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Title:
Implicit iteration approximation for a finite family of asymptotically quasi-pseudocontractive type mappings

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# IMPLICIT ITERATION APPROXIMATION FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-PSEUDOCONTRACTIVE TYPE MAPPINGS 

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#### Abstract

In this paper, strong convergence theorems of Ishikawa type implicit iteration process with errors for a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings in normed linear spaces are established by using a new analytical method, which essentially improve and extend some recent results obtained by Yang [Convergence theorems of implicit iteration process for asymptotically pseudocontractive mappings, Bulletin of the Iranian Mathematical Society, Available Online from 12 April 2011] and others. Keywords: Normed linear spaces, implicit iteration process, asymptotically quasi-pseudocontractive type mappings, nonexpansive mappings.


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## 1. Introduction

Let $E$ be an arbitrary real normed linear space with norm $\|\cdot\|$ and $E^{*}$ be the duality space of $E$. Let $\langle\cdot, \cdot\rangle$ denote the duality pairing between $E$ and $E^{*}$. For $1<p<\infty$, the mapping $J_{p}: E \rightarrow 2^{E^{*}}$ defined by

$$
J_{p}(x)=\left\{f \in E^{*}:\langle x, f\rangle=\|x\| \cdot\|f\|,\|f\|=\|x\|^{p-1}\right\}
$$

is called the duality mapping with gauge function $\varphi(t)=t^{p-1}$. In particular, for $p=2$, the duality mapping $J_{2}$ with gauge function $\varphi(t)=t$ is called the normalized duality mapping. It is well known that the duality

[^0]mapping $J_{p}$ has the following properties:
(i) $J_{p}(x)=\|x\|^{p-2} J_{2}(x)$ for all $x \in E(x \neq 0)$,
(ii) $J_{p}(\alpha x)=\alpha^{p-1} J_{p}(x)$ for all $\alpha \geq 0$,
(iii) $J_{p}$ can be equivalently defined as the subdifferential of the functional $\psi(x)=p^{-1}\|x\|^{p}$, i.e., $J_{p}(x)=\partial \psi(x)=\left\{f \in E^{*}: \psi(y)-\psi(x) \geq\right.$ $\langle y-x, f\rangle, \forall y \in E\}$ (Asplund [1]).

Definition 1.1. Let $K$ be a nonempty subset of $E$. A mapping $T$ : $K \rightarrow K$ is said to be (i) asymptotically nonexpansive, if there exists a sequence $\left\{k_{n}\right\} \subset[1, \infty)$ with $\lim _{n \rightarrow \infty} k_{n}=1$, such that

$$
\left\|T^{n} x-T^{n} y\right\| \leq k_{n}\|x-y\|, \quad \forall x, y \in K, n \geq 1
$$

(ii) asymptotically pseudo-contractive, if for all $x, y \in K$, there exist $j(x-y) \in J(x-y)$ and a sequence $\left\{k_{n}\right\} \subset[1, \infty)$ with $\lim _{n \rightarrow \infty} k_{n}=1$, such that

$$
\left\langle T^{n} x-T^{n} y, j(x-y)\right\rangle \leq k_{n}\|x-y\|^{2}, n \geq 1
$$

(iii) asymptotically quasi-pseudocontractive type if $F(T) \neq \emptyset$ and there exists a sequence $\left\{k_{n}\right\} \subset[1, \infty)$ with $\lim _{n \rightarrow \infty} k_{n}=1$, such that
$\limsup _{n \rightarrow \infty}\left\{\sup _{x \in K} \inf _{j_{p}\left(x-x^{*}\right) \in J_{p}\left(x-x^{*}\right)}\left\langle T^{n} x-x^{*}, j_{p}\left(x-x^{*}\right)\right\rangle-k_{n}\left\|x-x^{*}\right\|^{p}\right\} \leq 0$,
(iv) asymptotically nonexpansive in the intermediat sense if

$$
\limsup _{n \rightarrow \infty}\left\{\sup _{x, y \in K}\left(\left\|T^{n} x-T^{n} y\right\|-\|x-y\|\right)\right\} \leq 0
$$

It is easy to see that an asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense if the domain of $T$ is bounded. Every asymptotically nonexpansive mapping is asymptotically pseudocontractive, and every asymptotically pseudocontractive mapping is asymptotically quasi-pseudocontractive type mapping. But the inverse is not true, in general.

The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5], while the concept of asymptotically pseudocontractive mapping was introduced by Schu [12] in 1991. The iterative approximation problems for asymptotically nonexpansive mappings and asymptotically pseudo-contractive mappings were studied extensively by

Schu [12], Chang [3], Khan et al. [7], Ofoedu [8], Plubtieng et al [10], Xu and Ori [14], Zhou [19], Sun [13], Yang and Hu [15] and Yang [16] in the setting of Hilbert spaces or Banach spaces.

Let $K$ be a nonempty closed convex subset of $E$ and $\left\{T_{i}\right\}_{i=1}^{m}$ be a finite family of nonexpansive mappings from $K$ into itself (i.e., $\left\|T_{i} x-T_{i} y\right\| \leq$ $\|x-y\|$ for $x, y \in K$ and $i=1,2, \ldots, m)$. In 2001, Xu and Ori [14] introduced the following implicit iteration process. For an arbitrary $x_{0} \in K$ and $\alpha_{n} \in[0,1]$, the sequence $\left\{x_{n}\right\}$ is generated as follows:

$$
\left\{\begin{array}{l}
x_{1}=\left(1-\alpha_{1}\right) x_{0}+\alpha_{1} T_{1} x_{1} \\
x_{2}=\left(1-\alpha_{2}\right) x_{1}+\alpha_{2} T_{2} x_{2} \\
\quad \vdots \\
x_{N}=\left(1-\alpha_{N}\right) x_{N-1}+\alpha_{N} T_{N} x_{N} \\
x_{N+1}=\left(1-\alpha_{N+1}\right) x_{N}+\alpha_{N+1} T_{N+1} x_{N+1} \\
\quad \vdots
\end{array}\right.
$$

The scheme is expressed in its compact form by

$$
x_{n}=\left(1-\alpha_{n}\right) x_{n}+\alpha_{n} T_{n(\bmod N)} x_{n}, n \geq 1
$$

Using this iteration, they proved that the sequence $\left\{x_{n}\right\}$ converges weakly to a common fixed point of a finite family of nonexpansive mappings $\left\{T_{i}\right\}_{i=1}^{N}$ in a Hilbert space under certain conditions.

In 2006, Chang et al.[3] introduced another implicit iteration process with error. In the sense of [3], the implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings $\left\{T_{i}\right\}_{i=1}^{m}$ is generated from an arbitrary $x_{0} \in K$ by

$$
x_{n}=\alpha_{n} x_{n-1}+\left(1-\alpha_{n}\right) T_{i(n)}^{k(n)} x_{n}+u_{n}, n \geq 1,
$$

where $n=(k-1) m+i, i=i(n) \in\{1,2, \ldots, m\}, k=k(n) \geq 1$ is a positive integer and $k(n) \rightarrow \infty$, as $n \rightarrow \infty$. $\left\{\alpha_{n}\right\}$ is a suitable sequence in $[0,1]$ and $\left\{u_{n}\right\} \subset K$ is such that $\sum_{n=0}^{\infty}\left\|u_{n}\right\|<\infty$. They extended the results of [14] from Hilbert spaces to more general uniformly convex Banach spaces and from nonexpansive mappings to asymptotically nonexpansive mappings.

Yang and Hu [15] proposed another implicit iteration process which appears to be more satisfactory as follows:

$$
\begin{equation*}
x_{n}=\alpha_{n} x_{n-1}+\beta_{n} T_{i(n)}^{k(n)} x_{n}+\gamma_{n} u_{n}, n \geq 1, \tag{1.1}
\end{equation*}
$$

where $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\}$ and $\left\{\gamma_{n}\right\} \subset[0,1]$ with $\alpha_{n}+\beta_{n}+\gamma_{n}=1$, and $\left\{u_{n}\right\}$ is bounded in $K$.

Since for each $n \geq 1$, it can be written as $n=(k-1) m+i$, where $i=i(n) \in\{1,2, \ldots, m\}, k=k(n)-1$ is a positive integer and $k(n) \rightarrow \infty$ as $n \rightarrow \infty$. Hence, (1.1) can be expressed in the following form:

$$
\begin{equation*}
x_{n}=\left(1-\alpha_{n}-\gamma_{n}\right) x_{n-1}+\alpha_{n} T_{i(n)}^{n} x_{n}+\gamma_{n} u_{n}, n \geq 1, \tag{1.2}
\end{equation*}
$$

where $\alpha_{n}+\gamma_{n} \leq 1$, and $\left\{u_{n}\right\}$ is bounded in $K$.
Very recently, Yang [16] proved the following result.

Theorem 1.2. ([16]). Let $E$ be a real normed linear space, $K$ be a nonempty convex subset of $E, T_{i}: K \rightarrow K, i=1,2, \ldots, m$ be a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically pseudo-contractive mappings with $\left\{k_{i n}\right\} \subset[1, \infty)$ such that $\lim _{n \rightarrow \infty} k_{n}=1$, where $k_{n}=\max _{1 \leq i \leq m}\left\{k_{i n}\right\}$. Assume that $F=$ $\bigcap_{i=1}^{m} F\left(T_{i}\right) \neq \emptyset$ denotes the set of common fixed points of $\left\{T_{i}\right\}_{i=1}^{m}$. Let $\left\{x_{n}\right\}$ be the sequence defined by (1.2). Suppose that $\left\{u_{n}\right\}$ is bounded in $K$ and that $\left\{\alpha_{n}\right\},\left\{\gamma_{n}\right\}$ are sequences in $[0,1]$ satisfying the following conditions:
(i) $\sum_{n=0}^{\infty} \gamma_{n}<\infty, \sum_{n=0}^{\infty} \alpha_{n}\left(k_{n}-1\right)<\infty$,
(ii) $\sum_{n=0}^{\infty} \alpha_{n}=\infty, \lim _{n \rightarrow \infty} \alpha_{n}=0$.

Assume that there exists a strict increasing function $\varphi:[0, \infty) \rightarrow[0, \infty)$ with $\varphi(0)=0$ such that
$\limsup _{n \rightarrow \infty}\left\{\left\langle T_{i}^{n} x_{n}-x^{*}, j\left(x_{n}-x^{*}\right)\right\rangle-k_{n}\left\|x_{n}-x^{*}\right\|^{2}+\varphi\left(\left\|x_{n}-x^{*}\right\|\right)\right\} \leq 0$ for $x^{*} \in F$ and $i=1,2, \cdots, m$. Then $\left\{x_{n}\right\}$ converges strongly to a common fixed point $p$ of $\left\{T_{i}\right\}_{i=1}^{m}$.
Remark 1.3. We point out here that the conditions (i) is not always true.
Example 1.4. Let $\alpha_{n}=\frac{1}{\sqrt{n+1}}$ and $k_{n}=1+\frac{1}{\sqrt{n+1}}$, then $\sum_{n=0}^{\infty} \alpha_{n}\left(k_{n}-1\right)=$ $\sum_{n=0}^{\infty} \frac{1}{n+1}=\infty$, which show that conditions (i) in Theorem 1.2 is not satisfied. Hence Theorem 1.2 need to be improved.

The purpose of this paper is, under the condition of removing the restriction $\sum_{n=0}^{\infty} \alpha_{n}\left(k_{n}-1\right)<\infty$, to prove strong convergence theorems of Ishikawa type implicit iteration process with errors for a finite family of
asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings in normed linear spaces by using a new analytical method. Our results essentially extend and improve some recent results obtained by Yang [16] and others.

Now we consider Ishikawa type implicit iteration process with errors for a finite family of asymptotically quasi-pseudocontractive type mappings as follows:

$$
\left\{\begin{array}{l}
x_{n}=\left(1-\alpha_{n}-\gamma_{n}\right) x_{n-1}+\alpha_{n} T_{i(n)}^{n} y_{n}+\gamma_{n} u_{n}  \tag{1.3}\\
y_{n}=\left(1-\beta_{n}-\mu_{n}\right) x_{n}+\beta_{n} T_{i(n)}^{n} x_{n}+\mu_{n} v_{n}, \quad(n \geq 0),
\end{array}\right.
$$

where $n=(k-1) m+i, i=i(n) \in\{1,2, \cdots, m\}, k=k(n) \geq 1$ is a positive integer and $k(n) \rightarrow \infty$, as $n \rightarrow \infty$. $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\},\left\{\mu_{n}\right\}$ are four suitable sequences in [0,1] with $\alpha_{n}+\gamma_{n} \leq 1, \beta_{n}+\mu_{n} \leq 1$ and $\left\{u_{n}\right\},\left\{v_{n}\right\}$ are bounded sequences in $K$.

The following lemmas plays an important role in this paper.

Lemma 1.5 ([17). ] Let $E$ be a real normed linear space and $J_{p}: E \rightarrow$ $2^{E^{*}}$ a duality mapping. Then

$$
\|x+y\|^{p} \leq\|x\|^{p}+p\left\langle y, j_{p}(x+y)\right\rangle
$$

for all $x, y \in E, 1<p<\infty$ and $j_{p}(x+y) \in J_{p}(x+y)$.
Lemma 1.6. Let $\varphi_{i}(i=1,2, \ldots, m):[0, \infty) \rightarrow[0, \infty)$ be strictly increasing functions with $\varphi_{i}(0)=0$ and let $\left\{a_{n}\right\},\left\{b_{n}\right\},\left\{c_{n}\right\},\left\{\delta_{n}\right\}$ be nonnegative real sequences such that $\sum_{n=1}^{\infty} \delta_{n}=\infty, \lim _{n \rightarrow \infty} \frac{b_{n}}{\delta_{n}}=0, \sum_{n=1}^{\infty} c_{n}<$ $\infty$. Suppose that

$$
\begin{equation*}
a_{n+1}^{p} \leq a_{n}^{p}-\delta_{n} \varphi_{i}\left(a_{n+1}\right)+b_{n}+c_{n}, n \geq n_{0}, \tag{1.4}
\end{equation*}
$$

where $n_{0}$ is some nonnegative integer and $p \in(1, \infty)$, then $\lim _{n \rightarrow \infty} a_{n}=0$.
Proof. Setting $\liminf _{n \rightarrow \infty} a_{n}=\tau$, then $\tau \geq 0$. Now we prove $\tau=0$. If $\tau>0$, then there exists a positive integer $N_{1}>0$ such that $a_{n} \geq \frac{\tau}{2}$ for all $n \geq N_{1}$. By the strictly increasing property of $\varphi_{i}$, we have $\varphi_{i}\left(a_{n+1}\right)>\varphi_{i}\left(\frac{\tau}{2}\right) \geq \min _{1 \leq i \leq m} \varphi_{i}\left(\frac{\tau}{2}\right)=: \sigma$. Since $\lim _{n \rightarrow \infty} \frac{b_{n}}{\delta_{n}}=0$, there exists a positive integer $N_{2}>N_{1}$ such that $\frac{b_{n}}{\delta_{n}} \leq \frac{1}{2} \sigma$ for all $n \geq N_{2}$. Taking $N_{3}=\max \left\{N_{2}, n_{0}\right\}$, then from (1.4), we have

$$
a_{n+1}^{p} \leq a_{n}^{p}-\delta_{n} \sigma+\delta_{n} \frac{\sigma}{2}+c_{n}=a_{n}^{p}-\delta_{n} \frac{\sigma}{2}+c_{n}
$$

for all $n \geq N_{3}$, which means that $\delta_{n} \frac{\sigma}{2} \leq a_{n}^{p}-a_{n+1}^{p}+c_{n}$. Hence for any positive integer $h \geq N_{3}$, we obtain

$$
\frac{\sigma}{2} \sum_{n=N_{3}}^{h} \delta_{n} \leq a_{N_{3}}^{p}-a_{h+1}^{p}+\sum_{n=N_{3}}^{h} c_{n} \leq a_{N_{3}}^{p}+\sum_{n=N_{3}}^{h} c_{n}
$$

and so

$$
\infty=\frac{\sigma}{2} \sum_{n=N_{3}}^{\infty} \delta_{n} \leq a_{N_{3}}^{p}+\sum_{n=N_{3}}^{\infty} c_{n},
$$

a contradition. This implies that $\tau>0$ is impossible. Therefore $\tau=$ 0 , which there exists a subsequence $\left\{a_{n_{j}}\right\} \subset\left\{a_{n}\right\}$ such that $a_{n_{j}} \rightarrow$ $0(j \rightarrow \infty)$. Since $\lim _{n \rightarrow \infty} \frac{b_{n}}{\delta_{n}}=0, \sum_{n=1}^{\infty} c_{n}<\infty$, for any given $\varepsilon>0$, there exist two positive integers $j_{0}>0$ and $N_{4}>0$, such that for all $n \geq N_{4}, \sum_{n=N_{4}}^{\infty} c_{n}<\varepsilon^{p}, \frac{b_{n}}{\delta_{n}}<\frac{1}{2} \omega$ and $a_{n_{j}}<\varepsilon$ for all $j \geq j_{0}$, where $\omega=\min \left\{\varphi_{1}(\varepsilon), \varphi_{2}(\varepsilon), \cdots, \varphi_{m}(\varepsilon)\right\}$. Let $N_{5}=\max \left\{j_{0}, N_{4}\right\}$. For fixed $j_{*}>N_{5}$ and all $k \geq 0$, we now want to show that $a_{n_{j_{*}}+k}<2 \varepsilon$. To see this consider two possible cases.

Case I: $a_{n_{j_{*}+1}}<\varepsilon$.
In this case, $a_{n_{j_{*}+1}}^{p}<\varepsilon^{p}+c_{n_{j_{*}}}+c_{n_{j_{*}+1}}$ and so we have the desired result.

Case II: $a_{n_{j_{*}}+1} \geq \varepsilon$.
In this case, $\varphi_{i}\left(a_{n_{j_{*}}+1}\right) \geq \varphi_{i}(\varepsilon) \geq \omega>0$ since for each $i=1,2, \cdots, m$, $\varphi_{i}$ is a strictly increasing function. From (1.4), we also have

$$
\begin{aligned}
a_{n_{j_{*}}+1}^{p} & \leq a_{n_{j_{*}}}^{p}-\delta_{n_{j_{*}}} \varphi_{i}\left(a_{n_{j_{*}}+1}\right)+b_{n_{j_{*}}}+c_{n_{j_{*}}} \\
& \leq a_{n_{j_{*}}}^{p}-\delta_{n_{j_{*}}}\left(\omega-\frac{\omega}{2}\right)+c_{n_{j_{*}}} \\
& <\varepsilon^{p}+c_{n_{j_{*}}}+c_{n_{j_{*}}+1} .
\end{aligned}
$$

By using induction, we have

$$
a_{n_{j_{*}+k}}^{p}<\varepsilon^{p}+\sum_{i=n_{j_{*}}}^{n_{j_{*}}+k} c_{i}<\varepsilon^{p}+\varepsilon^{p}=2 \varepsilon^{p}<(2 \varepsilon)^{p}
$$

for all $k \geq 0$. This shows $a_{n} \rightarrow 0$ as $n \rightarrow \infty$. The proof of Lemma 1.6 is completed.

## 2. Main results

Theorem 2.1. Let $E$ be a real normed linear space, $K$ a nonempty convex subset of $E$ and $T_{i}: K \rightarrow K(i=1,2, \ldots, m)$ a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings with $\left\{k_{n}^{(i)}\right\} \subset[1, \infty)$ such that $\lim _{n \rightarrow \infty} k_{n}=1$, where $k_{n}=\max _{1 \leq i \leq m}\left\{k_{n}^{(i)}\right\}$. Assume that $F=$ $\bigcap_{i=1}^{m} F\left(T_{i}\right) \neq \emptyset$ denotes the set of common fixed points of $\left\{T_{i}\right\}_{i=1}^{m}$. Let $\left\{x_{n}\right\}$ be the sequence defined by (1.3). Suppose that $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$ are bounded sequences in $K$ and that $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\}$ and $\left\{\mu_{n}\right\}$ are sequences in [0, 1] satisfying the following conditions:
(i) $\sum_{n=0}^{\infty} \alpha_{n}=\infty$,
(ii) $\alpha_{n} \rightarrow 0, \beta_{n} \rightarrow 0(n \rightarrow \infty)$,
(iii) $\sum_{n=0}^{\infty} \gamma_{n}<\infty, \mu_{n} \rightarrow 0(n \rightarrow \infty)$.

Assume that there exist strict increasing functions $\varphi_{i}:[0, \infty) \rightarrow[0, \infty)$ with $\varphi_{i}(0)=0$ such that

$$
\limsup _{n \rightarrow \infty}\left\{\inf _{j_{p}\left(x_{n}-x^{*}\right) \in J_{p}\left(x_{n}-x^{*}\right)}\left\langle T_{i}^{n} x_{n}-x^{*}, j_{p}\left(x_{n}-x^{*}\right)\right\rangle, \begin{array}{l}
\left.-k_{n}\left\|x_{n}-x^{*}\right\|^{p}+\varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right)\right\} \leq 0 \tag{2.1}
\end{array}\right.
$$

for $x^{*} \in F$ and $i=1,2, \ldots, m$. Then $\left\{x_{n}\right\}$ converges strongly to a common fixed point $x^{*}$ of $\left\{T_{i}\right\}_{i=1}^{m}$.

Proof. For $i=1,2, \ldots, m$, let

$$
\begin{aligned}
\sigma_{n}^{(i)} & =\inf _{j_{p}\left(x_{n}-x^{*}\right) \in J_{p}\left(x_{n}-x^{*}\right)}\left\langle T_{i}^{n} x_{n}-x^{*}, j_{p}\left(x_{n}-x^{*}\right)\right\rangle \\
& -k_{n}\left\|x_{n}-x^{*}\right\|^{p}+\varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right)
\end{aligned}
$$

then there exist $j_{p}^{(i)}\left(x_{n}-x^{*}\right) \in J_{p}\left(x_{n}-x^{*}\right)$, such that

$$
\begin{align*}
\left\langle T_{i}^{n} x_{n}-x^{*}, j_{p}^{(i)}\left(x_{n}-x^{*}\right)\right\rangle & -k_{n}\left\|x_{n}-x^{*}\right\|^{p}+\varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right) \\
& <\sigma_{n}^{(i)}+\varepsilon_{n}^{(i)} \leq \xi_{n} \tag{2.2}
\end{align*}
$$

where $\varepsilon_{n}^{(i)} \in(0,1)$ with $\varepsilon_{n}^{(i)} \rightarrow 0(n \rightarrow \infty)$, and $\xi_{n}=\max _{1 \leq i \leq m}\left\{\sigma_{n}^{(i)}, 0\right\}+$ $\max _{1 \leq i \leq m}\left\{\varepsilon_{n}^{(i)}\right\}$. It is easy see (using (2.1)) that $\lim _{n \rightarrow \infty} \xi_{n}=0$. Since $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$ are bounded sequences in $K, M=\underset{n \geq 0}{\operatorname{Sup}}\left\{\left\|u_{n}-x^{*}\right\|+\left\|v_{n}-x^{*}\right\|\right\}<$ $\infty$. Also, since for each $i=1,2, \ldots, m, T_{i}: K \rightarrow K$ is an asymptotically
nonexpansive in the intermediate sense, there exists $n_{0} \geq 1$ such that $\sup _{x, y \in K}\left(\left\|T_{i}^{n} x-T_{i}^{n} y\right\|-\|x-y\|\right) \leq 1$ for all $n \geq n_{0}, i=1,2, \ldots, m$.

It follows from (1.3) that

$$
\begin{aligned}
& \frac{\left\|x_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|} \\
= & \frac{\left\|\left(1-\alpha_{n}-\gamma_{n}\right)\left(x_{n-1}-x^{*}\right)+\alpha_{n}\left(T_{i}^{n} y_{n}-x^{*}\right)+\gamma_{n}\left(u_{n}-x^{*}\right)\right\|}{1+\left\|x_{n-1}-x^{*}\right\|} \\
\leq & \frac{\left\|x_{n-1}-x^{*}\right\|+\alpha_{n}\left\|T_{i}^{n} y_{n}-x^{*}\right\|+\left\|u_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|} \\
\leq & \frac{\left\|x_{n-1}-x^{*}\right\|+\sup _{x, y \in K}\left(\left\|T_{i}^{n} x-T_{i}^{n} y\right\|-\|x-y\|\right)}{1+\left\|x_{n-1}-x^{*}\right\|} \\
+ & \frac{\alpha_{n}\left\|y_{n}-x^{*}\right\|+\left\|u_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|} \\
\leq & \frac{\left\|x_{n-1}-x^{*}\right\|+1+\alpha_{n}\left[\left\|x_{n}-x^{*}\right\|+\left\|T_{i}^{n} x_{n}-x^{*}\right\|\right.}{1+\left\|x_{n-1}-x^{*}\right\|} \\
+ & \frac{\left.\left\|v_{n}-x^{*}\right\|\right]+\left\|u_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|} \\
\leq & \frac{\left\|x_{n-1}-x^{*}\right\|+1+2 \alpha_{n}\left\|x_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|} \\
+ & \frac{\sup _{x, y \in K}\left(\left\|T_{i}^{n} x-T_{i}^{n} y\right\|-\|x-y\|\right)+2 M}{1+\left\|x_{n-1}-x^{*}\right\|} \\
\leq & 3+2 M+\frac{2 \alpha_{n}\left\|x_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|}
\end{aligned}
$$

for all $n \geq n_{0}$.
Since $1-2 \alpha_{n} \rightarrow 1(n \rightarrow \infty)$, there exists $n_{1} \geq n_{0}$ such that $1-2 \alpha_{n}>$ $\frac{1}{2}>0$ for all $n \geq n_{1}$, which together with (2.3) gives that

$$
\begin{equation*}
\frac{\left\|x_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|} \leq \frac{3+2 M}{1-2 \alpha_{n}} \leq 6+4 M \tag{2.4}
\end{equation*}
$$

Let $c_{n}^{(i)}=\sup _{x, y \in K}\left(\left\|T_{i}^{n} x-T_{i}^{n} y\right\|-\|x-y\|\right), d_{n}=\max \left\{0, \max _{1 \leq i \leq m} c_{n}^{(i)}\right\}$, then $\lim _{n \rightarrow \infty} d_{n}=0$.

By (1.3), we have

$$
\begin{align*}
\left\|x_{n}-y_{n}\right\| & \leq \beta_{n}\left\|T_{i}^{n} x_{n}-x_{n}\right\|+\mu_{n}\left\|v_{n}-x_{n}\right\| \\
& \leq \beta_{n}\left(\left\|T_{i}^{n} x_{n}-x^{*}\right\|-\left\|x_{n}-x^{*}\right\|\right) \\
& +\left(2 \beta_{n}+\mu_{n}\right)\left\|x_{n}-x^{*}\right\|+\mu_{n}\left\|v_{n}-x^{*}\right\| \\
& \leq \beta_{n} \sup _{x, y \in K}\left(\left\|T_{i}^{n} x-T_{i}^{n} y\right\|-\|x-y\|\right) \\
& +\left(2 \beta_{n}+\mu_{n}\right)\left\|x_{n}-x^{*}\right\|+\mu_{n} M \\
& \leq\left(2 \beta_{n}+\mu_{n}\right)\left\|x_{n}-x^{*}\right\|+\beta_{n} d_{n}+\mu_{n} M \tag{2.5}
\end{align*}
$$

for all $n \geq n_{1}$.
For $j_{p}^{(i)}\left(x_{n}-x^{*}\right) \in J_{p}\left(x_{n}-x^{*}\right), \forall n \geq 0$, we have from (1.3) and Lemma 1.5 that

$$
\begin{aligned}
& \left(\frac{\left\|x_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|}\right)^{p} \\
= & \frac{\left\|\left(1-\alpha_{n}-\gamma_{n}\right)\left(x_{n-1}-x^{*}\right)+\alpha_{n}\left(T_{i}^{n} y_{n}-x^{*}\right)+\gamma_{n}\left(u_{n}-x^{*}\right)\right\|^{p}}{\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}} \\
\leq & \frac{\left(1-\alpha_{n}\right)^{p}\left\|x_{n-1}-x^{*}\right\|^{p}+p \alpha_{n}\left\langle T_{i}^{n} x_{n}-x^{*}, j_{p}^{(i)}\left(x_{n}-x^{*}\right)\right\rangle}{\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}} \\
+ & \frac{p \alpha_{n}\left\langle T_{i}^{n} y_{n}-T_{i}^{n} x_{n}, j_{p}^{(i)}\left(x_{n}-x^{*}\right)\right\rangle}{\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}} \\
(2.6)+ & \frac{p \gamma_{n}\left\langle u_{n}-x^{*}, j_{p}^{(i)}\left(x_{n}-x^{*}\right)\right\rangle}{\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}}
\end{aligned}
$$

for all $n \geq n_{1}, i=1,2, \ldots, m$.
Next we consider the second and third term on the right side of (2.6). From (2.4) and (2.5), we obtain that

$$
\begin{aligned}
& \frac{p \alpha_{n}\left\langle T_{i}^{n} y_{n}-T_{i}^{n} x_{n}, j_{p}^{(i)}\left(x_{n}-x^{*}\right)\right\rangle}{\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}} \\
\leq & p \alpha_{n} \frac{\left\|T_{i}^{n} y_{n}-T_{i}^{n} x_{n}\right\|\left\|x_{n}-x^{*}\right\|^{p-1}}{\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}} \\
= & p \alpha_{n} \frac{\left(\left\|T_{i}^{n} x_{n}-T_{i}^{n} y_{n}\right\|-\left\|x_{n}-y_{n}\right\|\right)+\left\|x_{n}-y_{n}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|} \\
\cdot & \left(\frac{\left\|x_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|}\right)^{p-1}
\end{aligned}
$$

$$
\begin{align*}
& \text { Implicit iteration approximation } \\
& \leq p \alpha_{n}\left(\frac{d_{n}}{1+\left\|x_{n-1}-x^{*}\right\|}+\frac{\left\|x_{n}-y_{n}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|}\right) \\
& \\
& \cdot\left(\frac{\left\|x_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|}\right)^{p-1} \\
& \\
& \leq p \alpha_{n}\left(d_{n}+\frac{\left(2 \beta_{n}+\mu_{n}\right)\left\|x_{n}-x^{*}\right\|+\beta_{n} d_{n}+\mu_{n} M}{1+\left\|x_{n-1}-x^{*}\right\|}\right) \\
& \\
& \cdot \quad\left(\frac{\left\|x_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|}\right)^{p-1} \\
& (2.7) \leq p \alpha_{n}(6+4 M)^{p-1}\left[d_{n}+\left(2 \beta_{n}+\mu_{n}\right)(6+4 M)+\beta_{n} d_{n}+\mu_{n} M\right] \\
& \text { for all } n \geq n_{1} \cdot  \tag{2.8}\\
& \text { In view of }(2.4), \text { we deduce that } \\
& \\
& \quad p \gamma_{n}\left\langle u_{n}-x^{*}, j_{p}^{(i)}\left(x_{n}-x^{*}\right)\right\rangle \\
& \left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}
\end{align*} \leq \frac{p \gamma_{n}\left\|u_{n}-x^{*}\right\|\left\|x_{n}-x^{*}\right\|^{p-1}}{\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}} . \leq p \gamma_{n} M(6+4 M)^{p-1} .
$$

Substituting (2.2), (2.7) and (2.8) into (2.6) yields that

$$
\begin{aligned}
\left(\frac{\left\|x_{n}-x^{*}\right\|}{1+\left\|x_{n-1}-x^{*}\right\|}\right)^{p} & \leq \frac{\left(1-\alpha_{n}\right)^{p}\left\|x_{n-1}-x^{*}\right\|^{p}+p \alpha_{n} \xi_{n}}{\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}} \\
& +\frac{p \alpha_{n}\left(k_{n}\left\|x_{n}-x^{*}\right\|^{p}-\varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right)\right)}{\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p}} \\
& +p \alpha_{n}(6+4 M)^{p-1}\left[d_{n}+\left(2 \beta_{n}+\mu_{n}\right)(6+4 M)\right. \\
& \left.+\beta_{n} d_{n}+\mu_{n} M\right]+p \gamma_{n} M(6+4 M)^{p-1}
\end{aligned}
$$

for all $n \geq n_{1}, i=1,2, \ldots, m$.
Since $1-p \alpha_{n} k_{n} \rightarrow 1(n \rightarrow \infty)$, there exists $n_{2} \geq n_{1}$ such that $0<\frac{1}{2}<1-p \alpha_{n} k_{n}<1$ for all $n \geq n_{2}$. It follows from (2.9) that

$$
\begin{align*}
& \left\|x_{n}-x^{*}\right\|^{p} \leq \frac{\left(1-\alpha_{n}\right)^{p}\left\|x_{n-1}-x^{*}\right\|^{p}+p \alpha_{n} \xi_{n}-p \alpha_{n} \varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right)}{1-p \alpha_{n} k_{n}} \\
&  \tag{2.10}\\
& \hline 2.10)
\end{align*}
$$

for all $n \geq n_{2}, i=1,2, \ldots, m$,
where $A_{n}=(6+4 M)^{p-1}\left[d_{n}+\left(2 \beta_{n}+\mu_{n}\right)(6+4 M)+\beta_{n} d_{n}+\mu_{n} M\right] \rightarrow$
$0(n \rightarrow \infty)$. Note that $\left(1+\left\|x_{n-1}-x^{*}\right\|\right)^{p} \leq 2^{p-1}\left(1+\left\|x_{n-1}-x^{*}\right\|^{p}\right)$,

$$
\begin{aligned}
\left(1-\alpha_{n}\right)^{p} & =1-p \alpha_{n}+\frac{p(p-1) \alpha_{n}^{2}}{2!}-\frac{p(p-1)(p-2) \alpha_{n}^{3}}{3!}+\cdots+\left(-\alpha_{n}\right)^{p} \\
& =1-p \alpha_{n}+\alpha_{n} B_{n},
\end{aligned}
$$

where

$$
B_{n}=\frac{p(p-1) \alpha_{n}}{2!}-\frac{p(p-1)(p-2) \alpha_{n}^{2}}{3!}+\cdots+\left(-\alpha_{n}\right)^{p-1} \rightarrow 0(n \rightarrow \infty)
$$

In virtue of (2.10), we conclude that

$$
\begin{align*}
\left\|x_{n}-x^{*}\right\|^{p} & \leq\left[1+\frac{p \alpha_{n}\left(k_{n}-1\right)+\alpha_{n} B_{n}+p 2^{p-1} \alpha_{n} A_{n}}{1-p \alpha_{n} k_{n}}\right. \\
& \left.+\frac{p M 2^{p-1}(6+4 M)^{p-1} \gamma_{n}}{1-p \alpha_{n} k_{n}}\right]\left\|x_{n-1}-x^{*}\right\|^{p} \\
& +\frac{p \alpha_{n}\left(\xi_{n}+2^{p-1} A_{n}\right)+p M 2^{p-1}(6+4 M)^{p-1} \gamma_{n}}{1-p \alpha_{n} k_{n}} \\
& -\frac{p \alpha_{n} \varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right)}{1-p \alpha_{n} k_{n}} \\
& \leq\left[1+2 p \alpha_{n}\left(k_{n}-1\right)+2 \alpha_{n} B_{n}+p 2^{p} \alpha_{n} A_{n}\right. \\
& \left.+p M 2^{p}(6+4 M)^{p-1} \gamma_{n}\right]\left\|x_{n-1}-x^{*}\right\|^{p} \\
& +2 p \alpha_{n}\left(\xi_{n}+2^{p-1} A_{n}\right)+p M 2^{p}(6+4 M)^{p-1} \gamma_{n} \\
& -p \alpha_{n} \varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right) \tag{2.11}
\end{align*}
$$

for all $n \geq n_{2}, i=1,2, \ldots, m$.
Now we take a nonnegative integer $n_{3} \geq n_{2}$ such that $x_{n_{3}} \neq x^{*}$ (if not, $x_{n}=x^{*}$ for all $n \geq n_{2}$, then $x_{n} \rightarrow x^{*}(n \rightarrow \infty)$, and so we have done). Since $k_{n} \rightarrow 1, \xi_{n} \rightarrow 0, B_{n} \rightarrow 0, A_{n} \rightarrow 0(n \rightarrow \infty), \sum_{n=0}^{\infty} \gamma_{n}<\infty$, there exists a positive integer $N>n_{3}$ such that, for all $n \geq N,\left(k_{n}-1+\right.$ $\left.\frac{1}{p} B_{n}\right)(2 G)^{p}+\left(\xi_{n}+2^{p-1} A_{n}\left(1+(2 G)^{p}\right)\right)<\frac{\min _{1 \leq i \leq m}\left\{\varphi_{i}(G)\right\}}{4}$ and $\sum_{n=N}^{\infty} \gamma_{n}<$ $\frac{G^{p}}{p M 2^{p}(6+4 M)^{p-1}\left(1+(2 G)^{p}\right)}$, where $G=\max \left\{\left\|x_{n_{3}}-x^{*}\right\|,\left\|x_{n_{3}+1}-x^{*}\right\|, \ldots\right.$, $\left.\left\|x_{N-1}-x^{*}\right\|,\left\|x_{N}-x^{*}\right\|\right\}$, and obviously $0<G<\infty$.

Next we proceed by induction to show $\left\|x_{N+k}-x^{*}\right\| \leq 2 G$ for all $k \geq 1$.
To see this consider two possible cases.
Case III: $\left\|x_{N+1}-x^{*}\right\| \leq G$.

In this case, $\left\|x_{N+1}-x^{*}\right\|^{p} \leq G^{2}+p M 2^{p}(6+4 M)^{p-1}\left(1+(2 G)^{p}\right) \gamma_{N+1}$ and so we have the desired result.

Case IV: $\left\|x_{N+1}-x^{*}\right\|>G$.
In this case, $\varphi_{i}\left(\left\|x_{N+1}-x^{*}\right\|\right)>\varphi_{i}(G) \geq \min _{1 \leq i \leq m}\left\{\varphi_{i}(G)\right\}>0$ since for each $i=1,2, \cdots, m, \varphi_{i}$ is a strictly increasing function. From (2.11), we also have

$$
\begin{aligned}
&\left\|x_{N+1}-x^{*}\right\|^{2} \leq\left\|x_{N}-x^{*}\right\|^{p} \\
&+ {\left[2 p \alpha_{N+1}\left(k_{N+1}-1\right)+2 \alpha_{N+1} B_{N+1}+p M 2^{p}(6+4 M)^{p-1} \gamma_{N+1}\right.} \\
&+\left.p 2^{p} \alpha_{N+1} A_{N+1}\right](2 G)^{p}+2 p \alpha_{N+1}\left(\xi_{N+1}+2^{p-1} A_{N+1}\right) \\
&+ p M 2^{p}(6+4 M)^{p-1} \gamma_{N+1}-p \alpha_{N+1} \min _{1 \leq i \leq m}\left\{\varphi_{i}(G)\right\} \\
&=\left\|x_{N}-x^{*}\right\|^{p} \\
&- p \alpha_{N+1}\left[\min _{1 \leq i \leq m}\left\{\varphi_{i}(G)\right\}-2\left(\xi_{N+1}+2^{p-1} A_{N+1}\left(1+(2 G)^{p}\right)\right)\right. \\
&-\left.2\left(k_{N+1}-1+\frac{1}{p} B_{N+1}\right)(2 G)^{p}\right]+p M 2^{p}(6+4 M)^{p-1}\left(1+(2 G)^{p}\right) \gamma_{N+1} \\
& \leq\left\|x_{N}-x^{*}\right\|^{p}-p \alpha_{N+1}\left(\min _{1 \leq i \leq m}\left\{\varphi_{i}(G)\right\}-\frac{\min _{1 \leq i \leq m}\left\{\varphi_{i}(G)\right\}}{4}\right) \\
&+ p M 2^{p}(6+4 M)^{p-1}\left(1+(2 G)^{p}\right) \gamma_{N+1} \\
& \leq\left\|x_{N}-x^{*}\right\|^{p}+p M 2^{p}(6+4 M)\left(1+(2 G)^{p}\right) \gamma_{N+1} \\
& \leq G^{p}+p M 2^{p}(6+4 M)^{p-1}\left(1+(2 G)^{p}\right) \gamma_{N+1} .
\end{aligned}
$$

By using induction, we get that

$$
\begin{aligned}
\left\|x_{N+k}-x^{*}\right\|^{2} & \leq G^{p}+p M 2^{p}(6+4 M)^{p-1}\left(1+(2 G)^{p}\right) \sum_{i=N+1}^{N+k} \gamma_{i} \\
& \leq G^{p}+G^{p}=2 G^{p} \leq(2 G)^{p}
\end{aligned}
$$

for all $k \geq 1$.
This shows $\left\|x_{n}-x^{*}\right\| \leq 2 G$ for all $n \geq N$. Therefore, it follows from $(2,11)$ that

$$
\left\|x_{n}-x^{*}\right\|^{p} \leq\left\|x_{n-1}-x^{*}\right\|^{p}+2 p \alpha_{n}\left[\left(k_{n}-1+\frac{B_{n}}{p}+2^{p-1} A_{n}\right)(2 G)^{p}\right.
$$

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$$
\begin{align*}
& \left.+\xi_{n}+2^{p-1} A_{n}\right]+p M 2^{p}(6+4 M)^{p-1}\left(1+(2 G)^{p}\right) \gamma_{n} \\
& -p \alpha_{n} \varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right) \tag{2.12}
\end{align*}
$$

for all $n \geq N, i=1,2, \cdots, m$.
Taking $\delta_{n}=p \alpha_{n}, c_{n}=p M 2^{p}(6+4 M)^{p-1}\left(1+(2 G)^{p}\right) \gamma_{n}, a_{n}=\left\|x_{n}-x^{*}\right\|$ and $b_{n}=2 p \alpha_{n}\left[\left(k_{n}-1+\frac{B_{n}}{p}+2^{p-1} A_{n}\right)(2 G)^{p}+\xi_{n}+2^{p-1} A_{n}\right]$ for all $n \geq N$. By (2.12) and Lemma 1.6 ensures that $\left\|x_{n}-x^{*}\right\| \rightarrow 0$ as $n \rightarrow \infty$, that is, $x_{n} \rightarrow x^{*}$ as $n \rightarrow \infty$. This completes the proof.

Remark 2.2. Theorem 2.1 improves and extends Theorem 1.2 (i.e., Theorem 2.1 of Yang [16]) in the following aspects:
(1) Extend asymptotically pseudocontractive mapping to asymptotically quasi-pseudocontractive type mappings.
(2) It abolishes the condition that $\sum_{n=0}^{\infty} \alpha_{n}\left(k_{n}-1\right)<\infty$.
(3) The proof of sequence $\left\{x_{n}\right\}$ boundedness is entirely different from what it was before.
(4) Extend implicit iterative scheme (1.2) to Ishikawa type implicit iteration process (1.3).
(5) Condition
$\limsup _{n \rightarrow \infty}\left\{\left\langle T_{i}^{n} x_{n}-x^{*}, j\left(x_{n}-x^{*}\right)\right\rangle-k_{n}\left\|x_{n}-x^{*}\right\|^{2}+\varphi\left(\left\|x_{n}-x^{*}\right\|\right)\right\} \leq 0$ is replaced by the condition

$$
\begin{aligned}
& \limsup _{n \rightarrow \infty}\left\{\inf _{j_{p}\left(x_{n}-x^{*}\right) \in J_{p}\left(x_{n}-x^{*}\right)}\left\langle T_{i}^{n} x_{n}-x^{*}, j_{p}\left(x_{n}-x^{*}\right)\right\rangle\right. \\
& \left.-k_{n}\left\|x_{n}-x^{*}\right\|^{p}+\varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right)\right\} \leq 0 .
\end{aligned}
$$

From Theorem 2.1, we obtain the following result immediately.

Theorem 2.3. Let $E$ be a real normed linear space, $K$ a nonempty bounded convex subset of $E$ and $T_{i}: K \rightarrow K(i=1,2, \cdots, m)$ a finite family of asymptotically nonexpansive mappings with $\left\{k_{n}^{(i)}\right\} \subset[1, \infty)$ such that $\lim _{n \rightarrow \infty} k_{n}=1$, where $k_{n}=\max _{1 \leq i \leq m}\left\{k_{n}^{(i)}\right\}$. Assume that $F=$
$\bigcap_{i=1}^{m} F\left(T_{i}\right) \neq \emptyset$ denotes the set of common fixed points of $\left\{T_{i}\right\}_{i=1}^{m}$. Let $\left\{x_{n}\right\}$ be the sequence defined by (1.3). Suppose that $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$ are bounded sequences in $K$ and that $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\}$ and $\left\{\mu_{n}\right\}$ are sequences in [0, 1] satisfying the following conditions:
(i) $\sum_{n=0}^{\infty} \alpha_{n}=\infty$,
(ii) $\alpha_{n} \rightarrow 0, \beta_{n} \rightarrow 0(n \rightarrow \infty)$,
(iii) $\sum_{n=0}^{\infty} \gamma_{n}<\infty, \mu_{n} \rightarrow 0(n \rightarrow \infty)$.

Assume that there exist strict increasing functions $\varphi_{i}:[0, \infty) \rightarrow[0, \infty)$ with $\varphi_{i}(0)=0$ such that

$$
\limsup _{n \rightarrow \infty}\left\{\begin{array}{l}
\inf _{j_{p}\left(x_{n}-x^{*}\right) \in J_{p}\left(x_{n}-x^{*}\right)}\left\langle T_{i}^{n} x_{n}-x^{*}, j_{p}\left(x_{n}-x^{*}\right)\right\rangle \\
\left.-k_{n}\left\|x_{n}-x^{*}\right\|^{p}+\varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right)\right\} \leq 0
\end{array}\right.
$$

for $x^{*} \in F$ and $i=1,2, \cdots, m$. Then $\left\{x_{n}\right\}$ converges strongly to a common fixed point $x^{*}$ of $\left\{T_{i}\right\}_{i=1}^{m}$.

Proof. Since $T_{i}$ is an asymptotically nonexpansive mapping with $\left\{k_{n}\right\} \subset[1, \infty)$ such that $\lim _{n \rightarrow \infty} k_{n}=1$, we have

$$
\begin{gathered}
\limsup _{n \rightarrow \infty}\left\{\sup _{x, y \in K}\left(\left\|T_{i}^{n} x-T_{i}^{n} y\right\|-\|x-y\|\right)\right\} \\
\leq \limsup _{n \rightarrow \infty}\left[\left(k_{n}-1\right) \operatorname{diam}(K)\right]=0
\end{gathered}
$$

where $\operatorname{diam}(K)=\sup _{x, y \in K}\|x-y\|$. This implies that every asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediate sense. Also since every asymptotically nonexpansive mapping is asymptotically pseudo-contractive mapping. The conclusion now follows easily from Theorem 2.1.

If $\gamma_{n}=\mu_{n}=0(\forall n \geq 1)$ in Theorem 2.1 and Theorem 2.3, then we have the following results.

Theorem 2.4. Let $E$ be a real normed linear space, $K$ a nonempty convex subset of $E$ and $T_{i}: K \rightarrow K(i=1,2, \cdots, m)$ a finite family of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-pseudocontractive type mappings with $\left\{k_{n}^{(i)}\right\} \subset[1, \infty)$ such that $\lim _{n \rightarrow \infty} k_{n}=1$, where $k_{n}=\max _{1 \leq i \leq m}\left\{k_{n}^{(i)}\right\}$. Assume that $F=$
$\bigcap_{i=1}^{m} F\left(T_{i}\right) \neq \emptyset$ denotes the set of common fixed points of $\left\{T_{i}\right\}_{i=1}^{m}$. Let $\left\{x_{n}\right\}$ be the sequence defined by

$$
\left\{\begin{array}{l}
x_{n}=\left(1-\alpha_{n}\right) x_{n-1}+\alpha_{n} T_{i(n)}^{n} y_{n} \\
y_{n}=(1-\beta) x_{n}+\beta_{n} T_{i(n)}^{n} x_{n}, \quad(n \geq 0) .
\end{array}\right.
$$

Suppose that $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\}$ are sequences in [0, 1] satisfying the following conditions:
(i) $\sum_{n=0}^{\infty} \alpha_{n}=\infty$,
(ii) $\alpha_{n} \rightarrow 0, \beta_{n} \rightarrow 0(n \rightarrow \infty)$.

Assume that there exist strict increasing functions $\varphi_{i}:[0, \infty) \rightarrow[0, \infty)$ with $\varphi_{i}(0)=0$ such that

$$
\begin{aligned}
& \limsup _{n \rightarrow \infty}\left\{\inf _{j_{p}\left(x_{n}-x^{*}\right) \in J_{p}\left(x_{n}-x^{*}\right)}\left\langle T_{i}^{n} x_{n}-x^{*}, j_{p}\left(x_{n}-x^{*}\right)\right\rangle\right. \\
& \left.-k_{n}\left\|x_{n}-x^{*}\right\|^{p}+\varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right)\right\} \leq 0
\end{aligned}
$$

for $x^{*} \in F$ and $i=1,2, \cdots, m$. Then $\left\{x_{n}\right\}$ converges strongly to $a$ common fixed point $x^{*}$ of $\left\{T_{i}\right\}_{i=1}^{m}$.

Theorem 2.5. Let $E$ be a real normed linear space, $K$ a nonempty bounded convex subset of $E$ and $T_{i}: K \rightarrow K(i=1,2, \cdots, m)$ a finite family of asymptotically nonexpansive mappings with $\left\{k_{n}^{(i)}\right\} \subset[1, \infty)$ such that $\lim _{n \rightarrow \infty} k_{n}=1$, where $k_{n}=\max _{1 \leq i \leq m}\left\{k_{n}^{(i)}\right\}$. Assume that $F=$ $\bigcap_{i=1}^{m} F\left(T_{i}\right) \neq \emptyset$ denotes the set of common fixed points of $\left\{T_{i}\right\}_{i=1}^{m}$. Let $\left\{x_{n}\right\}$ be the sequence defined by

$$
\left\{\begin{array}{l}
x_{n}=\left(1-\alpha_{n}\right) x_{n-1}+\alpha_{n} T_{i(n)}^{n} y_{n} \\
y_{n}=(1-\beta) x_{n}+\beta_{n} T_{i(n)}^{n} x_{n}, \quad(n \geq 0)
\end{array}\right.
$$

Suppose that $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\}$ are sequences in [0, 1] satisfying the following conditions:
(i) $\sum_{n=0}^{\infty} \alpha_{n}=\infty$,
(ii) $\alpha_{n} \rightarrow 0, \beta_{n} \rightarrow 0(n \rightarrow \infty)$.

Assume that there exist strict increasing functions $\varphi_{i}:[0, \infty) \rightarrow[0, \infty)$ with $\varphi_{i}(0)=0$ such that

$$
\begin{aligned}
& \limsup _{n \rightarrow \infty}\left\{\inf _{j_{p}\left(x_{n}-x^{*}\right) \in J_{p}\left(x_{n}-x^{*}\right)}\left\langle T_{i}^{n} x_{n}-x^{*}, j_{p}\left(x_{n}-x^{*}\right)\right\rangle\right. \\
& \left.-k_{n}\left\|x_{n}-x^{*}\right\|^{p}+\varphi_{i}\left(\left\|x_{n}-x^{*}\right\|\right)\right\} \leq 0
\end{aligned}
$$

for $x^{*} \in F$ and $i=1,2, \cdots, m$. Then $\left\{x_{n}\right\}$ converges strongly to $a$ common fixed point $x^{*}$ of $\left\{T_{i}\right\}_{i=1}^{m}$.

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