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IMPLICIT ITERATION APPROXIMATION FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-PSEUDOCONTRACTIVE TYPE MAPPINGS

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ABSTRACT. In this paper, strong convergence theorems of Ishikawa type implicit iteration process with errors for a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings in normed linear spaces are established by using a new analytical method, which essentially improve and extend some recent results obtained by Yang [Convergence theorems of implicit iteration process for asymptotically pseudocontractive mappings, Bulletin of the Iranian Mathematical Society, Available Online from 12 April 2011] and others. **Keywords:** Normed linear spaces, implicit iteration process, asymp-

Keywords: Normed linear spaces, implicit iteration process, asymptotically quasi-pseudocontractive type mappings, nonexpansive mappings.

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1. Introduction

Let E be an arbitrary real normed linear space with norm $\|\cdot\|$ and E^* be the duality space of E. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between E and E^* . For $1 , the mapping <math>J_p: E \to 2^{E^*}$ defined by

$$J_p(x) = \{ f \in E^* : \langle x, f \rangle = ||x|| \cdot ||f||, ||f|| = ||x||^{p-1} \}$$

is called the duality mapping with gauge function $\varphi(t) = t^{p-1}$. In particular, for p = 2, the duality mapping J_2 with gauge function $\varphi(t) = t$ is called the normalized duality mapping. It is well known that the duality

Article electronically published on February 25, 2014. Received: 6 December 2011, Accepted: 28 January 2013. mapping J_p has the following properties:

- (i) $J_p(x) = ||x||^{p-2} J_2(x)$ for all $x \in E(x \neq 0)$,
- (ii) $J_p(\alpha x) = \alpha^{p-1} J_p(x)$ for all $\alpha \ge 0$,
- (iii) J_p can be equivalently defined as the subdifferential of the functional $\psi(x) = p^{-1} ||x||^p$, i.e., $J_p(x) = \partial \psi(x) = \{ f \in E^* : \psi(y) \psi(x) \ge \langle y x, f \rangle, \forall y \in E \}$ (Asplund [1]).

Definition 1.1. Let K be a nonempty subset of E. A mapping $T: K \to K$ is said to be

(i) asymptotically nonexpansive, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$, such that

$$||T^n x - T^n y|| \le k_n ||x - y||, \quad \forall x, y \in K, n \ge 1,$$

(ii) asymptotically pseudo-contractive, if for all $x, y \in K$, there exist $j(x-y) \in J(x-y)$ and a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\langle T^n x - T^n y, j(x-y) \rangle \le k_n ||x-y||^2, \ n \ge 1,$$

(iii) asymptotically quasi-pseudocontractive type if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\limsup_{n \to \infty} \left\{ \sup_{x \in K} \inf_{j_p(x - x^*) \in J_p(x - x^*)} \langle T^n x - x^*, j_p(x - x^*) \rangle - k_n ||x - x^*||^p \right\} \le 0,$$

(iv) asymptotically nonexpansive in the intermediat sense if

$$\limsup_{n \to \infty} \left\{ \sup_{x,y \in K} (\|T^n x - T^n y\| - \|x - y\|) \right\} \le 0.$$

It is easy to see that an asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense if the domain of T is bounded. Every asymptotically nonexpansive mapping is asymptotically pseudocontractive, and every asymptotically pseudocontractive mapping is asymptotically quasi-pseudocontractive type mapping. But the inverse is not true, in general.

The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5], while the concept of asymptotically pseudocontractive mapping was introduced by Schu [12] in 1991. The iterative approximation problems for asymptotically nonexpansive mappings and asymptotically pseudo-contractive mappings were studied extensively by

Schu [12], Chang [3], Khan et al. [7], Ofoedu [8], Plubtieng et al [10], Xu and Ori [14], Zhou [19], Sun [13], Yang and Hu [15] and Yang [16] in the setting of Hilbert spaces or Banach spaces.

Let K be a nonempty closed convex subset of E and $\{T_i\}_{i=1}^m$ be a finite family of nonexpansive mappings from K into itself (i.e., $||T_ix - T_iy|| \le ||x - y||$ for $x, y \in K$ and i = 1, 2, ..., m). In 2001, Xu and Ori [14] introduced the following implicit iteration process. For an arbitrary $x_0 \in K$ and $\alpha_n \in [0, 1]$, the sequence $\{x_n\}$ is generated as follows:

$$\begin{cases} x_1 = (1 - \alpha_1) x_0 + \alpha_1 T_1 x_1, \\ x_2 = (1 - \alpha_2) x_1 + \alpha_2 T_2 x_2, \\ \vdots \\ x_N = (1 - \alpha_N) x_{N-1} + \alpha_N T_N x_N, \\ x_{N+1} = (1 - \alpha_{N+1}) x_N + \alpha_{N+1} T_{N+1} x_{N+1}, \\ \vdots \end{cases}$$

The scheme is expressed in its compact form by

$$x_n = (1 - \alpha_n)x_n + \alpha_n T_{n(modN)}x_n, n \ge 1.$$

Using this iteration, they proved that the sequence $\{x_n\}$ converges weakly to a common fixed point of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in a Hilbert space under certain conditions.

In 2006, Chang et al.[3] introduced another implicit iteration process with error. In the sense of [3], the implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^m$ is generated from an arbitrary $x_0 \in K$ by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{k(n)} x_n + u_n, n \ge 1,$$

where $n=(k-1)m+i, i=i(n)\in\{1,2,\ldots,m\}, k=k(n)\geq 1$ is a positive integer and $k(n)\to\infty$, as $n\to\infty$. $\{\alpha_n\}$ is a suitable sequence in [0,1] and $\{u_n\}\subset K$ is such that $\sum_{n=0}^{\infty}\|u_n\|<\infty$. They extended the results of [14] from Hilbert spaces to more general uniformly convex Banach spaces and from nonexpansive mappings to asymptotically nonexpansive mappings.

Yang and Hu [15] proposed another implicit iteration process which appears to be more satisfactory as follows:

(1.1)
$$x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n + \gamma_n u_n, n \ge 1,$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $\{u_n\}$ is bounded in K.

Since for each n > 1, it can be written as n = (k-1)m + i, where $i = i(n) \in \{1, 2, ..., m\}, k = k(n) - 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Hence, (1.1) can be expressed in the following form:

$$(1.2) x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^n x_n + \gamma_n u_n, n \ge 1,$$

where $\alpha_n + \gamma_n \leq 1$, and $\{u_n\}$ is bounded in K.

Very recently, Yang [16] proved the following result.

Theorem 1.2. ([16]). Let E be a real normed linear space, K be a nonempty convex subset of E, $T_i: K \to K, i = 1, 2, ..., m$ be a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically pseudo-contractive mappings with $\{k_{in}\}\subset[1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$, where $k_n = \max_{1\leq i\leq m} \{k_{in}\}$. Assume that $F = \sum_{n\to\infty} k_n = 1$ $\bigcap_{i=1}^m F(T_i) \neq \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^m$. Let $\{x_n\}$ be the sequence defined by (1.2). Suppose that $\{u_n\}$ is bounded in K and that $\{\alpha_n\}$, $\{\gamma_n\}$ are sequences in [0, 1] satisfying the following

(i)
$$\sum_{n=0}^{\infty} \gamma_n < \infty$$
, $\sum_{n=0}^{\infty} \alpha_n (k_n - 1) < \infty$, (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} \alpha_n = 0$.

Assume that there exists a strict increasing function $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(0) = 0$ such that

$$\limsup_{n \to \infty} \left\{ \langle T_i^n x_n - x^*, j(x_n - x^*) \rangle - k_n ||x_n - x^*||^2 + \varphi(||x_n - x^*||) \right\} \le 0$$

for $x^* \in F$ and $i = 1, 2, \dots, m$. Then $\{x_n\}$ converges strongly to a common fixed point p of $\{T_i\}_{i=1}^m$.

Remark 1.3. We point out here that the conditions (i) is not always

Example 1.4. Let
$$\alpha_n = \frac{1}{\sqrt{n+1}}$$
 and $k_n = 1 + \frac{1}{\sqrt{n+1}}$, then $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) = \frac{1}{n+1}$

 $\sum_{n=0}^{\infty} \frac{1}{n+1} = \infty$, which show that conditions (i) in Theorem 1.2 is not satisfied. Hence Theorem 1.2 need to be improved.

The purpose of this paper is, under the condition of removing the restriction $\sum_{n=0}^{\infty} \alpha_n(k_n-1) < \infty$, to prove strong convergence theorems of Ishikawa type implicit iteration process with errors for a finite family of

asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings in normed linear spaces by using a new analytical method. Our results essentially extend and improve some recent results obtained by Yang [16] and others.

Now we consider Ishikawa type implicit iteration process with errors for a finite family of asymptotically quasi-pseudocontractive type mappings as follows:

$$(1.3) \begin{cases} x_n = (1 - \alpha_n - \gamma_n) x_{n-1} + \alpha_n T_{i(n)}^n y_n + \gamma_n u_n \\ y_n = (1 - \beta_n - \mu_n) x_n + \beta_n T_{i(n)}^n x_n + \mu_n v_n, & (n \ge 0), \end{cases}$$

where $n=(k-1)m+i, i=i(n)\in\{1,2,\cdots,m\}, k=k(n)\geq 1$ is a positive integer and $k(n)\to\infty$, as $n\to\infty$. $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\mu_n\}$ are four suitable sequences in [0, 1] with $\alpha_n+\gamma_n\leq 1, \beta_n+\mu_n\leq 1$ and $\{u_n\},\{v_n\}$ are bounded sequences in K.

The following lemmas plays an important role in this paper.

Lemma 1.5 ([17).] Let E be a real normed linear space and $J_p: E \to 2^{E^*}$ a duality mapping. Then

$$||x + y||^p \le ||x||^p + p\langle y, j_p(x + y)\rangle$$

for all $x, y \in E, 1 and <math>j_p(x + y) \in J_p(x + y)$.

Lemma 1.6. Let $\varphi_i(i=1,2,\ldots,m):[0,\infty)\to[0,\infty)$ be strictly increasing functions with $\varphi_i(0)=0$ and let $\{a_n\},\{b_n\},\{c_n\},\{\delta_n\}$ be nonnegative real sequences such that $\sum_{n=1}^{\infty}\delta_n=\infty$, $\lim_{n\to\infty}\frac{b_n}{\delta_n}=0$, $\sum_{n=1}^{\infty}c_n<\infty$. Suppose that

(1.4)
$$a_{n+1}^p \le a_n^p - \delta_n \varphi_i(a_{n+1}) + b_n + c_n, \ n \ge n_0,$$

where n_0 is some nonnegative integer and $p \in (1, \infty)$, then $\lim_{n \to \infty} a_n = 0$.

Proof. Setting $\liminf_{n\to\infty} a_n = \tau$, then $\tau \geq 0$. Now we prove $\tau = 0$. If $\tau > 0$, then there exists a positive integer $N_1 > 0$ such that $a_n \geq \frac{\tau}{2}$ for all $n \geq N_1$. By the strictly increasing property of φ_i , we have $\varphi_i(a_{n+1}) > \varphi_i(\frac{\tau}{2}) \geq \min_{1 \leq i \leq m} \varphi_i(\frac{\tau}{2}) =: \sigma$. Since $\lim_{n\to\infty} \frac{b_n}{\delta_n} = 0$, there exists a positive integer $N_2 > N_1$ such that $\frac{b_n}{\delta_n} \leq \frac{1}{2}\sigma$ for all $n \geq N_2$. Taking $N_3 = \max\{N_2, n_0\}$, then from (1.4), we have

$$a_{n+1}^p \le a_n^p - \delta_n \sigma + \delta_n \frac{\sigma}{2} + c_n = a_n^p - \delta_n \frac{\sigma}{2} + c_n$$

for all $n \geq N_3$, which means that $\delta_n \frac{\sigma}{2} \leq a_n^p - a_{n+1}^p + c_n$. Hence for any positive integer $h \geq N_3$, we obtain

$$\frac{\sigma}{2} \sum_{n=N_3}^h \delta_n \le a_{N_3}^p - a_{h+1}^p + \sum_{n=N_3}^h c_n \le a_{N_3}^p + \sum_{n=N_3}^h c_n,$$

and so

$$\infty = \frac{\sigma}{2} \sum_{n=N_3}^{\infty} \delta_n \le a_{N_3}^p + \sum_{n=N_3}^{\infty} c_n,$$

a contradition. This implies that $\tau>0$ is impossible. Therefore $\tau=0$, which there exists a subsequence $\{a_{n_j}\}\subset\{a_n\}$ such that $a_{n_j}\to 0$ ($j\to\infty$). Since $\lim_{n\to\infty}\frac{b_n}{\delta_n}=0$, $\sum_{n=1}^\infty c_n<\infty$, for any given $\varepsilon>0$, there exist two positive integers $j_0>0$ and $N_4>0$, such that for all $n\geq N_4, \sum_{n=N_4}^\infty c_n<\varepsilon^p$, $\frac{b_n}{\delta_n}<\frac{1}{2}\omega$ and $a_{n_j}<\varepsilon$ for all $j\geq j_0$, where $\omega=\min\{\varphi_1(\varepsilon),\varphi_2(\varepsilon),\cdots,\varphi_m(\varepsilon)\}$. Let $N_5=\max\{j_0,N_4\}$. For fixed $j_*>N_5$ and all $k\geq 0$, we now want to show that $a_{n_{j_*}+k}<2\varepsilon$. To see this consider two possible cases.

Case I: $a_{n_{i_*}+1} < \varepsilon$.

In this case, $a_{n_{j_*}+1}^p < \varepsilon^p + c_{n_{j_*}} + c_{n_{j_*}+1}$ and so we have the desired result.

Case II: $a_{n_{i_*}+1} \geq \varepsilon$.

In this case, $\varphi_i(a_{n_{j_*}+1}) \ge \varphi_i(\varepsilon) \ge \omega > 0$ since for each $i = 1, 2, \dots, m$, φ_i is a strictly increasing function. From (1.4), we also have

$$a_{n_{j_*}+1}^p \le a_{n_{j_*}}^p - \delta_{n_{j_*}} \varphi_i(a_{n_{j_*}+1}) + b_{n_{j_*}} + c_{n_{j_*}}$$

$$\le a_{n_{j_*}}^p - \delta_{n_{j_*}} \left(\omega - \frac{\omega}{2}\right) + c_{n_{j_*}}$$

$$< \varepsilon^p + c_{n_{j_*}} + c_{n_{j_*}+1}.$$

By using induction, we have

$$a_{n_{j_*}+k}^p < \varepsilon^p + \sum_{i=n_{j_*}}^{n_{j_*}+k} c_i < \varepsilon^p + \varepsilon^p = 2\varepsilon^p < (2\varepsilon)^p$$

for all $k \ge 0$. This shows $a_n \to 0$ as $n \to \infty$. The proof of Lemma 1.6 is completed.

2. Main results

Theorem 2.1. Let E be a real normed linear space, K a nonempty convex subset of E and $T_i: K \to K (i=1,2,\ldots,m)$ a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings with $\{k_n^{(i)}\}\subset [1,\infty)$ such that $\lim_{n\to\infty} k_n=1$, where $k_n=\max_{1\leq i\leq m}\{k_n^{(i)}\}$. Assume that $F=\bigcap_{i=1}^m F(T_i)\neq\emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^m$. Let $\{x_n\}$ be the sequence defined by (1.3). Suppose that $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K and that $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ and $\{\mu_n\}$ are sequences in [0,1] satisfying the following conditions:

$$(i) \sum_{n=0}^{\infty} \alpha_n = \infty,$$

$$(ii) \alpha_n \to 0, \beta_n \to 0 (n \to \infty),$$

$$(iii) \sum_{n=0}^{\infty} \gamma_n < \infty, \ \mu_n \to 0 (n \to \infty).$$

Assume that there exist strict increasing functions $\varphi_i:[0,\infty)\to[0,\infty)$ with $\varphi_i(0)=0$ such that

(2.1)
$$\lim_{n \to \infty} \sup_{j_p(x_n - x^*) \in J_p(x_n - x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*) \rangle -k_n \|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) \} \le 0$$

for $x^* \in F$ and i = 1, 2, ..., m. Then $\{x_n\}$ converges strongly to a common fixed point x^* of $\{T_i\}_{i=1}^m$.

Proof. For $i = 1, 2, \ldots, m$, let

$$\sigma_n^{(i)} = \inf_{j_p(x_n - x^*) \in J_p(x_n - x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*) \rangle - k_n ||x_n - x^*||^p + \varphi_i(||x_n - x^*||),$$

then there exist $j_p^{(i)}(x_n - x^*) \in J_p(x_n - x^*)$, such that

$$\langle T_i^n x_n - x^*, j_p^{(i)}(x_n - x^*) \rangle - k_n ||x_n - x^*||^p + \varphi_i(||x_n - x^*||)$$

$$< \sigma_n^{(i)} + \varepsilon_n^{(i)} \le \xi_n,$$
(2.2)

where $\varepsilon_n^{(i)} \in (0,1)$ with $\varepsilon_n^{(i)} \to 0 (n \to \infty)$, and $\xi_n = \max_{1 \le i \le m} \{\sigma_n^{(i)}, 0\} + \max_{1 \le i \le m} \{\varepsilon_n^{(i)}\}$. It is easy see (using (2.1)) that $\lim_{n \to \infty} \xi_n = 0$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in $K, M = \sup_{n \ge 0} \{\|u_n - x^*\| + \|v_n - x^*\|\} < \infty$. Also, since for each $i = 1, 2, \ldots, m, T_i : K \to K$ is an asymptotically

nonexpansive in the intermediate sense, there exists $n_0 \geq 1$ such that $\sup_{x,y \in K} (\|T_i^n x - T_i^n y\| - \|x - y\|) \leq 1$ for all $n \geq n_0, i = 1, 2, \dots, m$. It follows from (1.3) that

$$\frac{\|x_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|}$$

$$= \frac{\|(1 - \alpha_{n} - \gamma_{n})(x_{n-1} - x^{*}) + \alpha_{n}(T_{i}^{n}y_{n} - x^{*}) + \gamma_{n}(u_{n} - x^{*})\|}{1 + \|x_{n-1} - x^{*}\|}$$

$$\leq \frac{\|x_{n-1} - x^{*}\| + \alpha_{n}\|T_{i}^{n}y_{n} - x^{*}\| + \|u_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|}$$

$$\leq \frac{\|x_{n-1} - x^{*}\| + \sup_{x,y \in K} (\|T_{i}^{n}x - T_{i}^{n}y\| - \|x - y\|)}{1 + \|x_{n-1} - x^{*}\|}$$

$$+ \frac{\alpha_{n}\|y_{n} - x^{*}\| + \|u_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|}$$

$$\leq \frac{\|x_{n-1} - x^{*}\| + 1 + \alpha_{n}[\|x_{n} - x^{*}\| + \|T_{i}^{n}x_{n} - x^{*}\|]}{1 + \|x_{n-1} - x^{*}\|}$$

$$+ \frac{\|v_{n} - x^{*}\| + \|u_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|}$$

$$\leq \frac{\|x_{n-1} - x^{*}\| + 1 + 2\alpha_{n}\|x_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|}$$

$$+ \frac{\sup_{x,y \in K} (\|T_{i}^{n}x - T_{i}^{n}y\| - \|x - y\|) + 2M}{1 + \|x_{n-1} - x^{*}\|}$$

$$(2.3) \leq 3 + 2M + \frac{2\alpha_{n}\|x_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|}$$

for all $n \geq n_0$.

Since $1-2\alpha_n \to 1 \ (n \to \infty)$, there exists $n_1 \ge n_0$ such that $1-2\alpha_n > \frac{1}{2} > 0$ for all $n \ge n_1$, which together with (2.3) gives that

$$(2.4) \frac{\|x_n - x^*\|}{1 + \|x_{n-1} - x^*\|} \le \frac{3 + 2M}{1 - 2\alpha_n} \le 6 + 4M.$$

Let
$$c_n^{(i)} = \sup_{x,y \in K} (\|T_i^n x - T_i^n y\| - \|x - y\|), d_n = \max \left\{0, \max_{1 \le i \le m} c_n^{(i)}\right\},$$
 then $\lim_{n \to \infty} d_n = 0.$

By (1.3), we have

$$||x_{n} - y_{n}|| \leq \beta_{n} ||T_{i}^{n}x_{n} - x_{n}|| + \mu_{n} ||v_{n} - x_{n}||$$

$$\leq \beta_{n} (||T_{i}^{n}x_{n} - x^{*}|| - ||x_{n} - x^{*}||)$$

$$+ (2\beta_{n} + \mu_{n}) ||x_{n} - x^{*}|| + \mu_{n} ||v_{n} - x^{*}||$$

$$\leq \beta_{n} \sup_{x,y \in K} (||T_{i}^{n}x - T_{i}^{n}y|| - ||x - y||)$$

$$+ (2\beta_{n} + \mu_{n}) ||x_{n} - x^{*}|| + \mu_{n}M$$

$$\leq (2\beta_{n} + \mu_{n}) ||x_{n} - x^{*}|| + \beta_{n}d_{n} + \mu_{n}M$$

$$(2.5)$$

for all $n \geq n_1$.

For $j_p^{(i)}(x_n - x^*) \in J_p(x_n - x^*)$, $\forall n \geq 0$, we have from (1.3) and Lemma 1.5 that

$$\left(\frac{\|x_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|}\right)^{p}$$

$$= \frac{\|(1 - \alpha_{n} - \gamma_{n})(x_{n-1} - x^{*}) + \alpha_{n}(T_{i}^{n}y_{n} - x^{*}) + \gamma_{n}(u_{n} - x^{*})\|^{p}}{(1 + \|x_{n-1} - x^{*}\|)^{p}}$$

$$\leq \frac{(1 - \alpha_{n})^{p} \|x_{n-1} - x^{*}\|^{p} + p\alpha_{n} \left\langle T_{i}^{n}x_{n} - x^{*}, j_{p}^{(i)}(x_{n} - x^{*}) \right\rangle}{(1 + \|x_{n-1} - x^{*}\|)^{p}}$$

$$+ \frac{p\alpha_{n} \left\langle T_{i}^{n}y_{n} - T_{i}^{n}x_{n}, j_{p}^{(i)}(x_{n} - x^{*}) \right\rangle}{(1 + \|x_{n-1} - x^{*}\|)^{p}}$$

$$(2.6) + \frac{p\gamma_{n} \left\langle u_{n} - x^{*}, j_{p}^{(i)}(x_{n} - x^{*}) \right\rangle}{(1 + \|x_{n-1} - x^{*}\|)^{p}}$$

for all $n \geq n_1, i = 1, 2, ..., m$.

Next we consider the second and third term on the right side of (2.6). From (2.4) and (2.5), we obtain that

$$\frac{p\alpha_{n}\left\langle T_{i}^{n}y_{n} - T_{i}^{n}x_{n}, j_{p}^{(i)}(x_{n} - x^{*})\right\rangle}{(1 + \|x_{n-1} - x^{*}\|)^{p}} \\
\leq p\alpha_{n} \frac{\|T_{i}^{n}y_{n} - T_{i}^{n}x_{n}\| \|x_{n} - x^{*}\|^{p-1}}{(1 + \|x_{n-1} - x^{*}\|)^{p}} \\
= p\alpha_{n} \frac{(\|T_{i}^{n}x_{n} - T_{i}^{n}y_{n}\| - \|x_{n} - y_{n}\|) + \|x_{n} - y_{n}\|}{1 + \|x_{n-1} - x^{*}\|} \\
\cdot \left(\frac{\|x_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|}\right)^{p-1}$$

$$\leq p\alpha_{n} \left(\frac{d_{n}}{1 + \|x_{n-1} - x^{*}\|} + \frac{\|x_{n} - y_{n}\|}{1 + \|x_{n-1} - x^{*}\|} \right)
\cdot \left(\frac{\|x_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|} \right)^{p-1}
\leq p\alpha_{n} \left(d_{n} + \frac{(2\beta_{n} + \mu_{n})\|x_{n} - x^{*}\| + \beta_{n}d_{n} + \mu_{n}M}{1 + \|x_{n-1} - x^{*}\|} \right)
\cdot \left(\frac{\|x_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|} \right)^{p-1}$$

$$(2.7) \le p\alpha_n(6+4M)^{p-1}[d_n+(2\beta_n+\mu_n)(6+4M)+\beta_nd_n+\mu_nM]$$

for all $n \geq n_1$.

In view of (2.4), we deduce that

$$\frac{p\gamma_n \left\langle u_n - x^*, j_p^{(i)}(x_n - x^*) \right\rangle}{(1 + \|x_{n-1} - x^*\|)^p} \leq \frac{p\gamma_n \|u_n - x^*\| \|x_n - x^*\|^{p-1}}{(1 + \|x_{n-1} - x^*\|)^p} \\
\leq p\gamma_n M(6 + 4M)^{p-1}.$$

Substituting (2.2), (2.7) and (2.8) into (2.6) yields that

$$\left(\frac{\|x_{n} - x^{*}\|}{1 + \|x_{n-1} - x^{*}\|}\right)^{p} \leq \frac{(1 - \alpha_{n})^{p} \|x_{n-1} - x^{*}\|^{p} + p\alpha_{n}\xi_{n}}{(1 + \|x_{n-1} - x^{*}\|)^{p}} + \frac{p\alpha_{n}\left(k_{n}\|x_{n} - x^{*}\|^{p} - \varphi_{i}(\|x_{n} - x^{*}\|)\right)}{(1 + \|x_{n-1} - x^{*}\|)^{p}} + p\alpha_{n}(6 + 4M)^{p-1}\left[d_{n} + (2\beta_{n} + \mu_{n})(6 + 4M)^{p-1}\right] + \beta_{n}d_{n} + \mu_{n}M + p\gamma_{n}M(6 + 4M)^{p-1}$$
(2.9)

for all $n > n_1, i = 1, 2, ..., m$.

Since $1 - p\alpha_n k_n \to 1$ $(n \to \infty)$, there exists $n_2 \ge n_1$ such that $0 < \frac{1}{2} < 1 - p\alpha_n k_n < 1$ for all $n \ge n_2$. It follows from (2.9) that

$$||x_{n} - x^{*}||^{p} \leq \frac{(1 - \alpha_{n})^{p} ||x_{n-1} - x^{*}||^{p} + p\alpha_{n}\xi_{n} - p\alpha_{n}\varphi_{i}(||x_{n} - x^{*}||)}{1 - p\alpha_{n}k_{n}}$$

$$(2.10) + \frac{(p\alpha_{n}A_{n} + p\gamma_{n}M(6 + 4M)^{p-1})(1 + ||x_{n-1} - x^{*}||)^{p}}{1 - p\alpha_{n}k_{n}}$$

for all
$$n \ge n_2, i = 1, 2, ..., m$$
,
where $A_n = (6 + 4M)^{p-1} [d_n + (2\beta_n + \mu_n)(6 + 4M) + \beta_n d_n + \mu_n M] \to$

$$0(n \to \infty)$$
. Note that $(1 + ||x_{n-1} - x^*||)^p \le 2^{p-1}(1 + ||x_{n-1} - x^*||^p)$,

$$(1 - \alpha_n)^p = 1 - p\alpha_n + \frac{p(p-1)\alpha_n^2}{2!} - \frac{p(p-1)(p-2)\alpha_n^3}{3!} + \dots + (-\alpha_n)^p$$

= 1 - p\alpha_n + \alpha_n B_n,

where

$$B_n = \frac{p(p-1)\alpha_n}{2!} - \frac{p(p-1)(p-2)\alpha_n^2}{3!} + \dots + (-\alpha_n)^{p-1} \to 0 (n \to \infty).$$

In virtue of (2.10), we conclude that

$$||x_{n} - x^{*}||^{p} \leq \left[1 + \frac{p\alpha_{n}(k_{n} - 1) + \alpha_{n}B_{n} + p2^{p-1}\alpha_{n}A_{n}}{1 - p\alpha_{n}k_{n}} + \frac{pM2^{p-1}(6 + 4M)^{p-1}\gamma_{n}}{1 - p\alpha_{n}k_{n}}\right] ||x_{n-1} - x^{*}||^{p}$$

$$+ \frac{p\alpha_{n}(\xi_{n} + 2^{p-1}A_{n}) + pM2^{p-1}(6 + 4M)^{p-1}\gamma_{n}}{1 - p\alpha_{n}k_{n}}$$

$$- \frac{p\alpha_{n}\varphi_{i}(||x_{n} - x^{*}||)}{1 - p\alpha_{n}k_{n}}$$

$$\leq \left[1 + 2p\alpha_{n}(k_{n} - 1) + 2\alpha_{n}B_{n} + p2^{p}\alpha_{n}A_{n} + pM2^{p}(6 + 4M)^{p-1}\gamma_{n}\right] ||x_{n-1} - x^{*}||^{p}$$

$$+ 2p\alpha_{n}(\xi_{n} + 2^{p-1}A_{n}) + pM2^{p}(6 + 4M)^{p-1}\gamma_{n}$$

$$- p\alpha_{n}\varphi_{i}(||x_{n} - x^{*}||)$$

$$(2.11)$$

for all $n \ge n_2, i = 1, 2, ..., m$.

Now we take a nonnegative integer $n_3 \geq n_2$ such that $x_{n_3} \neq x^*$ (if not, $x_n = x^*$ for all $n \geq n_2$, then $x_n \to x^*(n \to \infty)$, and so we have done). Since $k_n \to 1, \xi_n \to 0$, $B_n \to 0$, $A_n \to 0$ $(n \to \infty)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$, there exists a positive integer $N > n_3$ such that, for all $n \geq N$, $(k_n - 1 + \frac{1}{p}B_n)(2G)^p + (\xi_n + 2^{p-1}A_n(1 + (2G)^p)) < \frac{\min\limits_{1 \leq i \leq m} \{\varphi_i(G)\}}{4}$ and $\sum_{n=N}^{\infty} \gamma_n < \frac{G^p}{pM2^p(6+4M)^{p-1}(1+(2G)^p)}$, where $G = \max\{\|x_{n_3} - x^*\|, \|x_{n_3+1} - x^*\|, \ldots, \|x_{N-1} - x^*\|, \|x_N - x^*\|\}$, and obviously $0 < G < \infty$.

Next we proceed by induction to show $||x_{N+k}-x^*|| \le 2G$ for all $k \ge 1$. To see this consider two possible cases.

Case III: $||x_{N+1} - x^*|| \le G$.

In this case, $||x_{N+1} - x^*||^p \le G^2 + pM2^p (6 + 4M)^{p-1} (1 + (2G)^p) \gamma_{N+1}$ and so we have the desired result.

Case IV: $||x_{N+1} - x^*|| > G$.

In this case, $\varphi_i(\|x_{N+1} - x^*\|) > \varphi_i(G) \ge \min_{1 \le i \le m} \{\varphi_i(G)\} > 0$ since for each $i = 1, 2, \dots, m$, φ_i is a strictly increasing function. From (2.11), we also have

$$||x_{N+1} - x^*||^2 \le ||x_N - x^*||^p$$

$$+ [2p\alpha_{N+1}(k_{N+1} - 1) + 2\alpha_{N+1}B_{N+1} + pM2^p(6 + 4M)^{p-1}\gamma_{N+1}$$

$$+ p2^p\alpha_{N+1}A_{N+1}](2G)^p + 2p\alpha_{N+1}(\xi_{N+1} + 2^{p-1}A_{N+1})$$

$$+ pM2^p(6 + 4M)^{p-1}\gamma_{N+1} - p\alpha_{N+1} \min_{1 \le i \le m} \{\varphi_i(G)\}$$

$$= ||x_N - x^*||^p$$

$$- p\alpha_{N+1} \left[\min_{1 \le i \le m} \{\varphi_i(G)\} - 2(\xi_{N+1} + 2^{p-1}A_{N+1}(1 + (2G)^p)) \right]$$

$$- 2(k_{N+1} - 1 + \frac{1}{p}B_{N+1})(2G)^p \right] + pM2^p(6 + 4M)^{p-1}(1 + (2G)^p)\gamma_{N+1}$$

$$\le ||x_N - x^*||^p - p\alpha_{N+1} \left(\min_{1 \le i \le m} \{\varphi_i(G)\} - \frac{\min_{1 \le i \le m} \{\varphi_i(G)\}}{4} \right)$$

$$+ pM2^p(6 + 4M)^{p-1}(1 + (2G)^p)\gamma_{N+1}$$

$$\le ||x_N - x^*||^p + pM2^p(6 + 4M)(1 + (2G)^p)\gamma_{N+1}$$

$$\le ||x_N - x^*||^p + pM2^p(6 + 4M)^{p-1}(1 + (2G)^p)\gamma_{N+1}.$$

By using induction, we get that

$$||x_{N+k} - x^*||^2 \le G^p + pM2^p (6 + 4M)^{p-1} (1 + (2G)^p) \sum_{i=N+1}^{N+k} \gamma_i$$

$$\le G^p + G^p = 2G^p \le (2G)^p$$

for all $k \geq 1$.

This shows $||x_n - x^*|| \le 2G$ for all $n \ge N$. Therefore, it follows from (2,11) that

$$||x_n - x^*||^p \le ||x_{n-1} - x^*||^p + 2p\alpha_n \left[\left(k_n - 1 + \frac{B_n}{p} + 2^{p-1} A_n \right) (2G)^p \right]$$

$$+ \xi_n + 2^{p-1}A_n \bigg] + pM2^p(6+4M)^{p-1}(1+(2G)^p)\gamma_n$$

$$(2.12) \qquad - p\alpha_n\varphi_i(\|x_n - x^*\|)$$

for all $n \ge N, i = 1, 2, \dots, m$.

Taking $\delta_n = p\alpha_n$, $c_n = pM2^p(6+4M)^{p-1}(1+(2G)^p)\gamma_n$, $a_n = \|x_n-x^*\|$ and $b_n = 2p\alpha_n\left[\left(k_n-1+\frac{B_n}{p}+2^{p-1}A_n\right)(2G)^p+\xi_n+2^{p-1}A_n\right]$ for all $n \geq N$. By (2.12) and Lemma 1.6 ensures that $\|x_n-x^*\| \to 0$ as $n \to \infty$, that is, $x_n \to x^*$ as $n \to \infty$. This completes the proof.

Remark 2.2. Theorem 2.1 improves and extends Theorem 1.2 (i.e., Theorem 2.1 of Yang [16]) in the following aspects:

- (1) Extend asymptotically pseudocontractive mapping to asymptotically quasi-pseudocontractive type mappings.
- (2) It abolishes the condition that $\sum_{n=0}^{\infty} \alpha_n(k_n-1) < \infty$.
- (3) The proof of sequence $\{x_n\}$ boundedness is entirely different from what it was before.
- (4) Extend implicit iterative scheme (1.2) to Ishikawa type implicit iteration process (1.3).
- (5) Condition

$$\limsup_{n \to \infty} \left\{ \langle T_i^n x_n - x^*, j(x_n - x^*) \rangle - k_n ||x_n - x^*||^2 + \varphi(||x_n - x^*||) \right\} \le 0$$

is replaced by the condition

$$\limsup_{n \to \infty} \left\{ \inf_{j_p(x_n - x^*) \in J_p(x_n - x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*) \rangle -k_n \|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) \right\} \le 0.$$

From Theorem 2.1, we obtain the following result immediately.

Theorem 2.3. Let E be a real normed linear space, K a nonempty bounded convex subset of E and $T_i: K \to K(i = 1, 2, \cdots, m)$ a finite family of asymptotically nonexpansive mappings with $\{k_n^{(i)}\} \subset [1, \infty)$ such that $\lim_{n \to \infty} k_n = 1$, where $k_n = \max_{1 \le i \le m} \{k_n^{(i)}\}$. Assume that $F = \max_{n \to \infty} k_n = 1$

 $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^{m}$. Let $\{x_n\}$ be the sequence defined by (1.3). Suppose that $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K and that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\mu_n\}$ are sequences in [0, 1] satisfying the following conditions:

$$(i) \sum_{n=0}^{\infty} \alpha_n = \infty,$$

$$(ii) \alpha_n \to 0, \beta_n \to 0 (n \to \infty),$$

$$(iii) \sum_{n=0}^{\infty} \gamma_n < \infty, \ \mu_n \to 0 (n \to \infty).$$

Assume that there exist strict increasing functions $\varphi_i : [0, \infty) \to [0, \infty)$ with $\varphi_i(0) = 0$ such that

$$\limsup_{n \to \infty} \left\{ \inf_{j_p(x_n - x^*) \in J_p(x_n - x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*) \rangle - k_n \|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) \right\} \le 0$$

for $x^* \in F$ and $i = 1, 2, \dots, m$. Then $\{x_n\}$ converges strongly to a common fixed point x^* of $\{T_i\}_{i=1}^m$.

Proof. Since T_i is an asymptotically nonexpansive mapping with $\{k_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$, we have

$$\limsup_{n \to \infty} \left\{ \sup_{x,y \in K} \left(\|T_i^n x - T_i^n y\| - \|x - y\| \right) \right\}$$

$$\leq \limsup_{n \to \infty} \left[(k_n - 1) \operatorname{diam}(K) \right] = 0,$$

where $\operatorname{diam}(K) = \sup_{x,y \in K} \|x-y\|$. This implies that every asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense. Also since every asymptotically nonexpansive mapping is asymptotically pseudo-contractive mapping. The conclusion now follows easily from Theorem 2.1.

If $\gamma_n = \mu_n = 0 (\forall n \geq 1)$ in Theorem 2.1 and Theorem 2.3, then we have the following results.

Theorem 2.4. Let E be a real normed linear space, K a nonempty convex subset of E and $T_i: K \to K (i = 1, 2, \cdots, m)$ a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings with $\{k_n^{(i)}\}\subset [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$, where $k_n = \max_{1\leq i\leq m} \{k_n^{(i)}\}$. Assume that $F = \max_{n\to\infty} k_n = 1$, where $k_n = \max_{1\leq i\leq m} \{k_n^{(i)}\}$.

 $\bigcap_{i=1}^m F(T_i) \neq \emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^m$. Let $\{x_n\}$ be the sequence defined by

$$\begin{cases} x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T_{i(n)}^n y_n \\ y_n = (1 - \beta) x_n + \beta_n T_{i(n)}^n x_n, & (n \ge 0). \end{cases}$$

Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0, 1] satisfying the following conditions:

conditions:
(i)
$$\sum_{n=0}^{\infty} \alpha_n = \infty$$
,

(ii)
$$\alpha_n \to 0, \beta_n \to 0 (n \to \infty)$$
.

Assume that there exist strict increasing functions $\varphi_i : [0, \infty) \to [0, \infty)$ with $\varphi_i(0) = 0$ such that

$$\limsup_{n \to \infty} \left\{ \inf_{j_p(x_n - x^*) \in J_p(x_n - x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*) \rangle - k_n \|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) \right\} \le 0$$

for $x^* \in F$ and $i = 1, 2, \dots, m$. Then $\{x_n\}$ converges strongly to a common fixed point x^* of $\{T_i\}_{i=1}^m$.

Theorem 2.5. Let E be a real normed linear space, K a nonempty bounded convex subset of E and $T_i: K \to K(i=1,2,\cdots,m)$ a finite family of asymptotically nonexpansive mappings with $\{k_n^{(i)}\}\subset [1,\infty)$ such that $\lim_{n\to\infty} k_n=1$, where $k_n=\max_{1\leq i\leq m}\{k_n^{(i)}\}$. Assume that $F=\bigcap_{i=1}^m F(T_i)\neq\emptyset$ denotes the set of common fixed points of $\{T_i\}_{i=1}^m$. Let $\{x_n\}$ be the sequence defined by

$$\begin{cases} x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T_{i(n)}^n y_n \\ y_n = (1 - \beta) x_n + \beta_n T_{i(n)}^n x_n, & (n \ge 0). \end{cases}$$

Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0, 1] satisfying the following conditions:

conditions:
(i)
$$\sum_{n=0}^{\infty} \alpha_n = \infty$$
,

$$(ii)^n \alpha_n \to 0, \beta_n \to 0 (n \to \infty).$$

Assume that there exist strict increasing functions $\varphi_i : [0, \infty) \to [0, \infty)$ with $\varphi_i(0) = 0$ such that

$$\limsup_{n \to \infty} \left\{ \inf_{j_p(x_n - x^*) \in J_p(x_n - x^*)} \langle T_i^n x_n - x^*, j_p(x_n - x^*) \rangle -k_n \|x_n - x^*\|^p + \varphi_i(\|x_n - x^*\|) \right\} \le 0$$

for $x^* \in F$ and $i = 1, 2, \dots, m$. Then $\{x_n\}$ converges strongly to a common fixed point x^* of $\{T_i\}_{i=1}^m$.

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