thus, by construction of SKP, we must have $j = \alpha_i$, which is a contradiction.

(v) Given an SKP and $d' \leq d-1$, we have $(U_{i,j} \in k^{(d')})_{i=0..d',j=1..\alpha_i}$. Moreover, the data $[U_{i,j}, \beta_{i,j}]_{i=0..d',j=1..\alpha_i}$ is an SKP for the sequence of values $\Gamma' = (\beta_{i,j} \in \Phi)_{i=0..d',j=1..\tilde{\alpha}_i}$.

Definition 2.6. Let $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$ be an SKP. We define the semigroups $\Gamma_{i,j}$ and the groups $\Phi_{i,j}$, for $i = 0, \ldots, d, j = 1, \ldots, \alpha_i$, as follows:

$$\Gamma_{i,j} = \langle \beta_{i',j'} \rangle_{(i',j') \leq_{lex} (i,j)},$$
$$\Phi_{i,j} = (\Gamma_{i,j}),$$
$$\Phi^*_{i,j} = \Phi_{i,j} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Definition 2.7. Consider a power series ring $A = k^{(i)}$. The order of an element $M = \sum_{\mathbf{m}} c_{\mathbf{m}} X^{\mathbf{m}}$ of this ring is $\mathrm{ord}_A(M) = \mathrm{ord}(M) = \min_{\mathbf{m}, c_{\mathbf{m}} \neq 0} \{\sum_{q=0}^{i} \mathbf{m}_q\}$.

Let $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$ be an SKP. Fix an $i \leq d$. Consider the abelian ordered group Φ_{i,α_i}. This group is order isomorphic to a subgroup of the ordered group $(\mathbb{R}^n, <_{lex})$, for some large enough n (see [1], Proposition 2.10).

Let us fix such an embedding and suppose α_i is a limit ordinal. Consider the first index $t \leq d$, such that $\#\{(\beta_{i,j})_t\}_{1 \leq j < \alpha_i} = \infty$. The index t is independent of the choice of an ordered embedding of Φ_{i,α_i} into $\mathbb{R}^n$; it is called the *effective component for* i . Notice that this t exists, since otherwise, we have $\#\{(\beta_{i,j})_t\}_{1 \leq j < \alpha_i, t=1..n} < \infty$. On the other hand, we have $\beta_{i,1} <_{lex} \beta_{i,2} <_{lex} \cdots <_{lex} \beta_{i,\alpha_i}$. But this is impossible when all the components of the β_i come from a finite set. Thus, t is well-defined. In [2], it is shown that well-ordered semi-groups of ordinal type $\leq \omega^h$, $h \in \mathbb{N}$, have no accumulation points in $\mathbb{R}^n$, in Euclidean topology. We show that the semi-groups of positive type have a stronger property: The effective component of any sequence of the elements of the semi-group tends to infinity (Lemma 2.9, and Lemma 7.3)

Proposition 2.8. *With the notation of the last paragraph, we have:*

(*i*) *There exists $j_{(i)}$, $1 \leq j_{(i)} < \alpha_i$, such that the first $(t-1)$ components of $\beta_{i,j}$ are the same (componentwise), for $j \geq j_{(i)}$; i.e., $(\beta_{i,j})_{t'} = (\beta_{i,j'})_{t'}$, for $j, j' \geq j_{(i)}$ and $t' < t$.*

(*ii*) *For $j > j' > j_{(i)}$, we have $(\beta_{i,j})_t \geq (\beta_{i,j'})_t$.*

(*iii*) *If $U_{i,\alpha_i} = 0$ then:*

(1) $t = \min\{t' |\ 1 \leq t' \leq n,\ \exists j < \alpha_i :\ (\beta_{i,j})_{t'} \neq 0\}$.

(2) $(\beta_{i,j})_{t'} = 0$, *for any $j < \alpha_i$ and $t' < t$.*

(3) $(\beta_{i,j})_t \to +\infty\ (j \to \alpha_i)$.

Proof. (i) is a direct consequence of the definition. For (ii), by definition of the SKP, we have $\beta_{i,j} >_{lex} \beta_{i,j'}$. On the other hand, by (i), the first $t-1$ components of $\beta_{i,j}$ and $\beta_{i,j'}$ are the same. Thus, $(\beta_{i,j})_t \geq (\beta_{i,j'})_t$.

For (iii), set $t_1 = \min\{t' |\ 1 \leq t' \leq n,\ \exists j < \alpha_i :\ (\beta_{i,j})_{t'} \neq 0\}$. By definition of t_1, we have $(\beta_{i,j})_{t'} = 0$, for any $j < \alpha_i$ and $t' < t_1$. So, $t_1 \leq t$. From the definition of the SKP, we deduce that $\beta_{i,j+1} >_{lex} (\prod_{j_0 \leq j' \leq j} n_{i,j'})\beta_{i,j_0}$. We choose j_0 such that $(\beta_{i,j_0})_{t_1} \neq 0$ (note that necessarily $(\beta_{i,j_0})_{t_1} > 0$). As $U_{i,\alpha_i} = 0$, there is an infinite number of $j > j_0$ such that $n_{i,j} > 1$ $(j \to \alpha_i)$. This shows that $(\beta_{i,j})_{t_1} \to \infty$ $(j \to \alpha_i)$. Thus, $t = t_1$. □

Lemma 2.9. *Let $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$ be an* SKP. *Fix an $i \leq d$ and let t be the effective component for i. If α_i is a limit ordinal, then $(\beta_{i,j})_t \to +\infty\ (j \to \alpha_i)$.*

Proof. If $U_{i,\alpha_i} = 0$, then the claim is the content of Proposition 2.8 (iii). Assume $U_{i,\alpha_i} \neq 0$. Then, by definition of $U_{i,\alpha_i} \neq 0$, there exists j_0 such that $n_{i,j} = 1$ for $j > j_0$. Notice that in this case there is a finite number of j (in general) such that $n_{i,j} \neq 1$ (by definition of $U_{i,\alpha_i} \neq 0$). And we have, $(\beta_{i,j})_t = \sum_{(i',j') \in S_{i,j}} m^{(i,j)}_{i',j'} (\beta_{i',j'})_t$, for $j > j_0$. Define,

$$C_i = \{(i', j') \in S_{i,j},\ \max\{j_0, j_{(i)}\} \leq j < \alpha_i,\ (\beta_{i',j'})_t \neq 0\}.$$

If $\#C_i = \infty$, then there exists some $i_0 < i$ and an infinite number of j' such that $(i_0, j') \in C_i$, and so we can speak of $j' \to \infty$. For such (i_0, j') (which are infinite in number), we have $n_{i_0,j'} > 1$, and hence α_{i_0} is a limit ordinal and $U_{i_0,\alpha_{i_0}} = 0$. Let t' be the effective component for i_0. By definition of C_i, there is at least one j' such that $(\beta_{i_0,j'})_t \neq 0$. But $U_{i_0,\alpha_{i_0}} = 0$, and thus by Proposition 2.8$(iii)(2)$, we have $t' = t$. As $(\beta_{i_0,j'})_t \to \infty$ $(j' \to \infty)$, we have $(\beta_{i,j})_t \to \infty$ $(j \to \alpha_i)$.

If $\#C_i < \infty$, then the $(\beta_{i,j})_t$ are elements of the discrete lattice $L \subset \mathbb{R}$, generated by the finite set of generators $\{(\beta_{i',j'})_t |\ (i', j') \in C_i\}$. Thus, as any bounded region of $\mathbb{R}$ contains only a finite number of elements of the lattice L, the sequence $(\beta_{i,j})_t$ $(j \to \alpha_i)$ cannot be contained in any bounded region of $\mathbb{R}$. On the other hand, by Proposition 2.8(ii), this sequence is increasing, and so it goes to $+\infty$. $\square$

Lemma 2.10. *Consider an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$. *Suppose α_i is a limit ordinal. Then, we have the following:*

(i) *For any $n \in \mathbb{N}$ and $i < d$, there exists an ordinal $j_n^{(i)}$ such that* $\mathrm{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) > n$, *for any $j > j_n^{(i)}$.*

(ii) *If $U_{i,\alpha_i} = 0$, then one can choose the above $j_n^{(i)}$ such that in addition,* $\mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i,j}) > n$, *for any $j > j_n^{(i)}$.*

Proof. Suppose both (i) and (ii) are proved for any n' and $i' < i$, and also for $n' \leq n$ and i, and notice the result holds for $n = 0$. We prove them for $n+1$ and i. Suppose t is the effective component for i. For any vector $V \in \mathbb{R}^n$, we define $|V|$ to be its tth component; i.e., $|V| = (V)_t$. Let

$$M^* = \max\{\{|\beta_{i',j'}| : \ (i',j') \in S_{i,j}, \ j' \leq j_{n+1}^{(i')} \text{ if } i' < i, j' \leq j_n^{(i)} \text{ if } i' = i\}.$$

Notice that the cardinality of this set is finite, and so M^* is well-defined.

(i):

By Lemma 2.9, we have $|\beta_{i,j}| \to +\infty$ $(j \to \alpha_i)$. Hence, there exists $j_{n+1}^{(i)}$ such that $|\beta_{i,j}| > (n+1)M^*$, for $j \geq j_{n+1}^{(i)}$. The claim is that the number $j_{n+1}^{(i)}$ works. We can assume $j_{(i)} < j_n^{(i)}$ (see Proposition 2.8(ii)). Suppose $j > j_{n+1}^{(i)}$.

If there exists at least one $(i, j') \in S_{i,j}$ such that $j' \geq j_n^{(i)}$ then we are done. Indeed, if $m_{i,j'}^{(i,j)} > 1$, since $\mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i,j'}) > n$ (by induction assumption for (ii), in the case n), then $\mathrm{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) > nm_{i,j'}^{(i,j)} > n+1$. If $m_{i,j'}^{(i,j)} = 1$, since $|\beta_{i,j'}| < |\beta_{i,j}|$ (because $n_{i,j'} > 1$ and $\beta_{i,j} >_{lex} n_{i,j'}\beta_{i,j'}$, and $|.|$ preserves ordering for $j'' > j_{(i)}$), then there should be at least another element $(i'', j'') \in S_{i,j}$. But, $\mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i'',j''}) \geq 1$. Therefore, we have

$$\mathrm{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) > \mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i,j'}) + \mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i'',j''}) > n+1.$$

If there exists some $(i', j') \in S_{i,j}$ such that $i' < i$ and $j' > j_{n+1}^{(i')}$, then clearly we are done.

It remains the case that for all $(i',j') \in S_{i,j}$:

- If $i' < i$, then $j' < j_{n+1}^{(i')}$.
- If $i' = i$, then $j' < j_n^{(i)}$.

By definition of M^* and conditions above, we have $|\beta_{i',j'}| < M^*$, for any $(i',j') \in S_{i,j}$. Hence,

$$|\beta_{i,j_{n+1}^{(i)}}| \leq |\beta_{i,j}| \leq n_{i,j}|\beta_{i,j}| = \sum_{(i',j')\in S_{i,j}} m_{i',j'}^{(i,j)}|\beta_{i',j'}| < (\sum_{(i',j')\in S_{i,j}} m_{i',j'}^{(i,j)})M^*,$$

where the first inequality holds, because $|.|$ preserves ordering for $j' \geq j_n^{(i)} > j_{(i)}$ (Proposition 2.8(ii)).

But, by definition of M^*, we have $|\beta_{i,j_{n+1}^{(i)}}| > (n+1)M^*$. Thus, $n+1 < \sum_{(i',j')\in S_{i,j}} m_{i',j'}^{(i,j)}$. Finally,

$$\mathrm{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) \geq \sum_{(i',j')\in S_{i,j}} m_{i',j'}^{(i,j)} > n+1.$$

(ii):

As (i) holds for $n+1$ and using the induction assumption, we can find $j_{n+1}^{(i)}$ such that $\mathrm{ord}(U_{i,j}) > n$, and $\mathrm{ord}(U^{m^{(i,j)}}) > n+1$, for $j > j_{n+1}^{(i)}$. If this $j_{n+1}^{(i)}$ does not work for (ii), then find the first $j_0 > j_{n+1}^{(i)}$ such that $n_{i,j_0} \neq 1$ (as $U_{i,\alpha_i} = 0$, this j_0 exists) and set $j_{n+1}^{(i)} := j_0$. It is straightforward to check that this new $j_{n+1}^{(i)}$ works also for (ii). □

Proposition 2.11. *Fix an* SKP $[U_{i,j},\beta_{i,j}]_{i=0..d,j=1..\alpha_i}$. *Then, for any* (i,j), *we have* $U_{i,j} \in k^{(i-1)}[X_i]$.

Proof. The proof is by induction on i and j. For $i=0$, it is obvious. Suppose it is valid for indices less than i, and prove it for i. When j is not a limit ordinal, formula (P1) represents $U_{i,j}$ as a polynomial in terms of previous U's and the claim is obvious in this case by induction on j.

It remains the case when j is a limit ordinal. We can assume that $j = \alpha_i$ (considering the SKP $[U_{i',j'},\beta_{i',j'}]_{i'=0..j',j'=1..\alpha'_{i'}}$, where $\alpha'_{i'} = \alpha_{i'}$ for $i' < i$ and $\alpha'_i = j$). We must show that $\lim_{j'\to\alpha_i} U_{i,j'} \in k^{(i-1)}[X_i]$.

If there is an infinite number of j such that $n_{i,j} > 1$, then by Lemma 2.10(ii), we have $U_{i,\alpha_i} = 0 \in k^{(i-1)}[X_i]$. Thus, we can assume $n_{i,j} = 1$, except for a finite number of j. Then, by Lemma 2.10(i), we have

$\operatorname{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) \to \infty$ $(j \to \alpha_i)$. By Remark 2.5(iii), we have $\deg_{X_i}(U^{m^{(i,j)}})$ is bounded. Hence, $\operatorname{ord}_{k^{(i-1)}}(U^{m^{(i,j)}}) \to \infty$ $(j \to \alpha_i)$. Using this fact and the equality $U_{i,j+1} - U_{i,j} = -\theta_{i,j}U^{m^{(i,j)}}$, for $j \geq j_0$ (where $n_{i,j} = 1$, for $j \geq j_0$), we have,

$$\lim_{j \to \alpha_i} U_{i,j} = U_{i,j_0}^{n_{i,j_0}} - \sum_{j, j_0 \leq j < \alpha_i} \theta_{i,j} U^{m^{(i,j)}} \in k^{(i-1)}[X_i].$$

□

Remark 2.12. The proof of the proposition shows that for any two ordinals $j' < j''$ such that $n_{i,j} = 1$, for $j' < j < j''$, we have $U_{i,j''} = \lim_{j \to j''} U_{i,j} = U_{i,j'}^{n_{i,j'}} - \sum_{j, j' \leq j < j''} \theta_{i,j} U^{m^{(i,j)}}$.

Example 2.13. Consider the ring $k[X_0, X_1, X_2]$ and the group $\Phi = \mathbb{Z}^3$ with reverse lexicographical order. Consider the valuation ν centered on this ring defined by the SKP $(U_{0,1}, U_{1,1}, (U_{2,j})_{j=1}^{\omega^2})$ and $\beta_{0,1} = (1, 0, 0), \beta_{1,1} = (0, 1, 0), \beta_{2,\omega n+j} = (j, n+2, 0)$, for $n \in \mathbb{N}, 0 < j < \omega$ and $\beta_{2,\omega^2} = (0, 0, 1)$. Here, we have the relations,

$$U_{2,\omega n+j+1} = U_{2,\omega n+j} - U_{0,1}^{j} U_{1,1}^{n+2}.$$

In this example, we have $n_{2,j} = 1$, for any $1 < j < \omega^2$. We see that we cannot continue to define U_{2,ω^2+1} : The reason is that $(\beta_{2,\omega n})_2 = n + 2 \to \infty$ $(n \to \infty)$ and therefore necessarily $\beta_{2,\omega^2} \notin \mathbb{Z}^2 \oplus \{0\}$. Thus, as $\beta_{0,1}, \beta_{1,1} \in \mathbb{Z}^2 \oplus \{0\}$, there does not exist any relation between $\beta_{2,\omega^2}, \beta_{0,1}, \beta_{1,1}$ and we are forced to stop at this step.

Example 2.14. Consider the ring $k[X_0, X_1, X_2]$ and the group $\Phi = \mathbb{Q}$ with the usual order $\leq$. Consider the valuation ν centered on this ring by the SKP, $(U_{0,1}, (U_{1,j})_{j=1}^{\omega}, (U_{2,j})_{j=1}^{\omega}, \beta_{i,j})$, which is defined as follows: Let $\{p_i\}_{i=1}^{\infty}$ be an increasing sequence of prime numbers. Define $\beta_{0,1} = 1$, $\beta_{1,1} = \frac{1}{p_1}$, $\beta_{1,j} = m_j + \frac{1}{p_j}$, for $j \geq 2$, where $m_2 = 1$ and $m_{j+1} = p_j m_j + 1$, and $\beta_{2,j} = \beta_{1,j}$, for $j \geq 1$. Then, after setting $\theta_{i,j} = 1$, we have $U_{1,j+1} = U_{1,j}^{p_j} - U_{0,1}^{m_{j+1}}$ and $U_{2,j+1} = U_{2,j} - U_{1,j}$.

3. *Adic* expansions

Suppose an SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$ is given. In this section, we show that any element f of the power series ring $k^{(d)}$ has a unique expansion in terms of key-polynomials. We give an algorithm for computing this expansion. The algorithm is based on the notion of acceptable vectors $\alpha' \leq \alpha$ associated with the SKP. Any acceptable vector determines an SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha'_i}$. We define the notion of $(U)_{\alpha'} - adic$ expansion, and show how one can get $(U)_{\alpha''} - adic$ expansions for $\alpha'' \geq \alpha'$, using $(U)_{\alpha'} - adic$ expansion. In the next section, we use the *adic* expansion of the elements to define a valuation, associated with a given SKP.

Lemma 3.1. *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$. *When* $U_{i,j} \neq 0$, *it is of the form,*

$$U_{i,j} = X_i^{d_{i,j}} + a_{i,j,d_{i,j}-1} X_i^{d_{i,j}-1} + \cdots + a_{i,j,0},$$

where $a_{i,j,j'} \in k^{(i-1)}$ *is so that the constant term of* $a_{i,j,j'}$ *is zero. Moreover, when* j *is not a limit ordinal, we have* $d_{i,j} = n_{i,j-1} d_{i,j-1}$, *for* $1 \leq j < \alpha_i$. *If* j *is a limit ordinal, then there exists an ordinal* $j_0 < j$, *which is not a limit ordinal and for any* j' *such that* $j_0 \leq j' \leq j$, *we have* $d_{i,j'} = d_{i,j_0} = n_{i,j_0-1} d_{i,j_0-1}$.

Proof. The proofs are all by induction. We prove the last part. By definition of the SKP, it is clear that for any $j' = 1, \ldots, j-1$, we have $m_{i,j'}^{(i,j)} \in S_{i,j}$, and so we have $0 \leq m_{i,j'}^{(i,j)} < n_{i,j'}$. By induction, we have $n_{i,j'} = d_{i,j'+1}/d_{i,j'}$. Hence, $m_{i,j'}^{(i,j)} + 1 \leq d_{i,j'+1}/d_{i,j'}$. So, we have,

$$\sum_{j'=1}^{j-1} m_{i,j'}^{(i,j)} d_{i,j'} \leq \sum_{j'=1}^{j-1} (\frac{d_{i,j'+1}}{d_{i,j'}} - 1) d_{i,j'} = d_{i,j} - 1 < n_{i,j} d_{i,j}.$$

Hence, $\deg_{X_i}(U_{i,j+1}) = n_{i,j} d_{i,j}$. For the last claim, we note that when j is a limit ordinal, there exists a j_0 such that for any j', $j_0 \leq j' \leq j$, we have $n_{i,j'} = 1$. □

Lemma 3.2. *For any* SKP $[U_{i,j},\beta_{i,j}]_{i=0..d,j=1..\alpha_i}$*, if* $U_{i,j}\neq 0$*, then we have,*

$$\deg_{X_i}(U_{i,j}) > \deg_{X_i}(\prod_{j'<j} U_{i,j'}^{p_{i,j'}}),$$

when $0\leq p_{i,j'}<n_{i,j'}$*. In other words,* $\sum_{j'<j} p_{i,j'}d_{i,j'}<d_{i,j}$*. Notice that* $p_{i,j'}=0$*, except for a finite number of* j'*.*

Definition 3.3. Fix an SKP $[U_{i,j},\beta_{i,j}]_{i=0..d,j=1..\alpha_i}$. We say that a vector $(\alpha'_0,\ldots,\alpha'_d)$ such that $\alpha'_i\leq\alpha_i$ is an acceptable vector if for any $i=0,\ldots,d$ and any $j=1,\ldots,\alpha'_i$, and for any $(i',j')\in S_{i,j}$ we have, $(i',j')\leq_{lex}(i',\alpha_{i'})$ for $i'<i$, and $(i',j')<_{lex}(i,j)$, when $i'=i$. This means that in the equation (P1), defining $U_{i,j}$ in terms of the U with smaller indices, one needs only indices from α', not necessarily all of α. Notice that an acceptable vector α' determines an SKP; i.e., $[U_{i,j},\beta_{i,j}]_{i=0..d,j=1..\alpha'_i}$ is an SKP.

Given an SKP $[U_{i,j},\beta_{i,j}]_{i=0..d,j=1..\alpha_i}$, the vector α is an acceptable vector. Moreover, the vector $(1,\ldots,1)\in\mathbb{N}^d$ is an acceptable vector for an arbitrary SKP.

Definition 3.4. Given any SKP and any acceptable α', we define the new SKP by this acceptable vector and construct the power series ring $k_{((\alpha',i))}=k[[(U_{i',j'})_{i'\leq i,j'<\alpha'_i,n_{i',j'}\neq 1},(U_{i',\alpha'_{i'}})_{i'\leq i}]]\subseteq k^{(d)}$. We have $k^{(i)}=k_{((\alpha,i))}$.

Given an SKP $[U_{i,j},\beta_{i,j}]_{i=0..d,j=1..\alpha_i}$, and an acceptable vector $\alpha'=(\alpha'_0,\ldots,\alpha'_d)$, we want to expand an arbitrary element $f\in k^{(d)}$ in terms of U's as an element of the power series ring $k_{((\alpha',d))}$.

Definition 3.5. (*adic* **expansions)** Fix an SKP $[U_{i,j},\beta_{i,j}]_{i=0..d,j=1..\alpha_i}$. Let α' be an acceptable vector for this SKP. For an element $f\in k^{(d)}$, consider the expansion $f=\sum_{I(J)} c_{I(J)}U^{I(J)}\in k_{((\alpha',d))}$, where $I(J)\in\mathbb{N}^1\times\cdots\times\mathbb{N}^{\alpha'_i}\times\cdots\times\mathbb{N}^{\alpha'_d}$, and $c_{I(J)}\in k$. This expansion is called the $(U)_{\alpha'}-$ *adic* expansion of f, when for every monomial $U^{I(J)}$ we have $0\leq I(J)_{i,j}<n_{i,j}$, for any $0\leq j<\alpha'_i$ and $i=0,\ldots,d$. Notice that $I(J)_{i,j}=0$, except for a finite number of j.

Definition 3.6. Fix an SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$, and let α' be an acceptable vector. For any monomial $M(U) = U^{\mathbf{a}} \in k_{((\alpha',d))}$, we define,

$$\mathrm{Vdeg}(M) = (\deg_{X_0}(U_0^{\mathbf{a}_0}), \deg_{X_1}(U_1^{\mathbf{a}_1}), \ldots, \deg_{X_d}(U_d^{\mathbf{a}_d})) \in \mathbb{N}^d.$$

Definition 3.7. Fix an SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$, and let α' be an acceptable vector. Let $M(U) = cU^{\mathbf{a}}$ be a monomial of the ring $k_{((\alpha',d))}$. We say that it is a monomial of *adic* form if it satisfies the conditions of monomials of Definition 3.5.

Lemma 3.8. *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$*, and let* α' *be an acceptable vector. Let* $M(U) = cU^{\mathbf{a}} \in k_{((\alpha',d))}$ *be a monomial of adic form with respect to this* SKP*. Then,* $\mathrm{Vdeg}(M)$ *determines the vector* $\mathbf{a}$.

Proof. This is a simple consequence of Lemma 3.2. If we set $n = \deg_{X_i}(U_i^{\mathbf{a}_i})$, then we have $\mathbf{a}_{i,\alpha'_i} = [\frac{n}{d_{i,\alpha'_i}}]$. Suppose by induction we obtained $\mathbf{a}_{i,\alpha'_i}, \ldots, \mathbf{a}_{i,j+1}$. Then, we have, $\mathbf{a}_{i,j} = [\frac{n - \sum_{j'=j+1}^{\alpha'_i} \mathbf{a}_{i,j'}.d_{i,j'}}{d_{i,j}}]$. Note that if $\mathbf{a}_{i,j} \neq 0$, then for any $j' < j$ such that $d_{i,j} = d_{i,j'}$, we have $\mathbf{a}_{i,j'} = 0$. This shows that in the case that α'_i is infinite ordinal type the number of nonzero entries of $\mathbf{a}$ computed inductively above, is finite. □

Corollary 3.9. *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$*. Let* α' *be an acceptable vector. For any two different monomials* M *and* M' *of the power series ring* $k_{((\alpha',d))}$*, we say* $M < M'$ *if*

$$\mathrm{Vdeg}(M) <_{lex} \mathrm{Vdeg}(M').$$

This is a well ordering on the set of monomials of $k_{((\alpha,d))}$ *of adic form.*

The following proposition shows that the *adic* expansions are well defined elements of the ring $k_{((\alpha,d))}$ and they are unique and it gives an algorithm to compute them.

Proposition 3.10. (Algorithm for getting *adic* expansions) *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$*. Let* α' *and* α'' *be two acceptable vectors for this* SKP *such that* $\alpha' < \alpha''$*, with respect to the partial product order of* $\mathbb{Z}^{d+1}$*. Let* $f \in k^{(d)}$ *and suppose we know its* $(U)_{\alpha'}$ *− adic expansion. In order to obtain its* $(U)_{\alpha''}$ *− adic expansion, we do the following:*

Starting from $(U)_{\alpha'} - adic$ expansion of f, for any monomial $M(U)$ in the expansion, and for any $i = 0, \dots, d$ and $j < \alpha''_i$, do the following replacements, and iterate this process on the resulting expansion as far as possible.

- *If $n_{i,j+1} > 1$, then replace any occurrence of $U_{i,j}^{n_{i,j}}$ in $M(U)$ by $U_{i,j+1} + \theta_{i,j} U^{m^{(i,j)}}$ (cf. $(P1)$ of Definition 2.4).*
- *If $n_{i,j+1} = 1$, then let $j + 1 < j_0 \leq \alpha''_i$ be the first ordinal such that $n_{i,j_0} > 1$ or $j_0 = \alpha''_i$ and replace any occurrence of $U_{i,j}^{n_{i,j}}$ in $M(U)$ by*

$$U_{i,j}^{n_{i,j}} = U_{i,j_0} + \sum_{j \leq j' < j_0} \theta_{i,j'} U^{m^{(i,j')}},$$

(cf. Remark 2.12).

The resulting expansion is equal to the $(U)_{\alpha''} - adic$ expansion of the element f. Moreover, this expansion is unique.

Proof. For any element of $k_{((\alpha,d))}$, we define $\mathcal{M}_n$ to be those monomials with ord $= n$. By Lemma 2.10, we know that $\#\mathcal{M}_n$ is finite. We do the replacements of the algorithm (staring from $\alpha' - adic$ expansion of f) in the nth step only on the monomials of $\bigcup_{n' \leq n} \mathcal{M}_{n'}$ of the current expansion. Using Lemma 6.6 of [8], this process terminates after finitely many steps. In this step, all the monomials of $\bigcup_{n' \leq n} \mathcal{M}_{n'}$ of the current expansion are of $\alpha'' - adic$ form. Moreover, there exists a number $m(n) < n$, where $m(n) \to \infty$ $(n \to \infty)$, such that in the process of replacements on the monomials of $\bigcup_{n' \leq n} \mathcal{M}_{n'}$, the monomials of $\bigcup_{m' \leq m(n)} \mathcal{M}_{m'}$ do not change (Lemma 2.10). Doing this process, as $n \to \infty$, we get an expansion, which satisfies all the properties of $\alpha'' - adic$ expansion. Thus, we obtain a $(U)_{\alpha''} - adic$ expansion of f.

Now, we prove that this expansion is unique. Suppose an element $f \in k^{(d)}$ has two different *adic* expansions, $f = \sum_{I(J)} c_{I(J)} U^{I(J)} = \sum_{I''(J'')} c''_{I''(J'')} U^{I''(J'')}$. Assume by induction on d that the claim is valid for the power series ring $R \otimes_k k^{(d-1)}$, where $R = k[[(U_{d,j})_{j<\alpha_d, n_{d,j} \neq 1}, U_{d,\alpha_d}]]$ is considered to be the coefficient ring. Consider f as an element of the ring $R \otimes_k k^{(d-1)}$. The two *adic* expansions of f give two *adic* expansion of $f \in R \otimes_k k^{(d-1)}$ as follows. Setting $U = (U_{(d-1)}, U_d)$ and $I(J) = (I(J)_{(d-1)}, I(J)_d)$, we have,

$$\begin{aligned} f &= \sum_{I(J)_{(d-1)}} (\sum_{I'(J')_d, I(J)_{(d-1)} = I'(J')_{(d-1)}} c_{I'(J')} U_d^{I'(J')_d}) U_{(d-1)}^{I(J)_{(d-1)}} \\ &= \sum_{I''(J'')_{(d-1)}} (\sum_{I'(J')_d, I''(J'')_{(d-1)} = I'(J')_{(d-1)}} c''_{I'(J')} U_d^{I'(J')_d}) U_{(d-1)}^{I''(J'')_{(d-1)}}. \end{aligned}$$

By the induction hypothesis, these two *adic* expansions are the same. Suppose M be the least monomial of this expansion, with respect to the ordering of Corollary 3.9, which refers to the indices $I_0(J_0)$ and $I''_0(J''_0)$ (respectively). Then, equating the coefficient of M in two *adic* expansions, we have,

$$\begin{aligned} g &= \sum_{I'(J')_d, I_0(J_0)_{(d-1)} = I'(J')_{(d-1)}} c_{I'(J')} U_d^{I'(J')_d} \\ &= \sum_{I'(J')_d, I''_0(J''_0)_{(d-1)} = I'(J')_{(d-1)}} c''_{I'(J')} U_d^{I'(J')_d}. \end{aligned}$$

Write $g \mid_{X_0=0,\ldots,X_{d-1}=0} = \sum_{\alpha \in \mathbb{Z}} c_\alpha X_d^\alpha$. Let α_0 be the first α such that $c_\alpha \neq 0$. Then, by Lemma 3.1 and 3.8, there is a unique monomial in either of the expansions g (M and M', respectively) such that $\mathrm{Vdeg}(M) = \mathrm{Vdeg}(M') = \alpha$ (here, $\mathrm{Vdeg}(M) = \deg_{X_d}(M)$). Hence, $M = M'$. Thus, the least monomials of two expansions of g (with respect to the ordering of Corollary 3.9) are equal. Subtracting this monomial from two representations, and iterating the last procedure we deduce that these two expansions are the same and we are done (an argument similar to the last part works for the start of the induction, $d = 1$). □

Remark 3.11. Given an SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d, j=1..\alpha_i}$, and an element $f \in k^{(d)}$, in order to obtain its $(U)_\alpha - adic$ expansion, we can use the algorithm of Proposition 3.10 for the acceptable vectors $\alpha' = (1, \ldots, 1)$ and $\alpha'' = \alpha$. Notice that in this case, the $(U)_{\alpha'} - adic$ expansion of every element $f \in k^{(d)}$ is itself.

We also use the notation of $(\alpha') - adic$ expansion. When there is no stress on the specific acceptable vector α' or it is understood, we will talk about $U_d - adic$ or *adic* expansion.

4. Valuations associated with the SKP

Here, we show that with any SKP one can associate a valuation ν of the field $k((X_0, \ldots, X_d))$ centered on the ring $k[[X_0, \ldots, X_d]]$.

Definition 4.1. Let $[U_{i,j}, \beta_{i,j}]$ be an SKP. For an acceptable vector α', we define a map,

$$\nu_{\alpha'} : \ k^{(d)} \backslash \{0\} \to \Phi,$$

by:

- If M is any monomial $M(U)$ with $(U)_{\alpha'} - adic$ expansion $M = c.U^{\mathbf{p}}$, where $c \in k$, then

$$\nu_{\alpha'}(M) = \sum_{i=0}^{d} \sum_{j=0}^{\alpha'_i} p_{i,j}\beta_{i,j}.$$

- If $f \in k^{(d)}$ has the $(U)_{\alpha'} - adic$ expansion $f = \sum_{I(J)} c_{I(J)} U^{I(J)}$, then

$$\nu_{\alpha'}(f) = \min_{I(J)}\{\nu_{\alpha'}(U^{I(J)})\}.$$

For any SKP, we denote the mapping of Definition 4.1 by $\nu_\alpha = \text{val}[U_{i,j}, \beta_{i,j}]$. We will see that this mapping is a valuation (Theorem 4.7).

Definition 4.2. Let $[U_{i,j}, \beta_{i,j}]$ be an SKP, $f \in k^{(d)}$ be an arbitrary element and let (α') be an acceptable vector for this SKP. The initial form of f with respect to $\nu_{\alpha'}$ is defined as:

$$\text{in}_{\nu_{\alpha'}}(f) = \sum_{I(J^0)} c_{I(J^0)} U^{I(J^0)},$$

where $f = \sum_{I(J)} c_{I(J)} U^{I(J)}$ is the $(U)_{\alpha'} - adic$ expansion of f and $I(J^0)$ ranges over those indices with minimal $\nu_{\alpha'}-$value.

Definition 4.3. Let $[U_{i,j}, \beta_{i,j}]_{i=0..d, j=1..\alpha_i}$ be an SKP and consider the power series ring $k_{((\alpha,d))}$. For any monomial $M(U) = U^{\mathbf{a}} \in k_{((\alpha,d))}$, we define the vectors of the powers,

$$\text{VP}(M(U)) = (\mathbf{a}_{d,\alpha_d}, \mathbf{a}_{d-1,\alpha_{d-1}}, \ldots, \mathbf{a}_{0,\alpha_0}) \in \mathbb{N}^{d+1}.$$

Lemma 4.4. *Fix an* SKP *and suppose that* α' *is an acceptable vector for this* SKP. *Let* $f \in k^{(d)}$ *and suppose* $\text{in}_{\nu_{\alpha'}}(f) = \sum_{I(J)} c_{I(J)} U^{I(J)}$. *Then, the vectors of the powers* $\text{VP}(M)$ *of the monomials* M *of* $\text{in}_{\nu_{\alpha'}}(f)$ *are all different.*

Proof. Let $cU^{I(J)}$ and $c'U^{I'(J')}$ be two monomials of $\text{in}_{\nu_{\alpha'}}(f)$ with equal vectors of the powers. We show that for any $j = 1, \ldots, \alpha'_d$, the powers of the $U_{d,j}$ in the two monomials are the same. Indeed, let $j' < \alpha'_d$ be the greatest index such that $I(J)_{d,j'} \neq I'(J')_{d,j'}$. Note that this maximum index exists. We assume $I(J)_{d,j'} > I'(J')_{d,j'}$. By equating the $\nu_{\alpha'}-$values of the two monomials,

$$(I(J)_{d,j'} - I'(J')_{d,j'})\beta_{d,j'} = \sum_{(i'',j'')<_{lex}(d,j')} -(I(J)_{i'',j''} - I'(J')_{i'',j''})\beta_{i'',j''}.$$

But the right hand side is in the group $(\beta_{i'',j''})_{(i'',j'')<_{lex}(d,j')}$, which is clearly a contradiction, because $0 < I(J)_{d,j'} - I'(J')_{d,j'} < n_{d,j'}$. Giving similar arguments for $i < d$, we deduce that the two monomials are the same. $\square$

Corollary 4.5. *Fix an* SKP *and suppose* $U_{i,\alpha_i} = 0$, *for* $i = 1..d$. *For an arbitrary* $0 \neq f \in k^{(d)}$, *the initial* $\mathrm{in}_{\nu_\alpha}(f)$ *consists of just one monomial of adic form.*

Lemma 4.6. *Fix an* SKP $[U_{i,j}, \beta_{i,j}]$ *and let* α' *be an acceptable vector. For any arbitrary monomial* $M(U) \in k_{((\alpha',d))}$, *where* $M = c.U^{\mathbf{a}}$, *we have:*

- (i) *The initial form of* M *in its* $(U)_{\alpha'}-$ *adic expansion is just one monomial* $M' = c'U^{\mathbf{a}'}$. *In other words, we have* $\mathrm{in}_{\nu_{\alpha'}}(M) = M'$.
- (ii) *We have* $\mathbf{a}'_{d,\alpha'_d} = \mathbf{a}_{d,\alpha'_d}$.
- (iii) *For any two monomials* M *and* M' *of the power series ring* $k_{((\alpha',d))}$ *with equal* $\nu_{\alpha'}-$*values, if* $\mathrm{VP}(M) <_{lex} \mathrm{VP}(M')$ *then* $\mathrm{VP}(\mathrm{in}_{\nu_{\alpha'}}(M)) <_{lex} \mathrm{VP}(\mathrm{in}_{\nu_{\alpha'}}(M'))$.

Proof. For the first claim, let $U_{i,j}$ be a factor of M with power greater than $n_{i,j}$. Replace $U_{i,j}^{n_{i,j}}$ by its expression from the algorithm in proposition 3.10 for getting *adic* expansion. The claim is that after one such replacement there exists just one monomial with minimal $\nu_{\alpha'}-$value. We prove the claim for the replacements of the first type of the algorithm for getting *adic* expansion. For the second type, the argument is similar. After a replacement of type one, we get two monomials with different $\nu_{\alpha'}-$values:

$$\begin{aligned} M &= \tfrac{M}{U_{i,j}^{n_{i,j}}}(U_{i,j+1} + \theta_{i,j}U^{m^{(i,j)}}) \\ &= \underbrace{c\frac{U^{\mathbf{a}}.U_{i,j+1}}{U_{i,j}^{n_{i,j}}}}_{M_2} + \underbrace{c\theta_{i,j}\frac{U^{\mathbf{a}}U^{m^{(i,j)}}}{U_{i,j}^{n_{i,j}}}}_{M_1}. \end{aligned}$$

Then, $\nu_{\alpha'}(M_2) > \nu_{\alpha'}(M_1) = \nu_{\alpha'}(M)$. Therefore, we have $\text{in}_{\nu_{\alpha'}}(M) = \text{in}_{\nu_{\alpha'}}(M_1)$. We do the same for M_1. Finally, we get a monomial M' whose *adic* expansion is itself, and this proves (i).

For the proof of part (ii), we notice the that the proof of the first part shows the following general fact:
For the monomial $M(U)$, a replacement on $U_{i,j}^{n_{i,j}}$ cannot affect the power of $U_{i',j'}$, for $(i',j') >_{lex.} (i,j)$, of the unique monomial with minimal value of the expansion generated after replacement.

For the proof of (iii), suppose $M = U^{\mathbf{a}}$ and $M' = U^{\mathbf{a}'}$. Let $d' \leq d$ be the first index such that $\mathbf{a}_{d',\alpha'_{d'}} < \mathbf{a}'_{d',\alpha'_{d'}}$. Then, by Lemma 4.4, we have $\mathbf{a}_{i,j} = \mathbf{a}'_{i,j}$, for $i = d'+1,\ldots,d$ and $j = 1,\ldots,\alpha_i$. Thus, the algorithm given in proposition 3.10 for getting *adic* expansion for these two monomials for such i and j can be chosen the same. Hence, without loss of generality we can assume that $\mathbf{a}_{i,j} < n_{i,j}$ and $\mathbf{a}'_{i,j} < n_{i,j}$, for $i = d'+1,\ldots,d$ and $j < \alpha_i$. Then because $\mathbf{a}_{d',\alpha'_{d'}} < \mathbf{a}'_{d',\alpha'_{d'}}$, by part (ii) we are done. □

Theorem 4.7. *Given any* SKP $[U_{i,j},\beta_{i,j}]$, *for any acceptable vector* α', *the mapping* $\nu_{\alpha'} : k^{(d)}\backslash\{0\} \to \Phi$ *extends in an obvious way to a* $k-$*valuation of the field of power series* $k((X_0,\ldots,X_d))$. *Moreover, for any two acceptable vectors* α' *and* α'' *such that* $\alpha' \leq \alpha''$ *and for any* $f \in k^{(d)}$, *we have* $\nu_{\alpha'}(f) \leq \nu_{\alpha''}(f)$.

Proof. The extension to the field $k((X_0,\ldots,X_d))$ is a trivial task. We need only to prove that given any $f,g \in k^{(d)}\backslash\{0\}$, we have $\nu_{\alpha'}(f+g) \geq \min\{\nu_{\alpha'}(f),\nu_{\alpha'}(g)\}$ and $\nu_{\alpha'}(f.g) = \nu_{\alpha'}(f)+\nu_{\alpha'}(g)$. The first one is a direct consequence of the definition and the uniqueness of the *adic* expansions. For the second equality, let $\text{in}(f) = \sum_{I(J)} c_{I(J)}U^{I(J)}$ and $\text{in}(g) = \sum_{I'(J')} c'_{I'(J')}U^{I'(J')}$. Let $M = c_{I(J_0)}U^{I(J_0)}$ (respectively $M' = c_{I'(J'_0)}U^{I'(J'_0)}$) be the unique (Lemma 4.4) monomial of the expansion of $\text{in}(f)$ (respectively $\text{in}(g)$) with minimal vector of powers, with respect to the *lex.* order. Then, by Lemma 4.6 (iii), we see that $\text{in}(M.M') = M''$ is the unique monomial of $\text{in}(f.g)$, with minimal vector of the powers. But, $\nu_{\alpha'}(M'') = \nu_{\alpha'}(M)+\nu_{\alpha'}(M') = \nu_{\alpha'}(f)+\nu_{\alpha'}(g)$. By the definition of the mapping $\nu_{\alpha'}$, we have $\nu_{\alpha'}(M'') = \nu_{\alpha'}(f.g)$. For the last part, we note that in the algorithm in proposition 3.10 for getting $\alpha''-adic$ expansion of an element from its $\alpha'-adic$ expansion, at every step in the substitution we replace a monomial with two new monomials with values equal to or greater than the original monomial. □

Corollary 4.8. *Given an* SKP $[U_{i,j},\beta_{i,j}]_{i=0..d,j=1..\alpha_i}$*, all the* $U_{i,j}$ *are irreducible elements of the power series ring* $k^{(i-1)}[X_i]$.

Proof. We prove the claim for $U_{d,j}$. Consider the vector (α'), defined by $\alpha'_i=\alpha_i$, for $0\leq i<d$, and $\alpha'_d=j$. This is an acceptable vector. In this proof, all the *adic* expansions are $(U)_{\alpha'}-adic$ expansions. We give a proof by contradiction. Assume that $U_{d,j}$ is reducible and $U_{d,j}=f.g$, for some non-unit elements $f,g\in k^{(d-1)}[X_d]$. As the $\alpha'-adic$ expansion of $U_{d,j}$ is itself, we have $\mathrm{in}(U_{d,j})=U_{d,j}$. We can compute this initial in the other way, using initials of f and g. This gives us $U_{d,j}=\mathrm{in}(\mathrm{in}(f).\mathrm{in}(g))$.

On the other hand, $\beta_{d,j}=\nu_{\alpha'}(U_{d,j})=\nu_{\alpha'}(f)+\nu_{\alpha'}(g)$. Thus, the monomials of $\mathrm{in}(f)$ and $\mathrm{in}(g)$ do not have a factor $U_{d,j}$. By Lemma 4.6 (*ii*), this shows that the monomials $\mathrm{in}(\mathrm{in}(f).\mathrm{in}(g))$ do not have a factor $U_{d,j}$, which is a contradiction. □

Remark 4.9. One should note that in the definition of the SKP for the ring $k[[X_0,\ldots,X_d]]$, the ordering of the variables plays an important role. In other words, changing the coordinates of the rings (even with a permutation) may change totally the system of the SKP associated with the valuation, or even they may not exist. This phenomenon can be seen even in dimension two; for example, consider the valuation ν centered on the ring $k[X_0,X_1]$; defined by the SKP, $[(U_{0,1},U_{1,1},U_{1,2},U_{1,3}),(2,3,9,10)]$, where, we have $U_{1,2}=U_{1,1}^2-U_{0,1}^3$, $U_{1,3}=U_{1,2}-U_{0,1}^3U_{1,1}$. Note that the last two equations are given to us (up to the knowledge of the corresponding θ's) as soon as the sequence of β's $(2,3,9)$ is known. Now, changing the order of the coordinates, we consider the same ring as $k[Y_0,Y_1]$ with $Y_0=X_1,Y_1=X_0$. The same valuation is given by the following SKP's in the new coordinate $\nu=\mathrm{val}[(V_{0,1},V_{1,1},V_{1,2},V_{1,3}),(3,2,9,10)]$, where the SKP are as follows:

$$V_{1,2}=V_{1,1}^3-V_{0,1}^2,\ V_{1,3}=V_{1,2}+V_{0,1}^3.$$

The relation between two SKP's is as follows:

$$V_{0,1}=U_{1,1},\ V_{1,1}=U_{0,1},\ V_{1,2}=-U_{1,2}.$$

For $V_{1,3}$ we have,

$$\begin{aligned} V_{1,3} &= V_{1,2} + V_{0,1}^3 \\ &= -U_{1,2} + U_{1,1}^3 = -U_{1,2} + (U_{0,1}^3 + U_{1,2})U_{1,1} = -U_{1,3} + U_{1,1}U_{1,2}. \end{aligned}$$

As this example shows, the explicit relation between the U and the V is not, in general, trivial.

5. Euclidean expansion and other properties of the SKP

Here, we give another expansion in the ring $k_{(d-1)}[X_d]$, associated with an SKP of the power series ring $k[[X_0, \dots, X_d]]$ ($k_{(i)} := k((X_0, \dots, X_i)))$. We show that the valuation ν associated with this SKP, can be defined using this new expansion, plus the knowledge of the valuation ν on the field $k_{(d-1)}$. Moreover, we show that the Euclidean expansion can be obtained directly from the *adic* expansion. This is interesting in practice, because *adic* expansion is defined only with substitutions, while Euclidean expansion is defined using divisions.

Definition 5.1. (**Euclidean expansion**) Fix an **SKP** $[U_{i,j}, \beta_{i,j}]$, where $i = 0..d$ and $j = 1..\alpha_i$. For any $j = 1, \dots, \alpha_d$, we define the acceptable vector $\alpha^{(j)} = (\alpha_0, \dots, \alpha_{d-1}, j)$. Let $f \in k_{(d-1)}[X_d]$, and consider the expansion $f = \sum_J c_J U_d^J \in k_{(d-1)}[U_d]$ such that $0 \leq J_{j'} < n_{d,j'}$, for any $0 \leq j' < j$. This is called the jth Euclidean expansion of f.

Proposition 5.2. (Algorithm for getting Euclidean expansion) *With the notations of Definition 5.1, do the following:*

Consider the greatest index j_0 such that $\deg_{X_d}(f) > d_{d,j_0}$. Divide f by U_{d,j_0} in the ring $k_{(d-1)}[X_d]$ to obtain $f = qU_{d,j_0} + r$, where $q, r \in k_{(d-1)}[X_d]$ and $\deg_{X_d}(r) < d_{d,j_0}$. Iterate the same procedure for q as far as possible to obtain $f = \sum_t f_t U_{d,j_0}^t$, where $\deg_{X_d}(f_t) < d_{d,j_0}$. Iterate the same procedure for each of the f_t and the greatest index j', $j' < j_0$, such that $d_{d,j'} < d_{d,j_0}$. Continue as far as possible. This process terminates after finitely many steps. The resulting expansion is equal to the jth Euclidean expansion of f. Moreover, the Euclidean expansion is unique.

Proof. As the U_d which appear in the process are among the elements of the finite set $\{U_{d,j'} /\ n_{d,j'} \neq 1,\ \text{and}\ \deg_{X_d}(f) > d_{d,j'}\}$, the process stops after finitely many steps. We show that the resulting expansion is the jth Euclidean expansion of f. Let U_d^J be a monomial generated in the algorithm above. It is sufficient to show that this monomial is of Euclidean form. Indeed, let j' be the greatest index less than j such that $J_{j'} \geq n_{d,j}$. This means that $\deg_{X_d}(U_{d,1}^{J_1} \cdots U_{d,j'}^{J_{j'}}) \geq d_{d,j'+1}$, and we must divide it (in the monomial in the procedure above) by $U_{d,j'+1}$, which is a contradiction.

The uniqueness of Euclidean expansion comes from the fact that (by Lemma 3.8) the $\deg_{X_d}(U_d^J)$ of a monomial of Euclidean form determines the vector J. Therefore, there is a unique vector J_0 such that $\deg_{X_d}(U_d^{J_0}) = \deg_{X_d}(f)$. This monomial (plus its coefficient) is common in all the possible Euclidean expansions of f. Subtracting this monomial from f, then by induction on the degree of f we are done. □

Lemma 5.3. *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$*, and let f be an element in the ring of power series $k[[X_0, \ldots, X_d]]$. The* jth *Euclidean expansion of f can be obtained using the $(\alpha^{(j)})$ − adic expansion of it as follows. Let $f = \sum_{I(J)} c_{I(J)} U^{I(J)}$ be the $(\alpha^{(j)})$ − adic expansion of f. Then, the Euclidean expansion of f is equal to:*

$$\sum_{J'} \Big(\sum_{I(J), I(J)_d = J'} c_{I(J)} \frac{U^{I(J)}}{U_d^{J'}} \Big) U_d^{J'}.$$

Proof. It is clear that the above expansion satisfies all the properties of the jth Euclidean expansion of f. Thus, by uniqueness, it is the Euclidean expansion of f. □

Remark 5.4. Using Lemma 5.3, we extend the notion of Euclidean expansion to the power series ring $k^{(d)}$. An expansion of $f \in k^{(d)}$ of the form $f = \sum_J c_J U_d^J \in k^{(d-1)}[[U_d]]$, which satisfies the conditions of Definition 5.1, is called the Euclidean expansion of f. Lemma 5.3 shows that such an expansion can be obtained using *adic* expansion of f. An argument, similar to the proof of Proposition 3.10, shows that this expansion is unique.

Proposition 5.5. *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$*, and let ν be the $k-$valuation of the field $k_{(d)}$ associated with it. Set $\overline{\nu} = \nu \mid_{k_{(d-1)}}$. The valuation ν (as a valuation of the field $k_{(d-1)}(X_d)$) can be defined using the data $[\overline{\nu}, (U_{d,j})_{j=1}^{\alpha_d}, (\beta_{d,j})_{j=1}^{\alpha_d}]$ as follows. For any $f \in k^{(d-1)}[X_d]$, let*

$f = \sum_J f_J U_d^J$ *be its* α_dth *Euclidean expansion. Then,*

$$\nu(f) = \min_J\{\overline{\nu}(f_J) + \beta_d.J\}.$$

Proof. Lemma 5.3 shows that the equation of the proposition is just another way of writing $\nu(f)$, which is originally the minimum of the values of the monomials in the *adic* expansion of f. $\square$

Remark 5.6. With the notations of Proposition 5.5, write $f = \sum_t f_t U_{d,j}^t$, with $\deg_{X_d}(f_t) < d_{d,j}$. Then, with a similar argument, we have,

$$\nu(f) = \min_t\{\nu(f_t) + t\beta_{d,j}\}.$$

Definition 5.7. Fix an SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$. We consider the set of acceptable vectors $\alpha^{(j)} = (\alpha_0, \dots, \alpha_{d-1}, j)$, for $j = 1, \dots, \alpha_d$.
For any $f \in k_{(d-1)}[X_d]$, and any $\alpha^{(j)}$, we define,

$$\delta_{\alpha^{(j)}}(f) = \max\{\ell : \ \ell \text{ is power of } U_{d,j} \text{ in the monomials of } \mathrm{in}_{\nu^{\alpha^{(j)}}}(f)\}.$$

Remark 5.8. Let $u \in k^{(d-1)}$ and $f \in k_{(d-1)}[X_d]$. Then, $\delta_{\alpha^{(j)}}(f) = \delta_{\alpha^{(j)}}(uf)$.

Lemma 5.9. *For any* $f, g \in k_{(d-1)}[X_d]$, *we have,*

$$\delta_{\alpha^{(j)}}(f.g) = \delta_{\alpha^{(j)}}(f) + \delta_{\alpha^{(j)}}(g).$$

Proof. First, we find $u, v \in k^{(d-1)}$ such that $uf, vg \in k^{(d-1)}[X_d]$. This is always possible. By Remark 5.6, it suffices to prove the lemma for uf and vg; i.e., we can assume $f, g \in k^{(d-1)}[X_d]$. Lemma 4.4 shows that there are unique monomials $f_J U_d^J$ and $g_{J'} U_d^{J'}$ of in(f) and in(g) (respectively) that have maximal $U_{d,j}$ power. Write Euclidean expansion of in$(f.g)$ using the product in(f).in(g) and the algorithm in proposition 3.10 for getting *adic* expansion. We see in(f).in(g) has a unique monomial with $U_{d,j}-$degree equal $\delta_{\alpha^{(j)}}(f) + \delta_{\alpha^{(j)}}(g)$; i.e., $f_J g_{J'} U^J U^{J'}$. Now, Lemma 4.6 (ii) shows that after getting *adic* expansion from this product, the $U_{d,j}-$powers of the monomials do not change, which proves the equality. $\square$

The following lemma is an adaptation of the results of [3] to our situation.

Lemma 5.10. *Fix an* SKP $[U_{i,j},\beta_{i,j}]$*, and let* $\alpha^{(j)}$ *be defined as in Definition 5.7. Then:*

(*i*) *For* $f\in k_{(d-1)}[X_d]$*, we have* $\delta_{\alpha^{(j)}}(f)=0$ *if and only if* $\mathrm{in}_{\nu_{\alpha^{(j)}}}(f)$ *is a unit in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$.

(ii) *If* $f,g\in k_{(d-1)}[X_d]$ *then there exist* $Q,R\in k_{(d-1)}[X_d]$ *such that* $\mathrm{in}_{\nu_{\alpha^{(j)}}}(f)=\mathrm{in}_{\nu_{\alpha^{(j)}}}(Qg+R)$ *in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$ *and* $\delta_{\alpha^{(j)}}(R)<\delta_{\alpha^{(j)}}(g)$.

(*iii*) *The polynomials* $\mathrm{in}_{\nu_{\alpha^{(j)}}}(U_{d,\alpha^{(j)}_j})$ *and* $\mathrm{in}_{\nu_{\alpha^{(j)}}}(U_{d,\alpha^{(j)}_{j+1}})$ *are irreducible in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$.

(*iv*) *If* $j'<j$*, then* $\mathrm{in}_{\nu_{\alpha^{(j)}}}(U_{d,j'})$ *is a unit in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$.

(*v*) *If* $f=\sum_t f_tU_{d,j}^t$*, with* $\deg_{X_d}(f_t)<d_{d,j}$ *and* $\delta_{\alpha^{(j)}}(f)<n_{d,j}$*, then* $\mathrm{in}_{\nu_{\alpha^{(j)}}}(f)=\mathrm{in}_{\nu_{\alpha^{(j)}}}(f_tU_{d,j}^t)$ *in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$*, for some* $t<n_{d,j}$.

Proof. Throughout the proof, we fix the expansion $f=\sum_t f_tU_{d,j}^t$, with $\deg_{X_d}(f_t)<d_{d,j}$.
(*i*). If $\delta_{\alpha^{(j)}}(f)=0$, then $\mathrm{in}_{\nu_{\alpha^{(j)}}}(f)=\mathrm{in}_{\nu_{\alpha^{(j)}}}(f_0)$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$. As $U_{d,j}$ is irreducible and $\deg_{X_d}(f_0)<d_{d,j}$, the polynomial $U_{d,j}$ is prime with f_0. Hence, we can find $A,B\in k_{(d-1)}[X_d]$, $\deg_{X_d}(A),\deg_{X_d}(B)<d_{d,j}$, so that $Af_0=1-BU_{d,j}$. Then, $\nu_{\alpha^{(j)}}(Af_0)=\nu_{\alpha^{(j)}}(1)<\nu_{\alpha^{(j)}}(BU_{d,j})$. Therefore, $\mathrm{in}_{\nu_{\alpha^{(j)}}}(Af_0)=1$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$. So, $\mathrm{in}_{\nu_{\alpha^{(j)}}}(f_0)$, and hence $\mathrm{in}_{\nu_{\alpha^{(j)}}}(f)$ is a unit in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$. Conversely, if $\mathrm{in}_{\nu_{\alpha^{(j)}}}(f)$ is unit, say $\mathrm{in}_{\nu_{\alpha^{(j)}}}(Af)=1$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$ for some $A\in k_{(d-1)}[X_d]$, then $\delta_{\alpha^{(j)}}(f)+\delta_{\alpha^{(j)}}(A)=\delta_{\alpha^{(j)}}(1)=0$ and so $\delta_{\alpha^{(j)}}(f)=0$.
(*ii*). Write $g=\sum_t g_tU_{d,j}^t$. It suffices to prove the claim, when $g_t=0$ for $t>M:=\delta_{\alpha^{(j)}}(g)$ and using (*i*) we may assume $g_M=1$. As $\deg_{X_d}(g_t)<d_{d,j}$, for $t\le M$, we have $\deg_{X_d}(g)=Md_{d,j}$. Euclidean division in $k_{(d-1)}[X_d]$ yields $Q,R^1\in k_{(d-1)}[X_d]$ with $\deg_{X_d}(R^1)<\deg_{X_d}(g)$ so that $f=Qg+R^1$. Write $R^1=\sum_t R_tU_{d,j}^t$, set $N:=\delta_{\alpha^{(j)}}(R^1)$ and $R:=\sum_{t\le N}R_tU_{d,j}^t$. Then, $\mathrm{in}_{\nu_{\alpha^{(j)}}}(f)=\mathrm{in}_{\nu_{\alpha^{(j)}}}(Qg+R)$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$, and

$$\deg_{X_d}(R)=\deg_{X_d}(R_N)+Nd_{d,j}<Md_{d,j}=\deg_{X_d}(f).$$

Hence, $N<M$, and we are done.
(*iii*). We have $\delta_{\alpha^{(j)}}(U_{d,j})=1$ and so if $\mathrm{in}_{\nu_{\alpha^{(j)}}}(U_{d,j})=fg$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$, then $\delta_{\nu_{\alpha^{(j)}}}(f)=0$ or $\delta_{\alpha^{(j)}}(g)=0$. Hence, by (*i*), $\mathrm{in}_{\nu_{\alpha^{(j)}}}(f)$ or $\mathrm{in}_{\nu_{\alpha^{(j)}}}(g)$ is a unit in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$.

For $U_{d,j+1}$, we have $U_{d,j+1} = U_{d,j}^{n_{d,j}} - \theta_{d,j} U^{m^{(d,j)}}$. Let $\mathrm{in}_{\nu_{\alpha(j)}}(U_{d,j+1}) = \mathrm{in}_{\nu_{\alpha(j)}}(fg)$ in $\mathrm{gr}_{\nu_{\alpha(j)}} k_{(d-1)}[X_d]$, with $0 < \delta_{\alpha^{(j)}}(f), \delta_{\alpha^{(j)}} < n_{d,j}$. By (v), we can write $f = f_t U_{d,j}^t$, $g = g_{t'} U_{d,j}^{t'}$. Then, $U_{d,j+1} = f_t g_{t'} U_{d,j}^{n_{d,j}}$ so $(1 - f_t g_{t'}) U_{d,j}^{n_{d,j}} = \theta_{d,j} U^{m^{(d,j)}}$. As $U_{d,j}$ is irreducible and $U^{m^{(d,j)}}$ is a unit, we have $\mathrm{in}_{\nu_{\alpha(j)}}(f_t g_{t'}) = 1$ in $\mathrm{gr}_{\nu_{\alpha(j)}} k_{(d-1)}[X_d]$. But then, $\mathrm{in}_{\nu_{\alpha(j)}}(U^{m^{(d,j)}}) = 0$ in $\mathrm{gr}_{\nu_{\alpha(j)}} k_{(d-1)}[X_d]$, which is absurd. So, we can assume $\delta_{\alpha^{(j)}}(f) = n_{d,j}$ and $\delta_{\alpha^{(j)}} = 0$. Hence, g is a unit.
(iv). By (i) it suffices to show that $\delta_{\alpha^{(j)}}(U_{d,j'}) = 0$. If $d_{d,j'} < d_{d,j}$, then this is obvious. If $d_{d,j'} = d_{d,j}$, then $U_{d,j'} = (U_{d,j'} - U_{d,j}) + U_{d,j}$, where $\deg_{X_d}(U_{d,j'} - U_{d,j}) < d_{d,j}$. Now, $\nu_{\alpha^{(j)}}(U_{d,j'}) = \beta_{d,j'} < \beta_{d,j} = \nu_{\alpha^{(j)}}(U_{d,j})$, and so $\nu_{\alpha^{(j)}}(U_{d,j'} - U_{d,j}) < \nu_{\alpha^{(j)}}(U_{d,j})$ and $\delta_{\alpha^{(j)}}(U'_j) = 0$.
(v). Suppose $\nu_{\alpha^{(j)}}(f_t U_{d,j}^t) = \nu_{\alpha^{(j)}}(f_{t'} U_{d,j}^{t'})$, where $t \le t' < n_{d,j}$. Then, $(t' - t)\beta_{d,j} = \nu_{\alpha^{(j-1)}}(f_t) - \nu_{\alpha^{(j-1)}}(f_{t'})$. Hence, $n_{d,j} \mid t' - t$ and thus $t' = t$. □

Proposition 5.11. *The graded algebra* $\mathrm{gr}_{\nu_\alpha} k_{(d-1)}[X_d]$ *is a Euclidean domain.*

Proof. Lemma 5.10 (ii) proves the claim. □

Theorem 5.12. *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d, j=1..\alpha_i}$, *and let* ν *be its associated valuation. Consider* $0 \neq f \in k[[X_1, \ldots, X_d]]$. *Then, initial form of* f *has a unique decomposition of the form:*

(i) *If* $U_{d,\alpha_d} \neq 0$ *and* $n_{d,\alpha_d} = \infty$, *then,*

$$f = \tilde{f} U_d^J, \text{ in } \mathrm{gr}_\nu k_{(d-1)}[X_d],$$

where $\tilde{f} \in k_{(d-1)}$ *and* $0 \le J_j < n_{d,j}$, *for* $1 \le j < \alpha_d$.

(ii) *If* $U_{d,\alpha_d} \neq 0$ *and* $n_{d,\alpha_d} \neq \infty$, *then,*

$$f = p(T) U_d^{\hat{J}}, \text{ in } \mathrm{gr}_\nu k_{(d)},$$

where $p(T) \in k_{(d-1)}[T]$ *and* $0 \le J_j < n_{d,j}$, *for* $1 \le j \le \alpha_d$, *and* $T = U_{d,\alpha_d}^{n_{d,\alpha_d}} U^{-m^{(d,\alpha_d)}}$. *Moreover, all the coefficients of* $p(T)$ *have the same* $\nu-$*value.*

(iii) *If* $U_{d,\alpha_d} = 0$, *then,*

$$f = \tilde{f} U_d^J, \text{ in } \mathrm{gr}_\nu k_{(d-1)}[X_d],$$

where $0 \leq J_j < n_{d,j}$, *for* $1 \leq j \leq \alpha_d$, *and* $J_j = 0$, *except for a finite number of* j.

Proof. (i). Suppose $f = \sum_J f_J U_d^J$ is the Euclidean expansion of f (Remark 5.4), where $f_J \in k_{(d-1)}$, and $0 \leq J_j < n_{d,j}$ for $j < \alpha_d$. We claim that for any two J and J', we have $\nu(f_J U_d^J) \neq \nu(f_{J'} U_d^{J'})$. Indeed, if we have equality, consider the greatest index j_0 such that $J_{j_0} \neq J'_{j_0}$. We have $(J'_{j_0} - J_{j_0})\beta_{d,j_0} = \nu(f_J) - \nu(f_{J'}) + \sum j < j_0 (J_j - J'_j)\beta_{d,j}$. Then, as $j_0 < \alpha_d$ (because $n_{d,\alpha_d} = \infty$), we have $n_{d,j_0} \mid J_{j_0} - J'_{j_0}$. Thus, $J_{j_0} = J'_{j_0}$, which is absurd.
(ii). We show that any monomial $f_J U_d^J$ of the Euclidean expansion of $\mathrm{in}(f)$ is of the form $\hat{f}_J T^{r_{\alpha_d}} U_d^{\hat{J}}$, in $\mathrm{gr}_\nu k^{(d)}$, for a fixed $\hat{J}$ such that $0 \leq \hat{J}_j < n_{d,j}$, for any j.
Fix J, and make the Euclidean division $J_{\alpha_d} = r_{\alpha_d} n_{d,\alpha_d} + \hat{J}_{\alpha_d}$, $0 \leq \hat{J}_{\alpha_d} < n_{d,\alpha_d}$, and write $f_J U_d^J = \overline{f}_J U_{d,\alpha_d}^{\hat{J}_{\alpha_d}} T^{r_{\alpha_d}} U_d^{\mathbf{a}}$, with $\mathbf{a} := J + r_{\alpha_d}.m_d^{(d,\alpha_d)}$. As

$$U_{d,j}^{n_{d,j}} = \theta_{d,j}(U_{<d-1}^{m_{<d-1}^{(d,j)}}) U_d^{m_d^{(d,j)}}, \text{ in } \mathrm{gr}_\nu k_{(d-1)}[X_d],$$

making the Euclidean division $\mathbf{a}_j = r_j n_{d,j} + \hat{J}_j$, (with $0 \leq \hat{J}_j < n_{d,j}$) for the greatest index j such that $\mathbf{a}_j \neq 0$, we get $\prod_{j' \leq j} U_{d,j'}^{\mathbf{a}_{j'}} = U_{d,j}^{\hat{J}_j} \prod_{j' < j} U_{d,j'}^{\mathbf{a}'_{j'}}$ with $\mathbf{a}'_{j'} \in \mathbb{N}$. We finally get, by induction, a representation,

$$f_J U_d^J = \hat{f}_J T^{r_{\alpha_d}} U_d^{\hat{J}},$$

where $0 \leq \hat{J}_j < n_{d,j}$, for any j. As $\nu(T) = 0$, with an argument similar to the final part of case (i), one can argue to show that $\hat{J}$ is the same for all the J. Clearly, the coefficients of p has the same $\nu-$value.
(iii). This is similar to (i). □

Corollary 5.13. *Let* ν *be a valuation as above.*

- (i) *If* $U_{d,\alpha_d} \neq 0$ *and* $n_{d,\alpha_d} \neq \infty$, *then the only irreducible element of the graded ring* $\mathrm{gr}_\nu k_{(d-1)}[X_d]$ *is* U_{d,α_d}.
- (ii) *Assume that* k *is an algebraically closed field and suppose that* $U_{d,\alpha_d} \neq 0$, $n_{d,\alpha_d} < \infty$ *and that the following additional condition is satisfied: For every two monomials* $U^I, U^J \in k_{(d-1)}$ *of adic form, we have* $U^I = U^J$, *whenever* $\nu(U^I) = \nu(U^J)$. *Then, the*

irreducible elements of $\mathrm{gr}_\nu k_{(d-1)}[X_d]$ *are of the form* $U_{d,\alpha_d}^{n_{d,\alpha_d}} - \theta U^{m^{(d,\alpha_d)}}$, *for some* $\theta \in k$.

(iii) *If* $U_{d,\alpha_d} = 0$, *then* $\mathrm{gr}_\nu k_{(d-1)}[X_d]$ *is a field.*

Proof. (i). Assume $f \in \mathrm{gr}_\nu k_{(d-1)}[X_d]$ is irreducible. By Theorem 5.12(i), $f = \tilde{f} U_d^J$. But $U_{d,j}$ is a unit for $j < \alpha_d$ (by Lemma 5.10 (iv)), and so U_{d,α_d} is the only irreducible element in $\mathrm{gr}_\nu k_{(d-1)}[X_d]$ (Lemma 5.10, (iii)).

(ii). We use Theorem 5.12(ii). There, we construct a polynomial $p(T) \in k_{(d-1)}[T]$. As we are working in the graded ring, we can replace the coefficients of p with their initials, which by assumption is a unique monomial $U^{I_0} \in k_{(d-1)}$. Thus, $P(T) = U^{I_0} p'(T)$, where $p'(T) \in k[T]$. Factorizing $p'(T) = \prod (T - \theta_l)$, modulo unit factors, we get,

$$f = U^{I_0} U_{d,\alpha_d}^{\hat{J}_{\alpha_d} - L n_{d,\alpha_d}} \prod_l (U_{d,\alpha_d}^{n_{d,\alpha_d}} - \theta_l U^{m^{(d,\alpha_d)}}),$$

where $L = \deg(p)$. On the other hand, Lemma 5.10 (iii) shows that all the elements of the form $U_{d,\alpha_d}^{n_{d,\alpha_d}} - \theta_l U^{m^{(d,\alpha_d)}}$ are irreducible in the graded ring $\mathrm{gr}_\nu k_{(d-1)}[X_d]$. Thus, the decomposition above is the decomposition of f into prime factors in $\mathrm{gr}_\nu k_{(d-1)}[X_d]$.

(iii). It is a result of Theorem 5.12(iii) and Lemma 5.10 (iv). □

Remark 5.14. Consider a valuation ν as above. The strong condition of Corollary 5.13 (ii) is satisfied if and only if for any $i = 0, \ldots, d-1$, either we have $U_{i,\alpha_i} = 0$ or $U_{i,\alpha_i} \neq 0$, and $n_{i,\alpha_i} = \infty$.

Theorem 5.15. (Homogeneous decomposition) *Let* ν *be a valuation associated with an* SKP. *Consider the ring* $R = k_{((\alpha,d))}$ *and the restriction of* ν *to it. Every element* $f \in R$ *has a unique decomposition of the form,*

$$f = p(T_{i_1}, \ldots, T_{i_{d_1}}) U^J, \text{ in } \mathrm{gr}_\nu R_\nu,$$

where $d_1 \leq d+1$ *and* $A = \{i_1, \ldots, i_{d_1}\}$, *for any* $i \in A$, $n_{i,\alpha_i} \neq \infty$ *and* $T_i = U_{i,\alpha_i}^{n_{i,\alpha_i}} U^{-m^{(i,\alpha_i)}}$. *And* $0 \leq J_{i',j} < n_{i',j}$, *for* $1 \leq j \leq \alpha_{i'}$. *And* $p(V_1, \ldots, V_{d_1}) \in k[V_1, \ldots, V_{d_1}]$.

Proof. Use a simple induction on d, using Theorem 5.12. For example, if $U_{d,\alpha_d} \neq 0$ and $n_{d,\alpha_d} \neq 0$, then $f = p(T_d) U_d^{J_d}$, in $\mathrm{gr}_\nu k_{(d-1)}[X_d]$, where the coefficients of $p(T_d) = \sum_l p_j T_d^l$ have the same ν-value. By the induction hypothesis, we have $p_l = q_l(T_{i_1}, \ldots, T_{i_{d^l}})U_{d-1}^{J^l}$, in $\mathrm{gr}_\nu R_\nu$, where $d^l \leq d$. Now, all the p_l have the same ν-value and thus the vectors J^l are the same for any l (similar argument as proof of Theorem 5.12(*ii*)). We denote this vector by J_{d-1}. Hence, we have $f = (\sum_l p_j)U_{d-1}^{J_{d-1}}U_d^{J_d}$ and we are done. □

Theorem 5.16. *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$, *such that* $\alpha_d \geq \omega$. *Suppose there exists an infinite sequence of ordinals* $s_1 < \cdots < s_\omega = \alpha_d$ *such that* $n_{d,s_j} > 1$, *for any* $j < \omega$. *Consider the acceptable vectors* $\alpha^{(s_j)}$ *(see Definition 5.7). For any* $f \in k_{(d-1)}[[X_d]]$, *there exists* $j_* \in \mathbb{N}$ *such that for any* $j \geq j_*$, *we have,*

$$\nu_{\alpha^{(s_j)}}(f) = \nu_{\alpha^{(s_{j_*})}}(f).$$

Thus, the limit $\lim_{j\to\omega} \nu_{\alpha^{(s_j)}}$ *is well-defined and is equal to* $\nu_{\alpha^{(s_\omega)}} = \nu$.

Proof. Multiplying f by a suitable factor $u \in k^{(d-1)}$, we can assume $f \in k^{(d)}$. By assumptions, we have $U_{d,\alpha_d} = 0$. Thus, by Corollary 4.5, we have $\mathrm{in}_{\nu_\alpha}(f) = c_J U_d^J$, $c_J \in k^{(d-1)}$. Suppose j_* is the maximum index such that $J_{s_{j_*}} \neq 0$. Then by the algorithm in proposition 3.10 for getting *adic* expansion, this j_* satisfies the conclusion of the Theorem. □

6. SKP-valuations and numerical invariants

One way to classify valuations is through their numerical invariants. Here, we show how the arithmetic of the SKP's of an SKP-valuation determines the numerical invariants of the associated valuation on the field $k^{(d)}$.

We define the notion of pseudo-SKP. It allows us to avoid ordinal numbers greater than ω for α_i.

Definition 6.1. For an SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$, a pseudo-SKP is a subset of U and β which comes from dropping an arbitrary number of $U_{i,j}$'s (and associated $\beta_{i,j}$'s) for $j < \alpha_i$ such that $n_{i,j} = 1$. With any SKP, a minimal pseudo-SKP is associated which is obtained by dropping all the $U_{i,j}$ for $j < \alpha_i$ such that $n_{i,j} = 1$. This minimal associated pseudo-SKP is unique. We denote this minimal pseudo-SKP by $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha'_i}$, where $\alpha'_i \leq \omega$ (using the same notation as the SKP).

Proposition 6.2. *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$*, and let* ν *be the associated* $k-$*valuation. Let* $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha'_i}$ *be its minimal pseudo-SKP. The valuation* ν *can be defined using the data* $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha'_i}$.

Proof. To define the valuation ν, it is sufficient to know the *adic* expansion of elements. Moreover, in the *adic* expansion of an element, the $U_{i,j}$ with $n_{i,j} = 1$ cannot appear. Thus, the *adic* expansion of every element is defined using only the minimal pseudo-SKP associated with ν. □

The following lemma computes the rank and rational rank and value semigroup of an SKP valuation in terms of the arithmetic of the SKP.

Lemma 6.3. *Consider a* $k-$*valuation centered on the ring* $k^{(d)}$ *such that* $\nu = \mathrm{val}[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$*. Let* $\overline{\nu} = \nu\mid_{k^{(d-1)}}$*. By Remark 2.5(v), the data* $[U_{i,j}, \beta_{i,j}]_{i=0..d-1,j=1..\alpha_i}$ *is an* SKP.

- (i) *We have,* $\overline{\nu} = \mathrm{val}[U_{i,j}, \beta_{i,j}]_{i=0..d-1,j=1..\alpha_i}$.
- (i) *We have* $\mathrm{rk}(\nu) - \mathrm{rk}(\overline{\nu}) \in \{0, 1\}$*. More precisely,* $\mathrm{rk}(\nu) = \mathrm{rk}(\overline{\nu}) + 1$ *if and only if* $\beta_{d,\alpha_d} \notin \Delta$ *(*Δ *is the smallest isolated subgroup of* Φ *such that* $\Phi^*_{d-1,\alpha_{d-1}} \subset \Delta$*), and* $\mathrm{rk}(\nu) = \mathrm{rk}(\overline{\nu})$ *iff* $\beta_{d,\alpha_d} \in \Delta$.
- (ii) *We have,* $\mathrm{r.rk}(\nu) - \mathrm{r.rk}(\overline{\nu}) \in \{0, 1\}$*. More precisely,* $\mathrm{r.rk}(\nu) = \mathrm{r.rk}(\overline{\nu}) + 1$ *if and only if* $\beta_{d,\alpha_d} \notin \Phi^*_{d-1,\alpha_{d-1}}$*, and* $\mathrm{r.rk}(\nu) = \mathrm{r.rk}(\overline{\nu})$ *if and only if* $\beta_{d,\alpha_d} \in \Phi^*_{d-1,\alpha_{d-1}}$.
- (iii) *The semigroup* $\nu(k^{(d)} \setminus \{0\})$ *is equal to* Γ_{d,α_d}.

Proof. We only prove (i). It is the consequence of the fact that for any $f \in k^{(d-1)}$, the *adic* expansions of f with respect to the two SKP's $[U_{i,j}, \beta_{i,j}]_{i=0..d,j=1..\alpha_i}$ and $[U_{i,j}, \beta_{i,j}]_{i=0..d-1,j=1..\alpha_i}$ are the same. □

Theorem 6.4. *Consider a* $k-$*valuation centered on the ring of power series* $k[[X_0, X_1, X_2]]$*,* ν*, which is defined by an* SKP*; in other words, let* $\nu = \mathrm{val}[U_{i,j}, \beta_{i,j}]_{i=0,1,2,j=1..\alpha_i}$*. Moreover, we suppose* $\beta_{0,1} \in \Delta_1$*. Then, we can compute the numerical invariants of this valuation using the arithmetic of its minimal pseudo-*SKP*. This is summarized in* Table *1.*

	Arithmetic of minimal pseudo-SKP of the valuation ν		*rk*	*r.rk*	*tr.deg*
(I)	$\alpha'_1 < \infty,\ \alpha'_2 < \infty$	$\beta_{i,j} \in \mathbb{Q}\beta_{0,1}$	1	1	2
$(\mathrm{II})_1$ $(\mathrm{II})_2$	$\alpha'_1 < \infty,\ \alpha'_2 < \infty$ $\alpha'_1 < \infty,\ \alpha'_2 < \infty$	$\beta_{i,j} \in \Delta_1,\ \beta_{1,\alpha'_1} \in \mathbb{Q}\beta_{0,1},\ \beta_{2,\alpha'_2} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1}$ $\beta_{i,j} \in \Delta_1,\ \beta_{1,\alpha'_1} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1},\ \beta_{2,\alpha'_2} \in \mathbb{Q}\beta_{0,1}$	1	2	1
$(\mathrm{III})_1$ $(\mathrm{III})_2$	$\alpha'_1 = \infty,\ \alpha'_2 < \infty$ $\alpha'_1 < \infty,\ \alpha'_2 = \infty$	$\beta_{i,j} \in \mathbb{Q}\beta_{0,1}$ $\beta_{i,j} \in \mathbb{Q}\beta_{0,1}$	1	1	1
(IV)	$\alpha'_1 < \infty, \alpha'_2 < \infty$	$\beta_{1,\alpha'_1} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1},\ \beta_{2,\alpha'_2} \in \Delta_1 \backslash (\beta_{0,1}, \beta_{1,\alpha'_1}) \otimes \mathbb{Q}$	1	3	0
$(\mathrm{V})_1$ $(\mathrm{V})_2$	$\alpha'_1 = \infty, \alpha'_2 < \infty$ $\alpha'_1 < \infty, \alpha'_2 = \infty$	$\beta_{2,\alpha'_2} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1}$ $\beta_{1,\alpha'_1} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1}$	1	2	0
(VI)	$\alpha'_1 < \infty, \alpha'_2 < \infty$	$\max\{\beta_{i,\alpha'_i}\} \in \Delta_2 \backslash \Delta_1,\ \beta_{1,\alpha'_1} \in (\beta_{0,1}, \beta_{2,\alpha'_2}) \otimes \mathbb{Q}$	2	2	1
$(\mathrm{VII})_1$ $(\mathrm{VII})_2$	$\alpha'_1 < \infty, \alpha'_2 < \infty$ $\alpha'_1 < \infty, \alpha'_2 < \infty$	$\beta_{1,\alpha'_1} \in \Delta_2 \backslash \Delta_1,\ \beta_{2,\alpha'_2} \in \Phi \backslash \Delta_2$ $\beta_{2,\alpha'_2} \in \Delta_2 \backslash \Delta_1,\ \beta_{1,\alpha'_1} \in \Phi \backslash \Delta_2$	3	3	0
$(\mathrm{VIII})_1$ $(\mathrm{VIII})_2$	$\alpha'_1 = \infty, \alpha'_2 < \infty$ $\alpha'_1 < \infty, \alpha'_2 = \infty$	$\beta_{2,\alpha'_2} \in \Delta_2 \backslash \Delta_1$ $\beta_{1,\alpha'_1} \in \Delta_2 \backslash \Delta_1$	2	2	0
(IX)	$\alpha'_1 < \infty, \alpha'_2 < \infty$	$\max\{\beta_{i,\alpha'_i}\} \in \Delta_2 \backslash \Delta_1,\ \beta_{1,\alpha'_1} \in \Delta_2 \backslash (\beta_{0,1}, \beta_{2,\alpha'_2}) \otimes \mathbb{Q}$	2	3	0
(X)	$\alpha'_1 = \infty, \alpha'_2 = \infty$		1	1	0

TABLE 1. Numerical invariants via arithmetic of SKP of the valuation

Proof. The computation of the rank and the rational-rank is a simple task. The only nontrivial task is the computation of the transcendence degree or the dimension of valuation. It is a direct calculation using Theorem 5.15. For example, in the case (I), pick $f, g \in k_{(d-1)}$ with $\nu(f) = \nu(g)$. Then, by Theorem 5.15, we have $\mathrm{in}(f) = p(T_1, T_2)U^J$ and $\mathrm{in}(g) = q(T_1, T_2)U^{J'}$. Using the properties of J and J' in the theorem, we see that $J = J'$. Thus, $f/g = p(T_1, T_2)/q(T_1, T_2)$. This shows $k_\nu = R_\nu/\mathfrak{m}_\nu = k(T_1, T_2)$. We show that T_1 and T_2 are algebraically independent in k_ν. If T_2 is algebraic over $k(T_1)$, then there is a polynomial $0 \neq p(T) \in k(T_1)[T]$ such that $p = p(T_2) = \sum_i c_i T_2^i = 0$ in k_ν. Regarding T_1 and T_2 as elements of R_ν, we have $p(T_2) = \sum_i c_i T_2^i \in \mathfrak{m}_\nu$. Note that $T_1 = \frac{U_{1,\alpha_1}^{n_{1,\alpha_1}}}{U^{m^\cdot}}$ and $T_2 = \frac{U_{2,\alpha_2}^{n_{2,\alpha_2}}}{U^{m}}$. Multiplying p with a suitable power of $U^{m^\cdot + m}$, say n, we can assume that $U^{n(m^\cdot + m)}p \in k_{((\alpha,2))}$. The condition $p \in \mathfrak{m}_\nu$ implies that the cancelation should occur between initail monomials of monomials of $U^{n(m^\cdot + m)}p$ in the course of getting the *adic* expansion. We show that this is impossible.

Write $p = \sum_{i,j} r_{i,j} T_1^i T_2^j$, $r_{i,j} \in k$. Then, we have,

$$U^{n(m^\cdot+m)}p = \sum_{i,j} r_{i,j} U^{n_{[i,j]}} U_{1,\alpha_1}^i U_{2,\alpha_2}^j.$$

By Lemma 4.6(ii), no cancelation can occur between initial monomials of monomials of $U^{n(m^\cdot+m)}p$ with different j's (notice that index $(2, \alpha_2)$ does not occur in $U^{m_{[i,j]}}$). It remains to show that no cancelation can occur for a sum of the form $q_j = \sum_i r_{i,j} U^{n_{[i,j]}} U_{1,\alpha_1}^i U_{2,\alpha_2}^j$. Notice that the power of the U_{2,α_2}, are the same for different monomials of q and the power of the U_{1,α_1} are different for any two monomial of q. Now, the proof of Lemma 4.6(ii), shows that in the course of getting the *adic* expansion of the monomials of q, the powers of U_{1,α_1} in the initial monomials remain diffrent, for any two monomial of q. Thus, no cancelation can occur between the initial monomials of q. □

7. Realization of a certain class of semi-groups as value semi-groups of polynomial rings

Here, we give a result on the realization of a semi-group as the semi-group of values which takes a valuation on a polynomial ring.

Theorem 7.1. *Let Γ be a semigroup of an ordered abelian group $(\Psi, <)$, given by a minimal system of generators $\{\gamma_j\}_{j \leq \alpha} \subseteq \Psi^+$, where $\alpha = \omega n + j$, for $n, j \in \mathbb{N}$. Suppose Γ is of positive type (Definition 2.2), and $\gamma_{j+1} > n_j \gamma_j$, when $n_j \neq \infty$. Set $G = (\Gamma)$ and $d = \text{r.rk}(G)$. Then, there exists a zero-dimensional valuation ν of the field $k(X_1, \ldots, X_d)$, centered on the polynomial ring $R = k[X_1, \ldots, X_d]$, such that its value-semigroup is equal to Γ.*

Proof. Consider the semigroup Γ with the minimal systems of generators $\{\gamma_{j'}\}_{j' \leq \alpha}$, and suppose $\alpha = \omega n + j^*$, $j^* \in \mathbb{N}$. We give new names $(s_{t'}^t)$ to the indices of those γ's which are rationally independent from the previous ones (by Proposition 7.3, this includes all the indices which are limit ordinals; i.e., for $t = 1..n$, we have $n_{\omega t} = \infty$; see Lemma 2.1 for definition of n): For $t = 1..n+1$, let $f_t \in \mathbb{N}$ be the number of j' such that $\omega(t-1) \leq j' < \omega t$ and $n_{j'} = \infty$, and then set $s_{t'}^t := j'$, when j' is the t'th such j', for $t' = 1..f_t$. Then, we have,

$$\{\gamma_{j'}\}_{j'\leq\alpha} = \{\gamma_{s^t_{t'}+j}\}_{t=1..n+1,t'=1..f_t,j=0..j_{t,t'}},$$

where $j_{t,t'}$ is the number of indices j' such that $\gamma_{s^t_{t'}} \leq \gamma_{j'} < \gamma_{s^t_{t'+1}}$. Then, by Proposition 7.3, we have $\mathrm{r.rk}(G) = f_1 + \cdots + f_{n+1}$.

We define new indices $i_{t,t'}$ which will be the indices of the variables of the polynomial ring: For $t = 1..n+1$ and $t' = 1..f_t$ set $i_{t,t'} := f_0 + \cdots + f_{t-1} + t'$, where, by convention, $f_0 = 0$. The total number d of the $i_{t,t'}$ which have been defined is equal to:

$$d = f_1 + \cdots + f_{n+1} = \mathrm{r.rk}(G).$$

It is straightforward to check that the sequence,

$$\{\beta_{i_{t,t'},j} := \gamma_{s^t_{t'}+j-1}\}_{i_{t,t'}=1..d,j=1..j_{t,t'}},$$

is a sequence of values (note the index i starts from 1). The key-polynomials of the SKP associated with this sequence of values are elements of the ring $R = k[X_1, \ldots, X_d]$. The valuation ν associated with this SKP has value semi-group Γ.

Notice that we have $\mathrm{r.rk}(\nu) = \dim R = d$. Hence, we are in the case of equality of Abhyankar's inequality $\mathrm{r.rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \dim R = d$. Thus, the valuation ν is zero-dimensional. Moreover, one can not realize Γ as a value semi-group of a polynomial ring with $< d$ variables. □

Remark 7.2. The following remarks are in order:

- The positivity condition is quite restrictive, in general. However, in the case we restrict to the value semi-groups of polynomial rings of two variables, all the value semi-groups are of positive type (see Proposition 4.2 in [2]). Moreover, in this case, if the ordinal type of the group is ω^2, then we are in the equality case of Abhyankar's inequality and the semigroup has to be of positive type.
- The semigroup Γ is well ordered by [9], and it is of ordinal type $\leq \omega^{\mathrm{rk}(G)}$ (see [15] Vol. II, Appendix 3, Proposition 2).

Proposition 7.3. *With the notation of Theorem 7.1 and Lemma 2.1, for any limit ordinal* $\omega(i+1) \leq \alpha$, *we have* $\mathrm{rk}(G_{\omega(i+1)}) = \mathrm{rk}(G_{\omega(i+1)^-}) + 1$. *In particular,* $n_{\omega(i+1)} = \infty$.

Proof. We extend the notion of effective component to this situation. Consider an order embedding $(\Phi, <) \subseteq (\mathbb{R}^n, <_{lex})$ such that $\Gamma \subseteq \mathbb{R}^n_{\geq_{les} 0}$. By definition, the effective component for the limit ordinal ωi is the first index $t \leq n$ such that $\#\{(\gamma_j)_t\}_{\omega i \leq j < \omega(i+1)} = \infty$. Like in the case of effective components, one can prove that t is well-defined. Note that $(\gamma_j)_{t'} = 0$, for $t' < t$ and $j < \omega(i+1)$. Moreover, one can show that the conditions of Proposition 2.8(i) and (ii) hold in this case. Suppose the effective component for ωi is t. Then, an argument, similar to the proof given for Proposition 2.8(iii), shows that $(\gamma_j)_t \to +\infty$ $(j \to \omega(i+1))$. But, $\gamma_{\omega(i+1)} >_{lex} \gamma_{\omega i + j}$, for $j \in \mathbb{N}$. This is possible only if $(\gamma_{\omega(i+1)})_{t'} > 0$, for some $t' < t$. □

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