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EXISTENCE AND UNIQUENESS OF COMMON COUPLED FIXED POINT RESULTS VIA AUXILIARY FUNCTIONS

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Abstract. The purpose of this paper is to establish some coupled coincidence point theorems for mappings having a mixed g-monotone property in partially ordered metric spaces. Also, we present a result on the existence and uniqueness of coupled common fixed points. The results presented in the paper generalize and extend several well-known results in the literature.
Keywords: Coupled coincidence point, ordered sets, coupled fixed point, mixed monotone property.
MSC(2010): Primary: 46N40; Secondary: 47H10, 54H25, 46T99.

1. Introduction and preliminaries

The Banach contraction mapping principle and its applications can be given as a pivotal example to support the conclusion that fixed point theory is crucial to nonlinear analysis, which is frequently used not only in many fundamental fields of mathematics, but also in Economics, Computer Science, and many others. Hence, it is very sensible to investigate generalizations of the Banach's principle to better equip the quantitative scientists working in these areas to tackle their problems.

Currently there exist considerable number of generalizations of the Banach contraction principle in the literature. For instance, Ran and Reurings [18] extended the Banach contraction principle in the context of partially ordered sets with some applications to linear and nonlinear matrix equations. Then Nieto and Rodŕiguez-López [17] extended the

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result of Ran and Reurings and applied their main theorems to obtain a unique solution to a first order ordinary differential equation with periodic boundary conditions. As a continuation of the work of these mathematicians, Guo and Lakshmikantham [11] introduced the notion of coupled fixed point. Later, Gnana-Bhaskar and Lakshmikantham [6] suggested the concept of mixed monotone mappings and obtained some coupled fixed point results for these mappings. Also, they applied their results to certain first order differential equation with periodic boundary conditions.

Recently, many researchers have obtained further fixed point, common fixed point, coupled fixed point and coupled common fixed point results in metric spaces and partially ordered metric spaces (see [1]-[20]). In this paper, we are particularly interested in establishing some coupled coincidence point results in partially ordered metric spaces for mappings having the mixed g-monotone property. Additionally, we aim to apply our result on integral equations to get the existence and uniqueness of coupled common fixed points. First, we recall some basic definitions and notations:

Definition 1.1. *(See* [6]*) An element* $(x, y) \in X \times X$ *is called a coupled fixed point of a mapping* $F : X \times X \to X$ *if* $F(x, y) = x$, *and* $F(y, x) = y$.

Definition 1.2. *(See* [15]*) An element* $(x, y) \in X \times X$ *is called a coupled coincidence point of mappings* $F : X \times X \to X$ *and* $g : X \to X$ *if* $F(x, y) = gx$, *and* $F(y, x) = gy$.

Note that if g is the identity mapping in Definition 1.2, then $(x, y) \in X \times X$ is called a *coupled fixed point.*

Definition 1.3. *Let* (X, d) *be a metric space and* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings. Then,* F *and* g *are said to commute if* $F(gx, gy) = g(F(x, y))$ *for all* $x, y \in X$.

Definition 1.4. *(See* [6]*) Let* $(X, \preceq)$ *be a partially ordered set and* $F : X \times X \to X$ *be a mapping. Then* F *is said to be non-decreasing if* $x \preceq y$ *implies* $F(x) \preceq F(y)$ *and non-increasing if* $x \preceq y$ *implies* $F(x) \succeq F(y)$ *for every* $x, y \in X$.

Definition 1.5. *(See* [15]*) Let* $(X, \preceq)$ *be a partially ordered set and* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings. The mapping* F *is said to be* g*-non-decreasing if* $gx \preceq gy$ *implies* $F(x) \preceq F(y)$ *and* g*-non-increasing if* $gx \preceq gy$ *implies* $F(x) \succeq F(y)$ *for every* $x, y \in X$.

Notice that if g is the identity map in the definition above, then Definition 1.5 reduces to Definition 1.4.

Definition 1.6. *(See [6]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 1.7. *(See [15]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. The mapping F is said to have the mixed g-monotone property if $F(x, y)$ is monotone g-non-decreasing in x and monotone g-non-increasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Similar to the remarks above, if g is the identity mapping in Definition 1.7, then the mapping F is said to have the *mixed monotone property.*

2. **Main results**

In this section, we aim to prove some coupled common fixed point theorems in the context of ordered metric spaces. First, we will give the following definition that we shall need in our results:

Definition 2.1. *Let Θ denote the class of functions $\theta : [0, \infty) \times [0, \infty) \to [0, 1)$ which satisfy the statement*

$$\theta(t_n, s_n) \to 1 \quad \textit{implies} \quad t_n, s_n \to 0$$

for any sequences $\{t_n\}$ and $\{s_n\}$ of positive real numbers.

As examples, consider $\theta_2(s, t) = \dfrac{\ln(1 + ks + lt)}{ks + lt}$, for all $(s, t) \in [0, \infty) \times [0, \infty) - \{(0, 0)\}$, $\theta_2(0, 0) \in [0, 1)$, where $k, l > 0$, and $\theta_3(s, t) = \dfrac{\ln(1 + \max\{s, t\})}{\max\{s, t\}}$, for all $(s, t) \in [0, \infty) \times [0, \infty) - \{(0, 0)\}$, $\theta_3(0, 0) \in [0, 1)$. Both θ_2 and θ_3 are in Θ. For more examples see [16].

Remark 2.2. *The condition*

$$\theta(t_n, s_n) \to 1 \text{ implies } (t_n, s_n) \to (0,0) \quad (1)$$

is equivalent to

$$(t_n, s_n) \nrightarrow (0,0) \text{ implies } \theta(t_n, s_n) \nrightarrow 1 \quad (2)$$

Every constant function satisfies the condition (2). So, constant functions are in Θ. *That is,* $\theta_1 \in \Theta$ *where* $\theta_1(s,t) = k$, *for all* $(s,t) \in [0,\infty) \times [0,\infty)$, *for* $k \in [0,1)$.

Our main theorem is stated as follows:

Theorem 2.3. *Let* $(X, \preceq)$ *be a partially ordered set and* (X, d) *be a metric space. Suppose that* $F : X \times X \to X$ *and* $g : X \to X$ *are two mappings such that* F *has the mixed* g*-monotone property on* X. *Assume that there exist two elements* $x_0, y_0 \in X$ *with* $g(x_0) \preceq F(x_0, y_0)$ *and* $g(y_0) \succeq F(y_0, x_0)$. *Suppose also that there exists* $\theta \in \Theta$ *such that*

(2.1)
$$\begin{aligned} d(F(x,y), F(u,v)) \quad &+ d(F(y,x), F(v,u)) \\ &\leq \theta(d(gx,gu), d(gy,gv))(d(gx,gu) + d(gy,gv)) \end{aligned}$$

for all $x, y, u, v \in X$ *with* $gx \succeq gu$ *and* $gy \preceq gv$. *Further suppose that* $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X. *Also, assume that* X *has the following properties:*

(i) if $\{g(x_n)\} \subset X$ *is a non-decreasing sequence with* $gx_n \to gx$ *in* $g(X)$, *then* $gx_n \preceq gx$, *for every* n;

(ii) if $\{g(y_n)\} \subset X$ *is a non-increasing sequence with* $gy_n \to gy$ *in* $g(X)$, *then* $gy_n \succeq gy$, *for every* n.

Then there exist $x, y \in X$ *such that* $F(x,y) = g(x)$ *and* $gy = F(y,x)$, *that is,* F *and* g *have a coupled coincidence point* $(x,y) \in X \times X$.

Proof. Let x_0, y_0 be two elements in X such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(2.2) $$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n), \forall n \geq 0.$$

We claim that for all $n \geq 0$,

(2.3) $$gx_n \preceq gx_{n+1},$$

and

(2.4) $$gy_n \succeq gy_{n+1}.$$

We shall use the mathematical induction. Let $n = 0$. Since $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, in the view of the facts $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$, that is, (2.3) and (2.4) hold for $n = 0$. Suppose that (2.3) and (2.4) hold for some $n > 0$. As F has the mixed g-monotone property and $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$, from (2.2), we get

$$gx_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \preceq F(x_{n+1}, y_{n+1}) = gx_{n+2}, \tag{2.5}$$

and

$$gy_{n+1} = F(y_n, x_n) \succeq F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = gy_{n+2}. \tag{2.6}$$

Now from (2.5) and (2.6), we obtain that $gx_{n+1} \preceq gx_{n+2}$ and $gy_{n+1} \succeq gy_{n+2}$. Thus by the mathematical induction, we conclude that (2.3) and (2.4) hold for all $n \geq 0$. Therefore

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq \ldots \preceq gx_n \preceq gx_{n+1} \preceq \ldots, \tag{2.7}$$

and

$$gy_0 \succeq gy_1 \succeq gy_2 \succeq \ldots \succeq gy_n \succeq gy_{n+1} \succeq \ldots. \tag{2.8}$$

Assume that there is some $r \in \mathbb{N}$ such that $d(gx_r, gx_{r-1}) + d(gy_r, gy_{r-1}) = 0$, that is, $gx_r = gx_{r-1}$ and $gy_r = gy_{r-1}$. Then $gx_{r-1} = F(x_{r-1}, y_{r-1})$ and $gy_{r-1} = F(y_{r-1}, x_{r-1})$, and hence we get the result.

For simplicity, we let $t_{n+1} := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)$ and also $\theta_n = \theta(d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}))$. Now, we assume that $t_n = d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) \neq 0$ for all n. Since $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$, from (2.1) and (2.2), we have

$$\begin{aligned} t_{n+1} = d(gx_{n+1}, gx_n) \quad & + d(gy_{n+1}, gy_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ & \qquad + d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ & \leq \theta_n (d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)), \\ & \leq \theta_n \cdot t_n \end{aligned} \tag{2.9}$$

which implies that $t_{n+1} < t_n$. It follows that the sequence $\{t_n\}$ is monotone decreasing. Therefore, there is some $t \geq 0$ such that $\lim_{n\to\infty} t_n = t$.

Now, we shall show that $t = 0$. Assume to the contrary that $t > 0$, then from (2.9), we have

$$\frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)} \leq \theta_n < 1,$$

which yields that $\lim_{n\to\infty} \theta_n = 1$.

This implies that $d(gx_{n-1}, gx_n) \to 0$ and $d(gy_{n-1}, gy_n) \to 0$ (since $\theta \in \Theta$) or $d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \to 0$, which is a contradiction. Therefore $t = 0$, that is,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] = 0. \tag{2.10}$$

Now, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. On the contrary, assume that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequences $\{gx_{m(k)}\}$ and $\{gx_{n(k)}\}$ of $\{gx_n\}$ and $\{gy_{m(k)}\}$ and $\{gy_{n(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) > k$ such that for every k,

$$d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) \geq \epsilon. \tag{2.11}$$

Furthermore, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (2.11). Then we get

$$d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)-1}, gy_{m(k)}) < \epsilon. \tag{2.12}$$

Using (2.11) and (2.12), we have

$$\begin{aligned} \epsilon \leq r_k &:= d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) \\ &\leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}) \\ &\quad + d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)}) \\ &< \epsilon + t_{n(k)}. \end{aligned}$$

On letting $k \to \infty$ and using (2.10), we obtain

$$\lim r_k = \lim[d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)})] = \epsilon. \tag{2.13}$$

Also, by the triangle inequality, we have

$$\begin{aligned} r_k &= d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) \\ &\leq d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)}) \\ &\quad + d(gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)}) \\ &= t_{n(k)} + t_{m(k)} + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gx_{m(k)+1}). \end{aligned}$$

Since $n(k) > m(k)$, $gx_{n(k)} \succeq gx_{m(k)}$ and $gy_{n(k)} \preceq gy_{m(k)}$, from (2.1) and (2.2), we derive

$$\begin{aligned} d(gx_{n(k)+1}, gx_{m(k)+1}) &+ d(gy_{n(k)+1}, gx_{m(k)+1}) = d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})) \\ &\leq \theta(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})) \\ &\quad (d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})) \\ &= \theta(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}))r_k. \end{aligned}$$

Therefore, we find
$r_k \leq t_{n(k)} + t_{m(k)} + \theta(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}))r_k$. This implies that

$$\frac{r_k - t_{n(k)} - t_{m(k)}}{r_k} \leq \theta(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})) < 1.$$

Letting $k \to \infty$, we get $\theta(d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})) = 1$. Since $\theta \in \Theta$, we conclude

$$\lim d(gx_{n(k)}, gx_{m(k)}) = \lim d(gy_{n(k)}, gy_{m(k)}) = 0,$$

a contradiction. This implies that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is a complete subspace of X, there exists $(x, y) \in X \times X$ such that $gx_n \to gx$ and $gy_n \to gy$. By the facts that $\{gx_n\}$ is a non-decreasing sequence with $gx_n \to gx$ and $\{gy_n\}$ is a non-increasing sequence with $gy_n \to gy$, it follows by the assumption of the theorem that $gx_n \preceq gx$ and $gy_n \succeq gy$ for all n. As a result, we derive the following inequality

$$\begin{aligned} d(F(x,y),gx)+d(F(y,x),gy) &\leq d(F(x,y),gx_{n+1})+d(gx_{n+1},gx) \\ &\quad +d(F(y,x),gy_{n+1})+d(gy_{n+1},gy) \\ &= d(gx_{n+1},gx)+d(gy_{n+1},gy)+d(F(x,y),F(x_n,y_n)) \\ &\qquad +d(F(y,x),F(y_n,x_n)) \\ &\leq d(gx_{n+1},gx)+d(gy_{n+1},gy) \\ &\qquad +\theta(d(gx,gx_n),d(gy,gy_n))(d(gx,gx_n)+d(gy,gy_n)). \end{aligned}$$

Taking the limit as $n \to \infty$, in the inequality above, we get $d(F(x, y), gx) + d(F(y, x), gy) = 0$. Hence, we find $gx = F(x, y)$ and $gy = F(y, x)$, which proves that F and g have a coupled coincidence point. □

Remark 2.4. *Theorem 2.3 is a generalization of Theorem 2.1 in* [16].

Theorem 2.5. *Let $(X, \preceq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ are two mappings such that F has the mixed g-monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose also that there exists a $\theta \in \Theta$ such that the inequality in (2.1) is satisfied for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Suppose further that g is a continuous, non-decreasing map which commutes with F such that $F(X \times X) \subseteq g(X)$. If either*

a) F is continuous or

b) X has the following properties:

(i) if a sequence $\{g(x_n)\} \subset X$ is a non-decreasing sequence with $gx_n \to gx$ in $g(X)$, then $gx_n \preceq gx$, for every n;

(ii) if a sequence $\{g(y_n)\} \subset X$ is a non-increasing sequence with $gy_n \to gy$ in $g(X)$, then $gy_n \succeq gy$, for every n,

then there exist $x, y \in X$ such that $F(x,y) = g(x)$ and $gy = F(y,x)$; that is, F and g have a coupled coincidence point $(x,y) \in X \times X$.

Proof. Following the proof of Theorem 2.3, we will get two Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that $\{gx_n\}$ is a non-decreasing sequence in X and $\{gy_n\}$ is a non-increasing sequence in X. Since X is a complete metric space, there is $(x,y) \in X \times X$ such that $gx_n \to x$ and $gy_n \to y$. Since g is continuous, we get $g(gx_n) \to gx$ and $g(gy_n) \to gy$.

Firstly, suppose that F is continuous. Then we know that $F(gx_n, gy_n) \to F(x,y)$ and $F(gy_n, gx_n) \to F(y,x)$. As, F commutes with g, we have $F(gx_n, gy_n) = gF(x_n, y_n) = g(gx_{n+1}) \to gx$ and $F(gy_n, gx_n) = gF(y_n, x_n) = g(gy_{n+1}) \to gy$. By the uniqueness of limit, we get $gx = F(x,y)$ and $gy = F(y,x)$.

Secondly, suppose that (b) holds. Since $\{gx_n\}$ is a non-decreasing sequence with $gx_n \to x$ and $\{gy_n\}$ is a non-increasing sequence with $gy_n \to y$, and g is a non-decreasing function, we find $g(gx_n) \preceq gx$ and $g(gy_n) \succeq gy$ hold for all $n \in \mathbb{N}$. Hence by (2.1), we have

$$\begin{aligned} d(g(gx_{n+1}), F(x,y)) \quad + \quad & d(g(gy_{n+1}), F(y,x)) = d(F(gx_n, gy_n), F(x,y)) \\ & +d(F(gy_n, gx_n)F(y,x)) \\ \leq \quad & \theta(d(g(gx_n), gx), d(g(gy_n), gy)) \\ & (d(g(gx_n), gx) + d(g(gy_n), gy)). \end{aligned}$$

Taking the limit as $n \to \infty$, we get $d(gx + F(x,y)) + d(gy, F(y,x)) = 0$. In other words, we obtain $gx = F(x,y)$ and $gy = F(y,x)$. Thus F and g have a coupled coincidence point. □

The following example illustrates that our result is more general than the main result of Lakshmikantham- Ćirić [15].

Example 2.6. *Let $X = [0, \infty)$ and $d(x,y) = |x - y|$.*

Let $F : X \times X \to X$ be defined as $F(x,y) = \begin{cases} \frac{3x^2 - 7y^2}{12} & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$

for $x, y \in X$

and

$g : X \to X$ be defined as $g(x) = x^2$.

Then the operator F has strict mixed monotone property and satisfies the condition of Theorem 2.1 in [15]*. But the main result of Lakshmikantham-Ćirić* [15] *is not applicable.*

Note that,

$$[d(gx,gu)+d(gy,gv)] = [|x^2-u^2|+|y^2-v^2|] \tag{2.14}$$

(2.15)

$$d(F(x,y),F(u,v))+d(F(y,x),F(v,u)) = \left|\frac{3x^2-7y^2}{12}-\frac{3u^2-7v^2}{12}\right|.$$

for $x,y,u,v \in X$ *with* $x \leq u,\ y \geq v$.

On the other hand, we derive that

(2.16)

$$d(F(x,y),F(u,v)) = \left|\frac{x^2-7y^2}{12}-\frac{u^2-7v^2}{12}\right| = \left|\frac{7v^2-7y^2}{12}\right| = \frac{7}{12}|v^2-y^2|$$

where $x=u$ *and* $y \leq v$. *By combining (2.15) and (2.16), we get*

$$\frac{7}{12}|v^2-y^2| \leq k\frac{1}{2}(|y^2-v^2|)$$

which is a contradiction. (Since k *is less than* 1*.)*

But, F *satisfied Theorem 2.3. Indeed, we have*

$$[d(x,u)+d(y,v)] = (|x^2-u^2|+|y^2-v^2|). \tag{2.17}$$

and also

$$\left|\frac{x^2-7y^2}{12}-\frac{u^2-7v^2}{12}\right| \leq \frac{1}{12}|x^2-u^2|+\frac{7}{12}|v^2-y^2|, \quad x \geq u, y \leq v \tag{2.18}$$

(2.19)

$$\left|\frac{7x^2-y^2}{12}-\frac{7u^2-v^2}{12}\right| \leq \frac{1}{12}|v^2-y^2|+\frac{7}{12}|x^2-u^2|, \quad x \geq u, y \leq v.$$

From (2.18) and (2.19), we obtain that

$$\begin{aligned} d(F(x,y),F(u,v))+d(F(y,x),F(v,u)) &= \left(\left|\frac{x^2-7y^2}{12}-\frac{u^2-7v^2}{12}\right|+\left|\frac{7x^2-y^2}{12}-\frac{7u^2-v^2}{12}\right|\right) \\ &\leq \frac{8}{12}\left(|x^2-u^2|+|y^2-v^2|\right). \end{aligned}$$

The claim follows by choosing $\theta(s,t)=k=\frac{2}{3}<1$. *Notice that* $(0,0)$ *is the coupled fixed point of* F.

Corollary 2.7. *Let* $(X,\preceq)$ *be a partially ordered set and* (X,d) *be a complete metric space. Suppose that* $F: X \times X \to X$ *and* $g: X \to X$ *are two mappings such that* F *has the mixed* g*-monotone property on* X.

Assume that there exists two elements $x_0, y_0 \in X$ *with* $g(x_0) \preceq F(x_0, y_0)$ *and* $g(y_0) \succeq F(y_0, x_0)$*. Suppose also that there exists a* $\zeta \in \Theta$ *such that* (2.20)

$$d(F(x,y), F(u,v)) \leq \frac{1}{2}\zeta(d(gx,gu), d(gy,gv))(d(gx,gu) + d(gy,gv))$$

for all $x, y, u, v \in X$ *with* $gx \succeq gu$ *and* $gy \preceq gv$*. Furthermore, suppose that* g *is continuous non-decreasing map which commutes with* F *such that* $F(X \times X) \subseteq g(X)$*. If either*

a) F *is continuous or*

b) X *has the following properties:*

(i) if a sequence $\{g(x_n)\} \subset X$ *is a non-decreasing sequence with* $gx_n \to gx$ *in* $g(X)$*, then* $gx_n \preceq gx$*, for every* n*;*

(ii) if a sequence $\{g(y_n)\} \subset X$ *is a non-increasing sequence with* $gy_n \to gy$ *in* $g(X)$*, then* $gy_n \succeq gy$*, for every* n*,*

then there exist $x, y \in X$ *such that* $F(x,y) = g(x)$ *and* $gy = F(y,x)$*; that is,* F *and* g *have a coupled coincidence point* $(x,y) \in X \times X$*.*

Proof. For $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, from (2.20), we have

$$d(F(x,y), F(u,v)) \leq \frac{1}{2}\zeta(d(gx,gu), d(gy,gv))(d(gx,gu) + d(gy,gv)),$$

and

$$\begin{aligned} d(F(y,x), F(v,u)) &= d(F(v,u), F(y,x)) \\ &\leq \frac{1}{2}\zeta(d(gv,gy), d(gu,gx))(d(gv,gy) + d(gu,gx)) \\ &= \frac{1}{2}\zeta(d(gv,gy), d(gu,gx))(d(gx,gu) + d(gy,gv)). \end{aligned}$$

Therefore, we derive the inequality

$$\begin{aligned} & d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\ \leq\; & \frac{1}{2}[\zeta(d(gv,gy), d(gu,gx)) + \zeta(d(gx,gu), d(gy,gv))](d(gx,gu) \\ & + d(gy,gv)) \\ =\; & \theta(d(gx,gu), d(gy,gv))(d(gx,gu) + d(gy,gv)), \end{aligned}$$

where $\theta(t_1, t_2) = \frac{1}{2}[\zeta(t_1, t_2) + \zeta(t_2, t_1)]$ for all $t_1, t_2 \in [0, \infty)$. It is easy to verify that $\theta \in \Theta$. Now, using the above theorem, we get the result. □

For the rest of the paper, we will need the definition below:

Definition 2.8. *Let Ω denote the class of functions $\psi : [0,\infty) \to [0,1)$ which satisfy the following condition: for any sequence $\{t_n\}$ of the positive real numbers, $\psi(t_n) \to 1$ implies $t_n \to 0$.*

Corollary 2.9. *Let $(X, \preceq)$ be a partially ordered set and (X,d) be a complete metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are two mappings such that F has the mixed g-monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$. Suppose that there exists an $\omega \in \Omega$ such that*

(2.21)
$$\begin{aligned} d(F(x,y),F(u,v)) \quad & +d(F(y,x),F(v,u)) \\ & \leq \omega(d(gx,gu)+d(gy,gv))(d(gx,gu)+d(gy,gv)), \end{aligned}$$

for all $x,y,u,v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Furthermore, suppose that g is a continuous non-decreasing map which commutes with F such that $F(X \times X) \subseteq g(X)$. If either

a) F is continuous or

b) X has the following properties:

(i) if a sequence $\{g(x_n)\} \subset X$ is a non-decreasing sequence with $gx_n \to gx$ in $g(X)$, then $gx_n \preceq gx$, for every n;

(ii) if a sequence $\{g(y_n)\} \subset X$ is a non-increasing sequence with $gy_n \to gy$ in $g(X)$, then $gy_n \succeq gy$, for every n,

then there exist $x, y \in X$ such that $F(x,y) = g(x)$ and $gy = F(y,x)$; that is, F and g have a coupled coincidence point $(x,y) \in X \times X$.

Proof. Taking $\theta(t_1,t_2) = \omega(t_1+t_2)$ for all $t_1,t_2 \in [0,\infty)$ in the theorem above, we get the result. □

If we let $\omega(t) = k$, where $k \in [0,1)$ for all $t \in [0,\infty)$ in the above corollary, we get the following result.

Corollary 2.10. *Let $(X, \preceq)$ be a partially ordered set and (X,d) be a complete metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are two mappings such that F has the mixed g-monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$. Suppose that there exists $k \in [0,1)$ such that*

(2.22)
$$d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) \leq k(d(gx,gu)+d(gy,gv)),$$

for all $x,y,u,v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Furthermore suppose that g is a continuous non-decreasing map which commutes with F such that $F(X \times X) \subseteq g(X)$. If either

a) F is continuous or

b) X has the following properties:

(i) if a sequence $\{g(x_n)\} \subset X$ is a non-decreasing sequence with $gx_n \to gx$ in $g(X)$, then $gx_n \preceq gx$, for every n;

(ii) if a sequence $\{g(y_n)\} \subset X$ is a non-increasing sequence with $gy_n \to gy$ in $g(X)$, then $gy_n \succeq gy$, for every n,

then there exist $x, y \in X$ such that $F(x,y) = g(x)$ and $gy = F(y,x)$; that is, F and g have a coupled coincidence point $(x,y) \in X \times X$.

Corollary 2.11. *Let $(X, \preceq)$ be a partially ordered set and (X,d) be a complete metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are two mappings such that F has the mixed g-monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists $k \in [0,1)$ such that*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}(d(gx,gu) + d(gy,gv)), \tag{2.23}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Furthermore suppose that g is a continuous non-decreasing map which commutes with F such that $F(X \times X) \subseteq g(X)$. If either

a) F is continuous or

b) X has the following properties:

(i) if a sequence $\{g(x_n)\} \subset X$ is a non-decreasing sequence with $gx_n \to gx$ in $g(X)$, then $gx_n \preceq gx$, for every n;

(ii) if a sequence $\{g(y_n)\} \subset X$ is a non-increasing sequence with $gy_n \to gy$ in $g(X)$, then $gy_n \succeq gy$, for every n,

then there exist $x, y \in X$ such that $F(x,y) = g(x)$ and $gy = F(y,x)$; that is, F and g have a coupled coincidence point $(x,y) \in X \times X$.

Now, we shall prove the existence and uniqueness of a coupled common fixed point. Note that, if $(X, \preceq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order relation:

$$\text{for } (x,y),(u,v) \in X \times X, (u,v) \preceq (x,y) \Leftrightarrow x \preceq u, y \succeq v.$$

Theorem 2.12. *In addition to the hypotheses of Theorem 2.5, suppose that for every $(x,y),(z,t) \in X \times X$, there exists $(u,v) \in X \times X$ such that $(F(u,v), F(v,u))$ is comparable to $(F(x,y), F(y,x))$ and $(F(z,t), F(t,z))$. Then F and g have a unique coupled common fixed point; that is, there exists a unique $(x,y) \in X \times X$ such that $x = gx = F(x,y)$ and $y = gy = F(y,x)$.*

Proof. From Theorem 2.5, the set of coupled coincidence points of F and g is non-empty. Suppose that (x,y) and (z,t) are coupled coincidence points of F and g; that is, $gx = F(x, y)$, $gy = F(y, x)$, $gz = F(z, t)$ and $gt = F(t, z)$. We shall show that $gx = gz$ and $gy = gt$. By the assumption, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. Then similarly as in the proof of Theorem 2.3, we can inductively define sequences $\{gu_n\}$, $\{gv_n\}$ as $gu_{n+1} = F(u_n, v_n)$ and $gv_{n+1} = F(v_n, u_n)$ for all n. Furthermore, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $t_0 = t$ and in the same way define the sequences $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$, $\{gt_n\}$. Then as in Theorem 2.3, we can show that $gx_n \to gx = F(x, y)$, $gy_n \to gy = F(y, x)$, $gz_n \to gz = F(z, t)$, $gt_n \to gt = F(t, z)$, for all $n \geq 1$. Since we have the fact that $(F(x, y), F(y, x)) = (gx_1, gy_1) = (gx, gy)$ and $(F(u, v), F(v, u)) = (gu_1, gv_1)$ are comparable, we derive $gx \succeq gu_1$ and $gy \preceq gv_1$. Now, we shall show that (gx, gy) and (gu_n, gv_n) are comparable; that is, $gx \succeq gu_n$ and $gy \preceq gv_n$ for all n. Suppose that it holds for some $n \geq 0$. Then by the mixed g-monotone property of F, we have $gu_{n+1} = F(u_n, v_n) \preceq F(x, y) = gx$ and $gv_{n+1} = F(v_n, u_n) \succeq F(y, x) = gy$. Hence $gx \succeq gu_n$ and $gy \preceq gv_n$ hold for all n. Thus from (2.1), we have

$$\begin{aligned} d(gx,gu_{n+1})+d(gy,gv_{n+1}) &= d(F(x,y),F(u_n,v_n))+d(F(y,x),F(v_n,u_n)) \\ &\leq \theta(d(gx,gu_n),d(gy,gv_n))(d(gx,gu_n)+d(gy,gv_n)) \\ &< d(gx,gu_n)+d(gy,gv_n) \end{aligned} \tag{2.24}$$

Consequently, the sequence $\{\delta_n := d(gx, gu_n) + d(gy, gv_n)\}$ is non-negative and decreasing and, therefore, $\lim \delta_n = \delta$, for some $\delta \geq 0$. We shall show that $\delta = 0$. On the contrary, assume that $\delta > 0$. By passing to the subsequences, if necessary, we may assume that $\lim \theta(d(gx, gu_n), d(gy, gv_n)) = \lambda$ exists (since $0 \leq \theta(d(gx, gu_n), d(gy, gv_n)) < 1$). From (2.24), taking the limit as $n \to \infty$, we obtain $\delta \leq \lambda\delta$ and so $\lambda = 1$. Since $\theta \in \Theta$, we get $\lim d(gx, gu_n) = 0 = \lim d(gy, gv_n)$; that is, $\lim d(gx, gu_n) + d(gy, gv_n) = 0$, which is contradiction. Thus, $\lim d(gx, gu_n) = 0 = \lim d(gy, gv_n)$. Similarly, we can prove that $\lim d(gz, gu_n) = 0 = \lim d(gt, gv_n)$. Finally, we have $d(gx, gz) \leq d(gx, gu_n) + d(gu_n, gz)$ and $d(gy, gt) \leq d(gy, gv_n) + d(gv_n, gt)$. Taking the limit as $n \to \infty$ in these inequalities, we get $d(gx, gz) = 0 = d(gy, gt)$; that is $gx = gz$ and $gy = gt$. Since $gx = F(x, y)$ and $gy = F(y, x)$, by the commutativity of F and g, we have

$$g(g(x)) = g(F(x, y)) = F(gx, gy), \text{ and } g(gy) = g(F(y, x)) = F(gy, gx). \tag{2.25}$$

Denote $gx = p$ and $gy = q$. Then we obtain $gp = F(p, q)$ and $gq = F(q, p)$. Thus (p, q) is a coupled coincidence point. Then from (2.25), with $z = p$ and $t = q$, it follows that $gp = gx$ and $gq = gy$; that is, $gp = p$ and $gq = q$. Hence $p = gp = F(p, q)$ and $q = gq = F(q, p)$. Therefore, (p, q) is a coupled common fixed point of F and g. To prove the uniqueness, assume that (r, s) is another coupled common fixed point. Then by (2.25), we have $r = gr = gp = p$ and $s = gs = gq = q$. Hence we get the result.

□

Theorem 2.13. *In addition to the hypotheses of Theorem 2.5, if we assume that gx_0 and gy_0 are comparable, then F and g have a coupled coincidence point; that is, there exists a $(x, y) \in X \times X$ such that $gx = F(x, y) = F(y, x) = gy$.*

Proof. By Theorem 2.3, we can construct two sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that $gx_n \to gx$ and $gy_n \to gy$, where (x, y) is a coincidence point of F and g. Suppose that $gx_0 \preceq gy_0$. We shall show that $gx_n \preceq gy_n$, where $gx_n = F(x_{n-1}, y_{n-1})$, $gy_n = F(y_{n-1}, x_{n-1})$, for all n. Suppose it holds for some $n \geq 0$. Then by mixed g-monotone property of F, we have $gx_{n+1} = F(x_n, y_n) \preceq F(y_n, x_n) = gy_{n+1}$. From 2.1, we have

$$\begin{aligned} d(F(y_n, x_n), \quad & F(x_n, y_n)) + d(F(x_n, y_n), F(y_n, x_n)) \\ & \leq \theta(d(gy_n, gx_n), d(gx_n, gy_n))(d(gy_n, gx_n) + d(gx_n, gy_n)) \end{aligned}$$

or

$$d(F(y_n, x_n), F(x_n, y_n)) \leq \theta(d(gy_n, gx_n), d(gx_n, gy_n))d(gy_n, gx_n).$$

By the triangle inequality, we have

(2.26)

$$\begin{aligned} d(gy, gx) \quad & \leq d(gy, gy_{n+1}) + d(gy_{n+1}, gx_{n+1}) + d(gx_{n+1}, gx) \\ & = d(F(y_n, x_n), F(x_n, y_n)) + d(gy, gy_{n+1}) + d(gx_{n+1}, gx) \\ & \leq \theta(d(gy_n, gx_n), d(gx_n, gy_n))d(gy_n, gx_n) \\ & + d(gy, gy_{n+1}) + d(gx_{n+1}, gx). \end{aligned}$$

Assume that $d(gy, gx) > 0$. Set $\alpha_n = d(gy_n, gx_n)$. By passing to the subsequences, if necessary, we may assume that $\lim \theta(\alpha_n, \alpha_n) = \alpha$ exists. Letting $n \to \infty$ in the above inequality, we obtain $d(gy, gx) \leq \alpha d(gy, gx)$ or $\alpha \geq 1$. Hence we get $\alpha = 1$. Since $\theta \in \Theta$, we find $d(gy, gx) = \lim d(gy_n, gx_n) = 0$, which is a contradiction. Therefore $d(gy, gx) = 0$. Hence $F(x, y) = gx = gy = F(y, x)$.

Similar arguments can be used if $gy_0 \preceq gx_0$. □

Other consequences of our results for the mappings involving contractions of integral type are the followings: Denote by Λ the set of functions $\mu : [0,\infty) \to [0,\infty)$ which satisfy the hypotheses below:

(h1) μ is a Lebesgue-integrable mapping on each compact of $[0,\infty)$;

(h2) for any $\epsilon > 0$, we have $\int_0^\epsilon \mu(t) > 0$.

Corollary 2.14. *Let $(X,\preceq)$ be a partially ordered set and (X,d) be a metric space. Suppose that $F : X\times X \to X$ and $g : X \to X$ are two mappings such that F has the mixed g-monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$. Suppose that there exists a $\theta \in \Theta$ such that*

$$\int_0^{[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))]} \alpha(s)ds \\ \leq \theta(d(gx,gu),d(gy,gv)) \int_0^{[(d(gx,gu)+d(gy,gv))]} \beta(s)ds$$

for all $x,y,u,v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, where $\alpha,\beta \in \Lambda$. Furthermore suppose that $g(X)$ is a complete subspace of X with $F(X\times X) \subseteq g(X)$. Also, assume that X has the following properties:

(i) if a sequence $\{g(x_n)\} \subset X$ is a non-decreasing sequence with $gx_n \to gx$ in $g(X)$, then $gx_n \preceq gx$, for every n;

(ii) if a sequence $\{g(y_n)\} \subset X$ is a non-increasing sequence with $gy_n \to gy$ in $g(X)$, then $gy_n \succeq gy$, for every n.

Then there exist $x,y \in X$ such that $F(x,y) = g(x)$ and $gy = F(y,x)$; that is, F and g have a coupled coincidence point $(x,y) \in X\times X$.

Corollary 2.15. *Let $(X,\preceq)$ be a partially ordered set and (X,d) be a metric space. Suppose that $F : X\times X \to X$ and $g : X \to X$ are two mappings such that F has the mixed g-monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$. Suppose that there exists $k \in [0,1)$ such that*

$$\int_0^{[d(F(x,y),F(u,v))+d(F(y,x),F(v,u))]} \alpha(s)ds \leq k \int_0^{[(d(gx,gu)+d(gy,gv))]} \beta(s)ds \tag{2.27}$$

for all $x,y,u,v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, where $\alpha,\beta \in \Lambda$. Furthermore suppose that $g(X)$ is a complete subspace of X with $F(X\times X) \subseteq g(X)$. Also, assume that X has the following properties:

(i) if a sequence $\{g(x_n)\} \subset X$ is a non-decreasing sequence with $gx_n \to gx$ in $g(X)$, then $gx_n \preceq gx$, for every n;

(ii) if a sequence $\{g(y_n)\} \subset X$ is a non-increasing sequence with $gy_n \to gy$ in $g(X)$, then $gy_n \succeq gy$, for every n.

Then there exist $x,y \in X$ such that $F(x,y) = g(x)$ and $gy = F(y,x)$, that is, F and g have a coupled coincidence point $(x,y) \in X\times X$.

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