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FINITE ITERATIVE METHODS FOR SOLVING SYSTEMS OF LINEAR MATRIX EQUATIONS OVER REFLEXIVE AND ANTI-REFLEXIVE MATRICES

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Abstract. A matrix $P \in \mathrm{C}^{n\times n}$ is called a generalized reflection matrix if $P^H = P$ and $P^2 = I$. An $n\times n$ complex matrix A is said to be a reflexive (anti-reflexive) matrix with respect to the generalized reflection matrix P if $A = PAP$ ($A = -PAP$). In this paper, we introduce two iterative methods for solving the pair of matrix equations $AXB = C$ and $DXE = F$ over reflexive and anti-reflexive matrices. The convergence of the iterative methods is also proposed. Finally, a numerical example is given to show the efficiency of the presented results.

Keywords: Matrix equation, reflexive matrix, anti-reflexive matrix.
MSC(2010): Primary: 15A06, 15A24; Secondary: 65F30, 65F10.

1. Introduction

In this paper, we shall adopt the following notations and concepts [21, 19]. Let $\mathrm{C}^{m\times n}$ denote the set of all $m \times n$ complex matrices. . We denote by I_n the $n \times n$ identity matrix. We also write it as I, when the dimension of this matrix is clear. Symbols A^T, A^H, $\mathrm{tr}(A)$ and $R(A)$ denote the transpose, the conjugate transpose, the trace and the column space of the matrix A respectively. The symbol

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$A \otimes B$ stands for the Kronecker product of matrices A and B. The inner product $\langle .,. \rangle$ in $\mathrm{C}^{m\times n}$ over the field R is defined as follows:

$$\langle A, B\rangle = \mathrm{Re}(\mathrm{tr}(B^H A)) \text{ for } A, B \in \mathrm{C}^{m\times n},$$

that is $\langle A, B\rangle$ is the real part of the trace of $B^H A$ [34]. The induced matrix norm is $||A|| = \sqrt{\langle A, A\rangle} = \sqrt{\mathrm{Re}(\mathrm{tr}(A^H A))} = \sqrt{\mathrm{tr}(A^H A)}$, which is the Frobenius norm. For $A = [a_1,\ a_2,\ ...,\ a_m] \in \mathrm{C}^{m\times n}$, the stretching $\mathrm{vec}(A)$ is defined by $\mathrm{vec}(A) = [\ a_1^T,\ a_2^T,\ ...,\ a_m^T\ ]^T$. In [1, 35], the subspaces

$$\mathrm{C}_r^{n\times n}(P) = \{A \in \mathrm{C}^{n\times n} : A = PAP\},$$

and

$$\mathrm{C}_a^{n\times n}(P) = \{A \in \mathrm{C}^{n\times n} : A = -PAP\},$$

are introduced where P of size n is generalized reflection matrix, that is $P^H = P$ and $P^2 = I$. The matrices $A \in \mathrm{C}_r^{n\times n}(P)$ and $B \in \mathrm{C}_a^{n\times n}(P)$ are, respectively, said to be reflexive and anti-reflexive matrices with respect to the generalized reflection matrix P. The reflexive and anti-reflexive matrices have many special properties and widely used in engineering and scientific computations [1].
Matrix equations are very useful in engineering problems, information theory, linear system theory, linear estimation theory, numerical analysis theory, and others. Hence various linear matrix equations have been investigated [5, 10, 23, 24, 32]. In [3, 18, 33], the matrix equation

$$AX = B,$$

was discussed over symmetric and positive definite matrices. In [29], the problem of solution to the matrix equation

$$AX + X^T C = B,$$

was considered by the Moore-Penrose generalized inverse matrix, and a general solution to this equation was obtained. Mitra [25, 26] provided conditions for the existence of a solution and a representation of the general common solution to the matrix equations

$$AX = C,\ XB = D,$$

and the matrix equations

$$A_1 X B_1 = C_1 \text{ and } A_2 X B_2 = C_2. \tag{1.1}$$

Navarra et al. [27] studied a representation of the general common solution to the matrix equations (1.1). In [30], Wang considered the matrix equations (1.1) over an arbitrary regular ring with identity and derived the necessary and sufficient conditions for the existence and the expression of the general solution to the system. Also Wang in [31], considered the system of four linear matrix equations over an arbitrary von Neumann regular ring with identity. Huang et al. [22] constructed a new iterative method for solving the linear matrix equation $AXB = C$ over skew-symmetric matrix X. Peng et al. [28] presented an iterative method to solve the minimum Frobenius norm residual problem

$$\min ||AXB - C||,$$

with unknown symmetric matrix X. Based on the hierarchical identification principle [14, 17], Ding and Chen presented the hierarchical gradient iterative (HGI) algorithms for general matrix equations [13] and hierarchical least-squares-iterative (HLSI) algorithms for generalized coupled Sylvester matrix equation and general coupled matrix equations [16, 14, 15]. In [6, 20, 7, 8, 9, 12], some efficient iterative algorithms were introduced to solve Sylvester and Lyapunov matrix equations. In this paper, we study the pair of matrix equations

$$AXB = C \text{ and } DXE = F, \tag{1.2}$$

over reflexive and anti-reflexive matrices, where $A \in \mathrm{C}^{p\times n}$, $B \in \mathrm{C}^{n\times q}$, $D \in \mathrm{C}^{s\times n}$, $E \in \mathrm{C}^{n\times t}$, $C \in \mathrm{C}^{p\times q}$, and $F \in \mathrm{C}^{s\times t}$ are known. First we consider the following four problems.

Problem 1.1. *For given matrices $A \in C^{p\times n}$, $B \in C^{n\times q}$, $D \in C^{s\times n}$, $E \in C^{n\times t}$, $C \in C^{p\times q}$, $F \in C^{s\times t}$ and the generalized reflection matrix $P \in C^{n\times n}$, find $X \in C_r^{n\times n}(P)$ such that (1.2) holds.*

Problem 1.2. *When Problem 1.1 is consistent, let its solution set be denoted by S_r. For a given $\widehat{X} \in C_r^{n\times n}(P)$, find $\widetilde{X} \in S_r$ such that*

$$||\widetilde{X} - \widehat{X}|| = \min_{X\in S_r} ||X - \widehat{X}||. \tag{1.3}$$

Problem 1.3. *For given matrices $A \in C^{p\times n}$, $B \in C^{n\times q}$, $D \in C^{s\times n}$, $E \in C^{n\times t}$, $C \in C^{p\times q}$, $F \in C^{s\times t}$ and the generalized reflection matrix $P \in C^{n\times n}$, find $X \in C_a^{n\times n}(P)$ such that (1.2) holds.*

Problem 1.4. *When Problem 1.3 is consistent, let its solution set be denoted by S_a. For a given $\widehat{X} \in C_a^{n\times n}(P)$, find $\widetilde{X} \in S_a$ such that (1.3) holds.*

This paper is organized as follows: In Section 2, first we obtain two new pairs of matrix equations equivalent to the pair of matrix equations (1.2) over reflexive and anti-reflexive matrices, respectively, and second we present new four problems. Second, two iterative methods are given for solving new problems. The convergence results of the iterative methods are proposed in Section 3. In Section 4, we employ a numerical example to support the theoretical results of this paper. Also we give some conclusions in Section 5 to end this paper.

2. **Preparatory knowledge**

In this section, first by using results in [35, 2], we give the structure and properties of P, $\mathrm{C}_r^{n\times n}(P)$ and $\mathrm{C}_a^{n\times n}(P)$.
Let $P \in \mathrm{C}^{n\times n}$ be a given generalized reflection matrix. From [2], there exists an unitary matrix $U \in \mathrm{C}^{n\times n}$ such that

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^H. \tag{2.1}$$

By applying (2.1), we can show [35]:
(a): $A \in \mathrm{C}_r^{n\times n}(P)$ if and only if there exist $Z_1 \in \mathrm{C}^{r\times r}$ and $Z_4 \in \mathrm{C}^{(n-r)\times(n-r)}$ such that

$$A = U \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix} U^H. \tag{2.2}$$

(b): $A \in \mathrm{C}_a^{n\times n}(P)$ if and only if there exist $Z_2 \in \mathrm{C}^{r\times(n-r)}$ and $Z_3 \in \mathrm{C}^{(n-r)\times r}$ such that

$$A = U \begin{pmatrix} 0 & Z_2 \\ Z_3 & 0 \end{pmatrix} U^H. \tag{2.3}$$

Now, for generalized reflection matrix $P \in \mathrm{C}^{n\times n}$ and the matrices $A \in \mathrm{C}^{p\times n}$, $B \in \mathrm{C}^{n\times q}$, $D \in \mathrm{C}^{s\times n}$, $E \in \mathrm{C}^{n\times t}$, $C \in \mathrm{C}^{p\times q}$, $F \in \mathrm{C}^{s\times t}$, we present the following decompositions:

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^H, \tag{2.4}$$

$$AU = (A_1, A_2),\ DU = (D_1, D_2), \tag{2.5}$$

$$U^H B = (B_1^T, B_2^T)^T,\ U^H E = (E_1^T, E_2^T)^T, \tag{2.6}$$

where $U \in \mathrm{C}^{n\times n}$ is an unitary matrix and $A_1, D_1 \in \mathrm{C}^{p\times r}$, $A_2, D_2 \in \mathrm{C}^{p\times(n-r)}$, $B_1, E_1 \in \mathrm{C}^{r\times q}$, $B_2, E_2 \in \mathrm{C}^{(n-r)\times q}$. In the rest of this paper, without loss of generality, we suppose that the matrices A, B, D and E have the above decompositions.

Theorem 2.1. *Let $A \in C^{p\times n}$, $B \in C^{n\times q}$, $D \in C^{s\times n}$, $E \in C^{n\times t}$, $C \in C^{p\times q}$, $F \in C^{s\times t}$, and P be a generalized reflection matrix of size n. For the pair of matrix equations (1.2), the following statements are equivalent:*
(1) The pair of matrix equations (1.2) has the reflexive solution $X \in C_r^{n\times n}(P)$.

(2) The following coupled matrix equations have a solution pair:

$$A_1X_1B_1 + A_2X_4B_2 = C \text{ and } D_1X_1E_1 + D_2X_4E_2 = F. \tag{2.7}$$

In that case, the reflexive solution of (1.2) can be expressed as the following

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^H, \tag{2.8}$$

where $X_1 \in C^{r\times r}$, $X_4 \in C^{(n-r)\times(n-r)}$ and U, U^H are as in (2.1).

Proof. By substituting (2.4)-(2.6) and (2.8) in (1.2), we can obtain

$$\begin{cases} AXB = C, \\ DXE = F, \end{cases}$$

$$\Leftrightarrow \begin{cases} (A_1, A_2)U^HU \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^HU \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = C, \\ (D_1, D_2)U^HU \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^HU \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = F, \end{cases}$$

$$\Leftrightarrow \begin{cases} A_1X_1B_1 + A_2X_4B_2 = C, \\ D_1X_1E_1 + D_2X_4E_2 = F. \end{cases}$$

Hence (1) is equivalent to (2). Now the proof is completed. □

Similar to the proof of Theorem 2.1, we can prove the following theorem.

Theorem 2.2. *Let $A \in C^{p\times n}$, $B \in C^{n\times q}$, $D \in C^{s\times n}$, $E \in C^{n\times t}$, $C \in C^{p\times q}$, $F \in C^{s\times t}$, and P be a generalized reflection matrix of size n. For the pair of matrix equations (1.2), the following statements are equivalent:*
(1) *The pair of matrix equations (1.2) has the anti-reflexive solution $X \in C_a^{n\times n}(P)$.*

(2) *The following coupled matrix equations have a solution pair:*

$$A_1X_2B_2 + A_2X_3B_1 = C \text{ and } D_1X_2E_2 + D_2X_3E_1 = F. \tag{2.9}$$

In that case, the anti-reflexive solution of (1.2) can be expressed as the following

$$X = U \begin{pmatrix} 0 & X_2 \\ X_3 & 0 \end{pmatrix} U^H, \tag{2.10}$$

where $X_2 \in C^{r\times(n-r)}$, $X_3 \in C^{(n-r)\times r}$ and U, U^H are as in (2.1).

From Theorems 2.1 and 2.2, we conclude that the matrix equations (2.7) and (2.9) are equivalent to the pair of matrix equations (1.2) over reflexive and anti-reflexive matrices, respectively. Therefore, in the rest of this paper, we study the matrix equations (2.7) and (2.9). Now the following four problems are considered:

Problem 2.1. *Given $A_1, D_1 \in C^{p\times r}$, $A_2, D_2 \in C^{p\times(n-r)}$, $B_1, E_1 \in C^{r\times q}$, $B_2, E_2 \in C^{(n-r)\times q}$, $C \in C^{p\times q}$ and $F \in C^{s\times t}$. Find matrix pair $[X_1, X_4]$ with $X_1 \in C^{r\times r}$, $X_4 \in C^{(n-r)\times(n-r)}$ such that*

$$A_1X_1B_1 + A_2X_4B_2 = C \text{ and } D_1X_1E_1 + D_2X_4E_2 = F. \tag{2.11}$$

Problem 2.2. *Let Problem 2.1 be consistent, and its solution pair set be denoted by S_1. For a given matrix pair $[\widehat{X}_1, \widehat{X}_4]$ with $\widehat{X}_1 \in$*

$C^{\,r\times r}$, $\widehat{X}_4 \in C^{\,(n-r)\times(n-r)}$ *find* $[\widetilde{X}_1, \widetilde{X}_4]$ *with* $\widetilde{X}_1 \in C^{\,r\times r}$, $\widetilde{X}_4 \in C^{\,(n-r)\times(n-r)}$ *such that*

(2.12)
$$||\widetilde{X}_1 - \widehat{X}_1||^2 + ||\widetilde{X}_4 - \widehat{X}_4||^2 = \min_{[X_1,X_4]\in S_1} ||X_1 - \widehat{X}_1||^2 + ||X_4 - \widehat{X}_4||^2.$$

Problem 2.3. *Given* $A_1, D_1 \in C^{p\times r}$, $A_2, D_2 \in C^{p\times(n-r)}$, $B_1, E_1 \in C^{r\times q}$, $B_2, E_2 \in C^{(n-r)\times q}$, $C \in C^{p\times q}$ *and* $F \in C^{s\times t}$. *Find matrix pair* $[X_2, X_3]$ *with* $X_2 \in C^{r\times(n-r)}$, $X_3 \in C^{(n-r)\times r}$ *such that*

$$A_1X_2B_2 + A_2X_3B_1 = C \text{ and } D_1X_2E_2 + D_2X_3E_1 = F. \tag{2.13}$$

Problem 2.4. *Let Problem 2.3 be consistent, and its solution pair set be denoted by* S_2. *For a given matrix pair* $[\widehat{X}_2, \widehat{X}_3]$ *with* $\widehat{X}_2 \in C^{r\times(n-r)}$, $\widehat{X}_3 \in C^{(n-r)\times r}$ *find* $[\widetilde{X}_2, \widetilde{X}_3]$ *with* $\widetilde{X}_2 \in C^{r\times(n-r)}$, $\widetilde{X}_3 \in C^{(n-r)\times r}$ *such that*

(2.14)
$$||\widetilde{X}_2 - \widehat{X}_2||^2 + ||\widetilde{X}_3 - \widehat{X}_3||^2 = \min_{[X_2,X_3]\in S_2} ||X_2 - \widehat{X}_2||^2 + ||X_3 - \widehat{X}_3||^2.$$

In fact, Problem 2.2 (2.4) is to find the optimal approximation solution pair to the given matrix pair $[\widehat{X}_1, \widehat{X}_4]$ ($[\widehat{X}_2, \widehat{X}_3]$) in the solution pair set of Problem 2.1 (2.3).
Now, two new iterative methods are given for solving Problems 2.1-2.4.

Method 1. (To solve Problems 2.1 and 2.2)

Step 1.1. Given $A_1, D_1 \in \mathrm{C}^{p\times r}$, $A_2, D_2 \in \mathrm{C}^{p\times(n-r)}$, $B_1, E_1 \in \mathrm{C}^{r\times q}$, $B_2, E_2 \in \mathrm{C}^{(n-r)\times q}$, $C \in \mathrm{C}^{p\times q}$, $F \in \mathrm{C}^{s\times t}$, $X_1^1 \in \mathrm{C}^{r\times r}$, $X_4^1 \in \mathrm{C}^{(n-r)\times(n-r)}$;

Step 1.2. Compute

$$R_1=\begin{pmatrix} C-A_1X_1^1B_1-A_2X_4^1B_2 & 0 \\ 0 & F-D_1X_1^1E_1-D_2X_4^1E_2 \end{pmatrix};$$

$$P_1=A_1^H(C-A_1X_1^1B_1-A_2X_4^1B_2)B_1^H+D_1^H(F-D_1X_1^1E_1-D_2X_4^1E_2)E_1^H;$$

$$Q_1=A_2^H(C-A_1X_1^1B_1-A_2X_4^1B_2)B_2^H+D_2^H(F-D_1X_1^1E_1-D_2X_4^1E_2)E_2^H;$$

$k := 1$;

Step 1.3. If $R_k = 0$ or $R_k \neq 0$, $P_k = 0$, $Q_k = 0$ then stop; else, $k := k + 1$;

Step 1.4. Compute

$$X_1^k = X_1^{k-1} + \frac{||R_{k-1}||^2}{||P_{k-1}||^2 + ||Q_{k-1}||^2} P_{k-1};$$
$$X_4^k = X_4^{k-1} + \frac{||R_{k-1}||^2}{||P_{k-1}||^2 + ||Q_{k-1}||^2} Q_{k-1};$$
$$R_k = \begin{pmatrix} C - A_1X_1^kB_1 - A_2X_4^kB_2 & 0 \\ 0 & F - D_1X_1^kE_1 - D_2X_4^kE_2 \end{pmatrix}$$
$$= R_{k-1} - \frac{||R_{k-1}||^2}{||P_{k-1}||^2 + ||Q_{k-1}||^2}$$
$$\times \begin{pmatrix} A_1P_{k-1}B_1 + A_2Q_{k-1}B_2 & 0 \\ 0 & D_1P_{k-1}E_1 + D_2Q_{k-1}E_2 \end{pmatrix};$$
$$P_k = A_1^H(C - A_1X_1^kB_1 - A_2X_4^kB_2)B_1^H$$
$$+D_1^H(F - D_1X_1^kE_1 - D_2X_4^kE_2)E_1^H + \frac{||R_k||^2}{||R_{k-1}||^2} P_{k-1};$$
$$Q_k = A_2^H(C - A_1X_1^kB_1 - A_2X_4^kB_2)B_2^H$$
$$+D_2^H(F - D_1X_1^kE_1 - D_2X_4^kE_2)E_2^H + \frac{||R_k||^2}{||R_{k-1}||^2} Q_{k-1};$$

Step 1.5. Go to Step 1.3.

Method 2. (To solve Problems 2.3 and 2.4)

Step 2.1. Given $A_1, D_1 \in \mathrm{C}^{p\times r}$, $A_2, D_2 \in \mathrm{C}^{p\times(n-r)}$, $B_1, E_1 \in \mathrm{C}^{r\times q}$, $B_2, E_2 \in \mathrm{C}^{(n-r)\times q}$, $C \in \mathrm{C}^{p\times q}$, $F \in \mathrm{C}^{s\times t}$, $X_2^1 \in \mathrm{C}^{r\times(n-r)}$, $X_3^1 \in \mathrm{C}^{(n-r)\times r}$;

Step 2.2. Compute

$$R_1 = \begin{pmatrix} C - A_1X_2^1B_2 - A_2X_3^1B_1 & 0 \\ 0 & F - D_1X_2^1E_2 - D_2X_3^1E_1 \end{pmatrix};$$
$$P_1 = A_1^H(C - A_1X_2^1B_2 - A_2X_3^1B_1)B_2^H + D_1^H(F - D_1X_2^1E_2 - D_2X_3^1E_1)E_2^H;$$
$$Q_1 = A_2^H(C - A_1X_2^1B_2 - A_2X_3^1B_1)B_1^H + D_2^H(F - D_1X_2^1E_2 - D_2X_3^1E_1)E_1^H;$$

$k := 1$;

Step 2.3. If $R_k = 0$ or $R_k \neq 0$, $P_k = 0$, $Q_k = 0$ then stop; else, $k := k+1$;

Step 2.4. Compute

$$X_2^k = X_2^{k-1} + \frac{||R_{k-1}||^2}{||P_{k-1}||^2 + ||Q_{k-1}||^2} P_{k-1};$$
$$X_3^k = X_3^{k-1} + \frac{||R_{k-1}||^2}{||P_{k-1}||^2 + ||Q_{k-1}||^2} Q_{k-1};$$
$$R_k = \begin{pmatrix} C - A_1 X_2^k B_2 - A_2 X_3^k B_1 & 0 \\ 0 & F - D_1 X_2^k E_2 - D_2 X_3^k E_1 \end{pmatrix}$$
$$= R_{k-1} - \frac{||R_{k-1}||^2}{||P_{k-1}||^2 + ||Q_{k-1}||^2}$$
$$\times \begin{pmatrix} A_1 P_{k-1} B_2 + A_2 Q_{k-1} B_1 & 0 \\ 0 & D_1 P_{k-1} E_2 + D_2 Q_{k-1} E_1 \end{pmatrix};$$
$$P_k = A_1^H (C - A_1 X_2^k B_2 - A_2 X_3^k B_1) B_2^H$$
$$+ D_1^H (F - D_1 X_2^k E_2 - D_2 X_3^k E_1) E_2^H + \frac{||R_k||^2}{||R_{k-1}||^2} P_{k-1};$$
$$Q_k = A_2^H (C - A_1 X_2^k B_2 - A_2 X_3^k B_1) B_1^H$$
$$+ D_2^H (F - D_1 X_2^k E_2 - D_2 X_3^k E_1) E_1^H + \frac{||R_k||^2}{||R_{k-1}||^2} Q_{k-1};$$

Step 2.5. Go to Step 2.3.

In the next section, the convergence results of the Methods 1 and 2 are given.

3. Convergence analysis

In this section, we first give some properties of the Methods 1 and 2. Second we show that these methods are convergent.

Lemma 3.1. *Assume that the sequences $\{R_i\}$, $\{P_i\}$ and $\{Q_i\}$ are generated by Method 1 and there exists a positive integer number k such that $R_i \neq 0$ for $i = 1, 2, ..., k$, then*

(3.1)
$$\langle R_i, R_j \rangle = 0, \ \langle P_i, P_j \rangle + \langle Q_i, Q_j \rangle = 0, \text{ for } i, j = 1, 2, ..., k, \ i \neq j.$$

Proof. Because $\langle R_i, R_j\rangle = \langle R_j, R_i\rangle$,$\langle P_i, P_j\rangle = \langle P_j, P_i\rangle$ and $\langle Q_i, Q_j\rangle = \langle Q_j, Q_i\rangle$ we only need to prove

(3.2) $\langle R_i, R_j\rangle = 0,$ and $\langle P_i, P_j\rangle + \langle Q_i, Q_j\rangle = 0,$ for $1 \le i < j \le k.$

We prove conclusion (3.2) by induction in two steps.

Step I: We firstly show that

(3.3)

$$\langle R_i, R_{i+1}\rangle = 0, \text{ and } \langle P_i, P_{i+1}\rangle + \langle Q_i, Q_{i+1}\rangle = 0, \text{ for } i = 1, 2, ..., k.$$

Let $i = 1$, we can get

$$\begin{aligned}
\langle R_1, R_2\rangle &= \mathrm{Re}(\mathrm{tr}(R_2^H R_1)) = \mathrm{Re}\Big(\mathrm{tr}\Big(\Big[R_1 - \frac{||R_1||^2}{||P_1||^2 + ||Q_1||^2} \\
&\times \begin{pmatrix} A_1P_1B_1 + A_2Q_1B_2 & 0 \\ 0 & D_1P_1E_1 + D_2Q_1E_2 \end{pmatrix}\Big]^H R_1\Big)\Big) \\
&= ||R_1||^2 - \frac{||R_1||^2}{||P_1||^2 + ||Q_1||^2} \\
&\times \mathrm{Re}\Big(\mathrm{tr}\Big(\begin{pmatrix} A_1P_1B_1 + A_2Q_1B_2 & 0 \\ 0 & D_1P_1E_1 + D_2Q_1E_2 \end{pmatrix}^H \\
&\times \begin{pmatrix} C - A_1X_1^1B_1 - A_2X_4^1B_2 & 0 \\ 0 & F - D_1X_1^1E_1 - D_2X_4^1E_2 \end{pmatrix}\Big)\Big) \\
&= ||R_1||^2 - \frac{||R_1||^2}{||P_1||^2 + ||Q_1||^2} \\
&\times \mathrm{Re}\Big(\mathrm{tr}\Big((B_1^HP_1^HA_1^H + B_2^HQ_1^HA_2^H)(C - A_1X_1^1B_1 - A_2X_4^1B_2) \\
&+ (E_1^HP_1^HD_1^H + E_2^HQ_1^HD_2^H)(F - D_1X_1^1E_1 - D_2X_4^1E_2)\Big)\Big) \\
&= ||R_1||^2 - \frac{||R_1||^2}{||P_1||^2 + ||Q_1||^2}\mathrm{Re}\Big(\mathrm{tr}\Big(P_1^H\Big[A_1^H(C - A_1X_1^1B_1 - A_2X_4^1B_2)B_1^H \\
&+ D_1^H(F - D_1X_1^1E_1 - D_2X_4^1E_2)E_1^H\Big] \\
&+ Q_1^H\Big[A_2^H(C - A_1X_1^1B_1 - A_2X_4^1B_2)B_2^H \\
&+ D_2^H(F - D_1X_1^1E_1 - D_2X_4^1E_2)E_2^H\Big]\Big)\Big) \\
&= ||R_1||^2 - \frac{||R_1||^2}{||P_1||^2 + ||Q_1||^2}\mathrm{Re}\Big(\mathrm{tr}\Big(P_1^HP_1 + Q_1^HQ_1\Big)\Big)
\end{aligned}$$

$$= 0. \tag{3.4}$$

And also we can obtain

$$\langle P_1, P_2\rangle + \langle Q_1, Q_2\rangle = \mathrm{Re}(\mathrm{tr}(P_2^H P_1)) + \mathrm{Re}(\mathrm{tr}(Q_2^H Q_1))$$
$$= \mathrm{Re}\Big(\mathrm{tr}\Big(\Big[A_1^H(C - A_1X_1^2B_1 - A_2X_4^2B_2)B_1^H + D_1^H(F - D_1X_1^2E_1$$
$$-D_2X_4^2E_2)E_1^H + \frac{||R_2||^2}{||R_1||^2}P_1\Big]^H P_1 + \Big[A_2^H(C - A_1X_1^2B_1 - A_2X_4^2B_2)B_2^H$$
$$+D_2^H(F - D_1X_1^2E_1 - D_2X_4^2E_2)E_2^H + \frac{||R_2||^2}{||R_1||^2}Q_1\Big]^H Q_1\Big)\Big)$$
$$= \mathrm{Re}\Big(\mathrm{tr}\Big((C - A_1X_1^2B_1 - A_2X_4^2B_2)^H A_1P_1B_1 + (F - D_1X_1^2E_1$$
$$-D_2X_4^2E_2)^H D_1P_1E_1 + (C - A_1X_1^2B_1 - A_2X_4^2B_2)^H A_2Q_1B_2$$
$$+(F - D_1X_1^2E_1 - D_2X_4^2E_2)^H D_2Q_1E_2\Big)\Big) + \frac{||R_2||^2}{||R_1||^2}(||P_1||^2 + ||Q_1||^2)$$
$$=\mathrm{Re}\Bigg(\mathrm{tr}\Bigg(\begin{pmatrix} C-A_1X_1^2B_1-A_2X_4^2B_2 & 0 \\ 0 & F-D_1X_1^2E_1-D_2X_4^2E_2 \end{pmatrix}^H$$
$$\times \begin{pmatrix} A_1P_1B_1 + A_2Q_1B_2 & 0 \\ 0 & D_1P_1E_1 + D_2Q_1E_2 \end{pmatrix}\Bigg)\Bigg)$$
$$+\frac{||R_2||^2}{||R_1||^2}(||P_1||^2 + ||Q_1||^2)$$
$$=\frac{||P_1||^2+||Q_1||^2}{||R_1||^2}\mathrm{Re}\Big(\mathrm{tr}\Big(R_2^H(R_1-R_2)\Big)\Big)+\frac{||R_2||^2}{||R_1||^2}(||P_1||^2+||Q_1||^2)$$
$$=\frac{||P_1||^2+||Q_1||^2}{||R_1||^2}\Big[\langle R_1,R_2\rangle-||R_2||^2\Big]+\frac{||R_2||^2}{||R_1||^2}(||P_1||^2+||Q_1||^2)$$
$$= 0. \tag{3.5}$$

Now suppose that (3.3) holds for $i = u-1$. For $i = u$, we can write

$$\langle R_u, R_{u+1}\rangle = \mathrm{Re}(\mathrm{tr}(R_{u+1}^H R_u)) = \mathrm{Re}\Big(\mathrm{tr}\Big(\Big[R_u - \frac{||R_u||^2}{||P_u||^2 + ||Q_u||^2}$$
$$\times \begin{pmatrix} A_1P_uB_1 + A_2Q_uB_2 & 0 \\ 0 & D_1P_uE_1 + D_2Q_uE_2 \end{pmatrix}\Big]^H R_u\Big)\Big)$$
$$= ||R_u||^2 - \frac{||R_u||^2}{||P_u||^2 + ||Q_u||^2}$$

$$
\begin{aligned}
&\times\mathrm{Re}\Big(\mathrm{tr}\Big(\begin{pmatrix} A_1P_uB_1+A_2Q_uB_2 & 0\\ 0 & D_1P_uE_1+D_2Q_uE_2\end{pmatrix}^H\\
&\quad\times\begin{pmatrix} C-A_1X_1^uB_1-A_2X_4^uB_2 & 0\\ 0 & F-D_1X_1^uE_1-D_2X_4^uE_2\end{pmatrix}\Big)\Big)\Big)\\
&=||R_u||^2-\frac{||R_u||^2}{||P_u||^2+||Q_u||^2}\mathrm{Re}\Big(\mathrm{tr}\Big(P_u^H\Big[A_1^H(C-A_1X_1^uB_1\\
&\quad -A_2X_4^uB_2)B_1^H+D_1^H(F-D_1X_1^uE_1-D_2X_4^uE_2)E_1^H\Big]\\
&\quad +Q_u^H\Big[A_2^H(C-A_1X_1^uB_1-A_2X_4^uB_2)B_2^H\\
&\quad +D_2^H(F-D_1X_1^uE_1-D_2X_4^uE_2)E_2^H\Big]\Big)\Big)\\
&=||R_u||^2-\frac{||R_u||^2}{||P_u||^2+||Q_u||^2}\mathrm{Re}\Big(\mathrm{tr}\Big(P_u^H\Big[P_u-\frac{||R_u||^2}{||R_{u-1}||^2}P_{u-1}\Big]\\
&\quad +Q_u^H\Big[Q_u-\frac{||R_u||^2}{||R_{u-1}||^2}Q_{u-1}\Big]\Big)\Big)\\
&=||R_u||^2-\frac{||R_u||^2}{||P_u||^2+||Q_u||^2}\Big(||P_u||^2+||Q_u||^2\Big)\\
&\quad +\frac{||R_u||^4}{(||P_u||^2+||Q_u||^2)||R_{u-1}||^2}\Big[\langle P_{u-1},P_u\rangle+\langle Q_{u-1},Q_u\rangle\Big]\\
&=0. \qquad (3.6)
\end{aligned}
$$

And we can obtain

$$
\begin{aligned}
&\langle P_u,P_{u+1}\rangle+\langle Q_u,Q_{u+1}\rangle=\mathrm{Re}(\mathrm{tr}(P_{u+1}^HP_u))+\mathrm{Re}(\mathrm{tr}(Q_{u+1}^HQ_u))\\
&=\mathrm{Re}\Big(\mathrm{tr}\Big(\Big[A_1^H(C-A_1X_1^{u+1}B_1-A_2X_4^{u+1}B_2)B_1^H\\
&\quad +D_1^H(F-D_1X_1^{u+1}E_1-D_2X_4^{u+1}E_2)E_1^H+\frac{||R_{u+1}||^2}{||R_u||^2}P_u\Big]^HP_u\\
&\quad +\Big[A_2^H(C-A_1X_1^{u+1}B_1-A_2X_4^{u+1}B_2)B_2^H+D_2^H(F-D_1X_1^{u+1}E_1\\
&\quad -D_2X_4^{u+1}E_2)E_2^H+\frac{||R_{u+1}||^2}{||R_u||^2}Q_u\Big]^HQ_u\Big)\Big)\\
&=\mathrm{Re}\Big(\mathrm{tr}\Big(\begin{pmatrix} C-A_1X_1^{u+1}B_1-A_2X_4^{u+1}B_2 & 0\\ 0 & F-D_1X_1^{u+1}E_1-D_2X_4^{u+1}E_2\end{pmatrix}^H
\end{aligned}
$$

$$\times \begin{pmatrix} A_1P_uB_1 + A_2Q_uB_2 & 0 \\ 0 & D_1P_uE_1 + D_2Q_uE_2 \end{pmatrix}\Big)\Big)$$

$$+\frac{||R_{u+1}||^2}{||R_u||^2}(||P_u||^2 + ||Q_u||^2)$$

$$= \frac{||P_u||^2 + ||Q_u||^2}{||R_u||^2}\mathrm{Re}\Big(\mathrm{tr}\Big(R_{u+1}^H(R_u - R_{u+1})\Big)\Big)$$

$$+\frac{||R_{u+1}||^2}{||R_u||^2}\Big(||P_u||^2 + ||Q_u||^2\Big)$$

$$=\frac{||P_u||^2+||Q_u||^2}{||R_u||^2}\Big[\langle R_u, R_{u+1}\rangle - ||R_{u+1}||^2\Big] + \frac{||R_{u+1}||^2}{||R_u||^2}\Big(||P_u||^2+||Q_u||^2\Big)$$

$$= 0. \tag{3.7}$$

This implies that the conclusion (3.3) holds for $i = u$. Hence, the conclusion(3.3) holds by principal of induction.

Step II: In this step, we assume that

$$\langle R_i, R_{i+m}\rangle = 0 \text{ and } \langle P_i, P_{i+m}\rangle + \langle Q_i, Q_{i+m}\rangle = 0 \text{ for } 1 \le i \le m \tag{3.8}$$

and $1 < m < k$.

Now we show $\langle R_i, R_{i+m+1}\rangle = 0$ and $\langle P_i, P_{i+m+1}\rangle + \langle Q_i, Q_{i+m+1}\rangle = 0$. Similar to the above results, first we can obtain

$$\langle P_1, P_{i+1}\rangle + \langle Q_1, Q_{i+1}\rangle = 0.$$

By using Step I, (3.8) and similar to the proofs of (3.4)-(3.7), we can write

$$\langle R_i,R_{i+m+1}\rangle=\mathrm{Re}(\mathrm{tr}(R_{i+m+1}^H R_i))=\mathrm{Re}\Big(\mathrm{tr}\Big(\Big[R_{i+m}-\frac{||R_{i+m}||^2}{||P_{i+m}||^2+||Q_{i+m}||^2}$$

$$\times \begin{pmatrix} A_1P_{i+m}B_1 + A_2Q_{i+m}B_2 & 0 \\ 0 & D_1P_{i+m}E_1 + D_2Q_{i+m}E_2 \end{pmatrix}\Big]^H R_i\Big)\Big)$$

$$= \langle R_i, R_{i+m}\rangle - \frac{||R_{i+m}||^2}{||P_{i+m}||^2 + ||Q_{i+m}||^2}$$

$$\times\mathrm{Re}\Big(\mathrm{tr}\Big(\begin{pmatrix} A_1P_{i+m}B_1 + A_2Q_{i+m}B_2 & 0 \\ 0 & D_1P_{i+m}E_1 + D_2Q_{i+m}E_2 \end{pmatrix}^H$$

$$\times \begin{pmatrix} C - A_1X_1^iB_1 - A_2X_4^iB_2 & 0 \\ 0 & F - D_1X_1^iE_1 - D_2X_4^iE_2 \end{pmatrix}\Big)\Big)$$

$$= -\frac{||R_{i+m}||^2}{||P_{i+m}||^2 + ||Q_{i+m}||^2}\mathrm{Re}\Big(\mathrm{tr}\Big(P_{i+m}^H\Big[P_i - \frac{||R_i||^2}{||R_{i-1}||^2}P_{i-1}\Big] + Q_{i+m}^H\Big[Q_i - \frac{||R_i||^2}{||R_{i-1}||^2}Q_{i-1}\Big]\Big)\Big)$$

$$= -\frac{||R_{i+m}||^2}{||P_{i+m}||^2 + ||Q_{i+m}||^2}\Big[\langle P_i, P_{i+m}\rangle + \langle Q_i, Q_{i+m}\rangle\Big]$$

$$+\frac{||R_{i+m}||^2||R_i||^2}{(||P_{i+m}||^2 + ||Q_{i+m}||^2)||R_{i-1}||^2}\Big[\langle P_{i-1}, P_{i+m}\rangle + \langle Q_{i-1}, Q_{i+m}\rangle\Big]$$

$$= \dots\dots\dots = n_1(\langle P_1, P_{i+1}\rangle + \langle Q_1, Q_{i+1}\rangle) = 0, \tag{3.9}$$

for certain n_1. Also we can write

$$\langle P_i, P_{i+m+1}\rangle + \langle Q_i, Q_{i+m+1}\rangle = \mathrm{Re}(\mathrm{tr}(P_{i+m+1}^H P_i)) + \mathrm{Re}(\mathrm{tr}(Q_{i+m+1}^H Q_i))$$

$$= \mathrm{Re}\Big(\mathrm{tr}\Big(\Big[A_1^H(C - A_1X_1^{i+m+1}B_1 - A_2X_4^{i+m+1}B_2)B_1^H$$

$$+D_1^H(F - D_1X_1^{i+m+1}E_1 - D_2X_4^{i+m+1}E_2)E_1^H + \frac{||R_{i+m+1}||^2}{||R_{i+m}||^2}P_{i+m}\Big]^H P_i$$

$$+\Big[A_2^H(C - A_1X_1^{i+m+1}B_1 - A_2X_4^{i+m+1}B_2)B_2^H + D_2^H(F - D_1X_1^{i+m+1}E_1$$

$$-D_2X_4^{i+m+1}E_2)E_2^H + \frac{||R_{i+m+1}||^2}{||R_{i+m}||^2}Q_{i+m}\Big]^H Q_i\Big)\Big)$$

$$= \mathrm{Re}\Big(\mathrm{tr}\Big(\begin{pmatrix} C - A_1X_1^{i+m+1}B_1 - A_2X_4^{i+m+1}B_2 & 0 \\ 0 & F - D_1X_1^{i+m+1}E_1 - D_2X_4^{i+m+1}E_2 \end{pmatrix}^H$$

$$\times \begin{pmatrix} A_1P_iB_1 + A_2Q_iB_2 & 0 \\ 0 & D_1P_iE_1 + D_2Q_iE_2 \end{pmatrix}\Big)\Big)$$

$$+\frac{||R_{i+m+1}||^2}{||R_{i+m}||^2}\Big(\langle P_i, P_{i+m}\rangle + \langle Q_i, Q_{i+m}\rangle\Big)$$

$$= \frac{||P_i||^2 + ||Q_i||^2}{||R_i||^2}\mathrm{Re}\Big(\mathrm{tr}\Big(R_{i+m+1}^H(R_i - R_{i+1})\Big)\Big)$$

$$= \frac{||P_i||^2 + ||Q_i||^2}{||R_i||^2}\Big[\langle R_i, R_{i+m+1}\rangle - \langle R_{i+1}, R_{i+m+1}\rangle\Big]$$

$$= \dots\dots\dots = n_2(\langle P_1, P_{i+1}\rangle + \langle Q_1, Q_{i+1}\rangle) = 0, \tag{3.10}$$

for certain n_2. By Steps I and II, the conclusion (3.2) holds by the principal of induction. $\square$

By a similar proof to the previous lemma we can prove the following lemma.

Lemma 3.2. *Assume that the sequences* $\{R_i\}$, $\{P_i\}$ *and* $\{Q_i\}$ *are generated by Method 2 and there exists a positive integer number* k *such that* $R_i \neq 0$ *for* $i = 1, 2, ..., k$, *then*

(3.11)
$$\langle R_i, R_j\rangle = 0, \ \langle P_i, P_j\rangle + \langle Q_i, Q_j\rangle = 0, \ \text{for } i, j = 1, 2, ..., k, \ i \neq j.$$

Lemma 3.3. *Suppose that the coupled matrix equations* (2.7) *are consistent and* $[X_1^*, X_4^*]$ *is a solution pair of* (2.7). *Then, for any initial matrix pair* $[X_1^1, X_4^1]$, *it holds that*

(3.12) $$\langle P_i, (X_1^* - X_1^i)\rangle + \langle Q_i, (X_4^* - X_4^i)\rangle = ||R_i||^2, \ \text{for } i = 1, 2, ...,$$

where the sequences $\{P_i\}$, $\{Q_i\}$, *and* $\{R_i\}$ *are generated by Method 1.*

Proof. We prove the conclusion by induction. When $i = 1$, we have

$$\begin{aligned}
&\langle P_1, (X_1^* - X_1^1)\rangle + \langle Q_1, (X_4^* - X_4^1)\rangle \\
&= \mathrm{Re}\Big(\mathrm{tr}\Big((X_1^* - X_1^1)^H P_1\Big)\Big) + \mathrm{Re}\Big(\mathrm{tr}\Big((X_4^* - X_4^1)^H Q_1\Big)\Big) \\
&= \mathrm{Re}\Big(\mathrm{tr}\Big((X_1^* - X_1^1)^H [A_1^H (C - A_1 X_1^1 B_1 - A_2 X_4^1 B_2) B_1^H \\
&\quad + D_1^H (F - D_1 X_1^1 E_1 - D_2 X_4^1 E_2) E_1^H]\Big)\Big) \\
&+ \mathrm{Re}\Big(\mathrm{tr}\Big((X_4^* - X_4^1)^H [A_2^H (C - A_1 X_1^1 B_1 - A_2 X_4^1 B_2) B_2^H \\
&\quad + D_2^H (F - D_1 X_1^1 E_1 - D_2 X_4^1 E_2) E_2^H]\Big)\Big) \\
&= \mathrm{Re}\Big(\mathrm{tr}\Big((C - A_1 X_1^1 B_1 - A_2 X_4^1 B_2)^H A_1 (X_1^* - X_1^1) B_1 \\
&\quad + (F - D_1 X_1^1 E_1 - D_2 X_4^1 E_2)^H D_1 (X_1^* - X_1^1) E_1\Big)\Big) \\
&+ \mathrm{Re}\Big(\mathrm{tr}\Big((C - A_1 X_1^1 B_1 - A_2 X_4^1 B_2)^H A_2 (X_4^* - X_4^1) B_2 \\
&\quad + (F - D_1 X_1^1 E_1 - D_2 X_4^1 E_2)^H D_2 (X_4^* - X_4^1) E_2\Big)\Big)
\end{aligned}$$

$$=\mathrm{Re}\Bigg(\mathrm{tr}\Bigg(\begin{pmatrix} (C-A_1X_1^1B_1-A_2X_4^1B_2)^H & 0 \\ 0 & (F-D_1X_1^1E_1-D_2X_4^1E_2)^H \end{pmatrix}$$

$$\times\begin{pmatrix} A_1(X_1^*-X_1^1)B_1+A_2(X_4^*-X_4^1)B_2 & 0 \\ 0 & D_1(X_1^*-X_1^1)E_1+D_2(X_4^*-X_4^1)E_2 \end{pmatrix}\Bigg)\Bigg)$$

$$=\mathrm{Re}\Bigg(\mathrm{tr}\Bigg(\begin{pmatrix} C-A_1X_1^1B_1-A_2X_4^1B_2 & 0 \\ 0 & F-D_1X_1^1E_1-D_2X_4^1E_2 \end{pmatrix}^H$$

$$\times\begin{pmatrix} C-A_1X_1^1B_1-A_2X_4^1B_2 & 0 \\ 0 & F-D_1X_1^1E_1-D_2X_4^1E_2 \end{pmatrix}\Bigg)\Bigg)$$

$$=||R_1||^2. \tag{3.13}$$

Now suppose that the conclusion (3.12) holds for $1 \le i \le l$. Then for $i = l+1$, we get

$$\langle P_{l+1},(X_1^*-X_1^{l+1})\rangle+\langle Q_{l+1},(X_4^*-X_4^{l+1})\rangle$$

$$=\mathrm{Re}\Big(\mathrm{tr}\Big((X_1^*-X_1^{l+1})^HP_{l+1}\Big)\Big)+\mathrm{Re}\Big(\mathrm{tr}\Big((X_4^*-X_4^{l+1})^HQ_{l+1}\Big)\Big)$$

$$=\mathrm{Re}\Big(\mathrm{tr}\Big((X_1^*-X_1^{l+1})^H[A_1^H(C-A_1X_1^{l+1}B_1-A_2X_4^{l+1}B_2)B_1^H$$

$$+D_1^H(F-D_1X_1^{l+1}E_1-D_2X_4^{l+1}E_2)E_1^H+\frac{||R_{l+1}||^2}{||R_l||^2}P_l]\Big)\Big)$$

$$+\mathrm{Re}\Big(\mathrm{tr}\Big((X_4^*-X_4^{l+1})^H[A_2^H(C-A_1X_1^{l+1}B_1-A_2X_4^{l+1}B_2)B_2^H$$

$$+D_2^H(F-D_1X_1^{l+1}E_1-D_2X_4^{l+1}E_2)E_2^H+\frac{||R_{l+1}||^2}{||R_l||^2}Q_l]\Big)\Big)$$

$$=\mathrm{Re}\Bigg(\mathrm{tr}\Bigg(\begin{pmatrix} C-A_1X_1^{l+1}B_1-A_2X_4^{l+1}B_2 & 0 \\ 0 & F-D_1X_1^{l+1}E_1-D_2X_4^{l+1}E_2 \end{pmatrix}^H$$

$$\times\begin{pmatrix} C-A_1X_1^{l+1}B_1-A_2X_4^{l+1}B_2 & 0 \\ 0 & F-D_1X_1^{l+1}E_1-D_2X_4^{l+1}E_2 \end{pmatrix}\Bigg)\Bigg)$$

$$+\frac{||R_{l+1}||^2}{||R_l||^2}\mathrm{Re}\Big(\mathrm{tr}\Big((X_1^*-X_1^{l+1})^HP_l+(X_4^*-X_4^{l+1})^HQ_{l+1}\Big)\Big)$$

$$=||R_{l+1}||^2+\frac{||R_{l+1}||^2}{||R_l||^2}\mathrm{Re}\Big(\mathrm{tr}\Big((X_1^*-X_1^l)^HP_l+(X_4^*-X_4^l)^HQ_l\Big)\Big)$$

$$-\frac{||R_{l+1}||^2}{||P_l||^2+||Q_l||^2}\mathrm{Re}\Big(\mathrm{tr}\Big(P_l^HP_l+Q_l^HQ_l\Big)\Big)$$

(3.14) $\qquad = ||R_{l+1}||^2.$

By induction principal, the conclusion (3.12) holds for $i = 1, 2, ...$ □

Similar to the proof of Lemma 3.3, we can prove the following lemma.

Lemma 3.4. *Suppose that the coupled matrix equations* (2.9) *are consistent and* $[X_2^*, X_3^*]$ *is a solution pair of* (2.9). *Then, for any initial matrix pair* $[X_2^1, X_3^1]$, *it holds that*

(3.15) $\langle P_i, (X_2^* - X_2^i)\rangle + \langle Q_i, (X_3^* - X_3^i)\rangle = ||R_i||^2, \text{ for } i = 1, 2, ...,$

where the sequences $\{P_i\}$, $\{Q_i\}$, *and* $\{R_i\}$ *are generated by Method 2.*

Remark 3.1. *From Lemma 3.4 (3.3), we can easily see if* $P_k = 0$ *or* $Q_k = 0$ *but* $R_k \neq 0$, *then the matrix equations (2.9) ((2.7)) are not consistent. Then, the solvability of the matrix equations (2.9) ((2.7)) can be determined by Method 1 (2) in the absence of roundoff errors. Hence the solvability of the pair of matrix equations (1.2) over reflexive and anti-reflexive matrices can be determined by Methods 1 and 2 in the absence of roundoff errors, respectively.*

Theorem 3.1. *Assume that Problem 2.1 is consistent. Then by Method 1 with any arbitrary initial matrix pair* $[X_1^1, X_4^1]$, *a solution pair of Problem 2.1 can be obtained with finite iteration steps in the absence of roundoff errors.*

Proof. If $R_i \neq 0$ $(i = 1, 2, ..., 2(pq + st))$, then from Lemma 3.3, we have $P_i \neq 0$ or $Q_i \neq 0$ $(i = 1, 2, ..., 2(pq + st))$. We can calculate $R_{pq+st+1}$ and $[X_1^{2(pq+st)+1}, X_4^{2(pq+st)+1}]$ by Method 1. We can obtain

$$\mathrm{Re}(\mathrm{tr}(R^H_{2(pq+st)+1} R_i)) = 0, \; i = 1, 2, ..., pq + st,$$

and

$$\mathrm{Re}(\mathrm{tr}(R_i^H R_j)) = 0, \; i, j = 1, 2, ..., 2(pq + st), \; i \neq j.$$

It can be seen that the set of $R_1, R_2, ..., R_{2(pq+st)}$ is an orthogonal basis of subspace

$$S = \left\{ M | M = \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix} \text{ where } M_1 \in \mathrm{C}^{p\times q} \text{ and } M_4 \in \mathrm{C}^{s\times t} \right\}.$$

Hence $R_{2(pq+st)+1} = 0$, i.e. the matrix pair $[X_1^{pq+st+1}, X_4^{2(pq+st)+1}]$ is a solution of Problem 2.1. Thus when Problem 2.1 is consistent, we can verify that the solution of Problem 2.1 can be obtained within finite iterative steps in the absence of roundoff errors. $\square$

Similar to the proof of Theorem 3.1, we can prove the following theorem.

Theorem 3.2. *Assume that Problem 2.3 is consistent. Then by Method 2 with any arbitrary initial matrix pair $[X_2^1, X_3^1]$, a solution pair of Problem 2.3 can be obtained with finite iteration steps in the absence of roundoff errors.*

Remark 3.2. *By using Method 1 with any arbitrary initial matrix pair $[X_1^1, X_4^1]$, a reflexive solution of Problem 1.1 can be obtained with finite iteration steps in the absence of roundoff errors by the following form*

$$X^* = U \begin{pmatrix} X_1^* & 0 \\ 0 & X_4^* \end{pmatrix} U^H,$$

where $[X_1^*, X_4^*]$ is a solution pair of Problem 2.1.

Remark 3.3. *By applying Method 2 with any arbitrary initial matrix pair $[X_2^1, X_3^1]$, an anti-reflexive solution of Problem 1.3 can be obtained with finite iteration steps in the absence of roundoff errors by the following form*

$$X^* = U \begin{pmatrix} 0 & X_2^* \\ X_3^* & 0 \end{pmatrix} U^H,$$

where $[X_2^*, X_3^*]$ is a solution pair of Problem 2.3.

Lemma 3.5. [22] *Assume that the consistent system of linear equations $Ay = b$ has a solution $y^* \in R(A^H)$. Then y^* is an unique least Frobenius norm solution of the system of linear equations.*

Theorem 3.3. *Suppose that Problem 2.1 is consistent and the initial matrix pair $[X_1^1, X_4^1]$ is considered as*

$$X_1^1 = A_1^H N B_1^H + D_1^H \widehat{N} E_1^H, \; X_4^1 = A_2^H N B_2^H + D_2^H \widehat{N} E_2^H, \tag{3.16}$$

where N and $\widehat{N}$ are arbitrary, or especially, $X_1^1 = 0$ and $X_4^1 = 0$. Then the solution pair $[X_1^*, X_4^*]$ generated by Method 1 is the least Frobenius norm solution pair of the coupled matrix equations (2.7).

Proof. It is obvious that the coupled matrix equations (2.7) are equivalent to the matrix equation

$$\begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 \\ E_1^T \otimes D_1 & E_2^T \otimes D_2 \end{pmatrix} \begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_4) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}(C) \\ \mathrm{vec}(F) \end{pmatrix}.$$

Now let N and $\widehat{N}$ be arbitrary matrices. We can write

$$\begin{aligned} \begin{pmatrix} \mathrm{vec}(A_1^H N B_1^H + D_1^H \widehat{N} E_1^H) \\ \mathrm{vec}(A_2^H N B_2^H + D_2^H \widehat{N} E_2^H) \end{pmatrix} &= \begin{pmatrix} \overline{B_1} \otimes A_1^H & \overline{E_1} \otimes D_1^H \\ \overline{B_2} \otimes A_2^H & \overline{E_2} \otimes D_2^H \end{pmatrix} \begin{pmatrix} \mathrm{vec}(N) \\ \mathrm{vec}(\widehat{N}) \end{pmatrix} \\ &= \begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 \\ E_1^T \otimes D_1 & E_2^T \otimes D_2 \end{pmatrix}^H \begin{pmatrix} \mathrm{vec}(N) \\ \mathrm{vec}(\widehat{N}) \end{pmatrix} \\ &\in R\Big(\begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 \\ E_1^T \otimes D_1 & E_2^T \otimes D_2 \end{pmatrix}^H \Big). \end{aligned}$$

Therefore, if we consider

$$X_1^1 = A_1^H N B_1^H + D_1^H \widehat{N} E_1^H \text{ and } X_4^1 = A_2^H N B_2^H + D_2^H \widehat{N} E_2^H,$$

then X_1^k and X_4^k, generated by Method 1 satisfy

$$\begin{pmatrix} \mathrm{vec}(X_1^k) \\ \mathrm{vec}(X_4^k) \end{pmatrix} \in R\Big(\begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 \\ E_1^T \otimes D_1 & E_2^T \otimes D_2 \end{pmatrix}^H \Big).$$

According to these results, with the initial matrices $X_1^1 = A_1^H N B_1^H + D_1^H \widehat{N} E_1^H$ and $X_4^1 = A_2^H N B_2^H + D_2^H \widehat{N} E_2^H$ where N and $\widehat{N}$ are arbitrary, or especially, $X_1^1 = 0$ and $X_4^1 = 0$, the solution pair $[X_1^*, X_4^*]$, generated by Method 1, is the least Frobenius norm solution pair of the coupled matrix equations (2.7). $\square$

Similar to the proof of Theorem 3.3, we can prove the following theorem.

Theorem 3.4. *Suppose that Problem 2.3 is consistent and the initial matrix pair* $[X_2^1, X_3^1]$ *is considered as*

$$X_2^1 = A_1^H N B_2^H + D_1^H \widehat{N} E_2^H, \ X_3^1 = A_2^H N B_1^H + D_2^H \widehat{N} E_1^H, \tag{3.17}$$

where N *and* $\widehat{N}$ *are arbitrary, or especially,* $X_2^1 = 0$ *and* $X_3^1 = 0$. *Then the solution pair* $[X_2^*, X_3^*]$ *generated by Method 2 is the least Frobenius norm solution pair of the coupled matrix equations* (2.9).

Remark 3.4. *Assume that Problem 1.1 is consistent. Then from Theorems 2.1 and 3.3 we conclude that the reflexive solution* X^* *as follows*

$$X^* = U \begin{pmatrix} X_1^* & 0 \\ 0 & X_4^* \end{pmatrix} U^H, \tag{3.18}$$

is the least Frobenius norm reflexive solution of the pair of matrix equations (1.2) *where* X_1^* *and* X_4^* *are generated by Method 1 with the initial matrices* $X_1^1 = A_1^H N B_1^H + D_1^H \widehat{N} E_1^H$ *and* $X_4^1 = A_2^H N B_2^H + D_2^H \widehat{N} E_2^H$ *where* N *and* $\widehat{N}$ *are arbitrary, or especially,* $X_1^1 = 0$ *and* $X_4^1 = 0$.

Remark 3.5. *Assume that Problem 1.3 is consistent. Then from Theorems 2.2 and 3.4 we conclude that the anti-reflexive solution* X^* *as follows*

$$X^* = U \begin{pmatrix} 0 & X_2^* \\ X_3^* & 0 \end{pmatrix} U^H, \tag{3.19}$$

is the least Frobenius norm anti-reflexive solution of the pair of matrix equations (1.2) *where* X_2^* *and* X_3^* *are generated by Method 2 with the initial matrices* $X_2^1 = A_1^H N B_2^H + D_1^H \widehat{N} E_2^H$ *and* $X_3^1 = A_2^H N B_1^H + D_2^H \widehat{N} E_1^H$ *where* N *and* $\widehat{N}$ *are arbitrary, or especially,* $X_2^1 = 0$ *and* $X_3^1 = 0$.

3.1. **Solutions of Problems 2.2 and 1.2.** Here we will study Problems 2.2 and 1.2 as follows.

When Problem 2.1 is consistent, its solution pair set S_1 is nonempty. For a given matrix pair $[\widehat{X}_1, \widehat{X}_4]$ with $\widehat{X}_1 \in \mathrm{C}^{r\times r}$, $\widehat{X}_4 \in \mathrm{C}^{(n-r)\times(n-r)}$ we can obtain

$$\left\{ \begin{array}{l} A_1X_1B_1 + A_2X_4B_2 = C, \\ D_1X_1E_1 + D_2X_4E_2 = F, \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} A_1(X_1 - \widehat{X}_1)B_1 + A_2(X_4 - \widehat{X}_4)B_2 = C - A_1\widehat{X}_1B_1 - A_2\widehat{X}_4B_2, \\ D_1(X_1 - \widehat{X}_1)E_1 + D_2(X_4 - \widehat{X}_4)E_2 = F - D_1\widehat{X}_1E_1 - D_2\widehat{X}_4E_2. \end{array} \right.$$

Now let $Y_1 = X_1 - \widehat{X}_1$, $Y_4 = X_4 - \widehat{X}_4$, $\widehat{C} = C - A_1\widehat{X}_1B_1 - A_2\widehat{X}_4B_2$ and $\widehat{F} = F - D_1\widehat{X}_1E_1 - D_2\widehat{X}_4E_2$. Then the matrix nearness Problem 2.2 is equivalent to finding the least Frobenius norm solution pair $[Y_1^*, Y_4^*]$ of the matrix equations

$$A_1Y_1B_1 + A_2Y_4B_2 = \widehat{C} \text{ and } D_1Y_1E_1 + D_2Y_4E_2 = \widehat{F}, \tag{3.20}$$

which can be obtained by using Method 1 with the initial matrices $Y_1^1 = A_1^H N B_1^H + D_1^H \widehat{N} E_1^H$ and $Y_4^1 = A_2^H N B_2^H + D_2^H \widehat{N} E_2^H$ where N and $\widehat{N}$ are arbitrary, or especially, $Y_1^1 = 0$ and $Y_4^1 = 0$. Also the solution pair of the matrix nearness Problem 2.2 can be represented as $\widetilde{X}_1 = Y_1^* + \widehat{X}_1$ and $\widetilde{X}_4 = Y_4^* + \widehat{X}_4$.

Now suppose that Problem 1.1 is consistent. Therefore the solution set S_r of matrix equation (1.2) is nonempty. For given reflexive matrix $\widetilde{X}$, we can write

$$\widehat{X} = U \begin{pmatrix} \widehat{X}_1 & 0 \\ 0 & \widehat{X}_4 \end{pmatrix} U^H. \tag{3.21}$$

From the discussion above, the solutions of the matrix nearness Problem 1.2 can be represented as

$$\widetilde{X} = U \begin{pmatrix} \widetilde{X}_1 & 0 \\ 0 & \widetilde{X}_4 \end{pmatrix} U^H, \tag{3.22}$$

where matrix pair $[\widetilde{X}_1, \widetilde{X}_4]$ is a solution pair of Problem 2.2.

3.2. **Solutions of Problems 2.4 and 1.4.** Let Problem 2.3 be consistent, therefore its solution pair set S_a is nonempty. For given matrix pair $[\widehat{X}_2, \widehat{X}_3]$ with $\widehat{X}_2 \in C^{r\times(n-r)}$, $\widehat{X}_3 \in C^{(n-r)\times r}$ we can obtain

$$\left\{ \begin{array}{l} A_1X_2B_2 + A_2X_3B_1 = C, \\ D_1X_2E_2 + D_2X_3E_1 = F, \end{array} \right.$$

$$\Leftrightarrow \begin{cases} A_1(X_2-\widehat{X}_2)B_2 + A_2(X_3-\widehat{X}_3)B_1 = C - A_1\widehat{X}_2B_2 - A_2\widehat{X}_3B_1, \\ D_1(X_2-\widehat{X}_2)E_2 + D_2(X_3-\widehat{X}_3)E_1 = F - D_1\widehat{X}_2E_2 - D_2\widehat{X}_3E_1. \end{cases}$$

Now let $Y_2 = X_2 - \widehat{X}_2$, $Y_3 = X_3 - \widehat{X}_3$, $\widehat{C} = C - A_1\widehat{X}_2B_2 - A_2\widehat{X}_3B_1$ and $\widehat{F} = F - D_1\widehat{X}_2E_2 - D_2\widehat{X}_3E_1$. Then the matrix nearness Problem 2.4 is equivalent to finding the least Frobenius norm solution pair $[Y_2^*, Y_3^*]$ of the matrix equations

$$A_1Y_2B_2 + A_2Y_3B_1 = \widehat{C} \text{ and } D_1Y_2E_2 + D_2Y_3E_1 = \widehat{F}, \tag{3.23}$$

which can be obtained by using Method 2 with the initial matrices $Y_2^1 = A_1^H NB_2^H + D_1^H \widehat{N} E_2^H$ and $Y_3^1 = A_2^H NB_1^H + D_2^H \widehat{N} E_1^H$ where N and $\widehat{N}$ are arbitrary, or especially, $Y_2^1 = 0$ and $Y_3^1 = 0$. Also the solution pair of the matrix nearness Problem 2.4 can be represented as $\widetilde{X}_2 = Y_2^* + \widehat{X}_2$ and $\widetilde{X}_3 = Y_3^* + \widehat{X}_3$.

Now suppose that Problem 1.3 is consistent. Therefore the solution set S_a of matrix equation (1.2) is nonempty. For given anti-reflexive matrix $\widetilde{X}$, we can write

$$\widehat{X} = U \begin{pmatrix} 0 & \widehat{X}_2 \\ \widehat{X}_3 & 0 \end{pmatrix} U^H. \tag{3.24}$$

By using the above results, the solutions of the matrix nearness Problem 1.4 can be represented as

$$\widetilde{X} = U \begin{pmatrix} 0 & \widetilde{X}_2 \\ \widetilde{X}_3 & 0 \end{pmatrix} U^H, \tag{3.25}$$

where matrix pair $[\widetilde{X}_2, \widetilde{X}_3]$ is a solution pair of Problem 2.4.

4. **Numerical results**

In this section, we give a numerical example to illustrate the performance of the proposed algorithms in Section 2. Computations were done on a PC Pentium IV using MATLAB 7. Because of the influence of the error of calculation, we consider the arbitrary matrix Z as a zero matrix if $||Z|| < 10^{-13}$.

Example 4.1. Consider the pair of matrix equations (1.2) with the parameters

$$A = \begin{pmatrix} 1+i & 3 & 1 & 2 \\ -1 & -1-2i & -1 & 2 \\ i & 4 & 5 & 3 \\ 1 & 2-3i & -4 & -4i \end{pmatrix}, \; B = \begin{pmatrix} i & -1-2i & -1 & 1 \\ 2 & 1 & 2i & 1 \\ 4+i & 3 & 2 & 1 \\ 1 & 4 & -5 & 4 \end{pmatrix},$$

$$D = \begin{pmatrix} 2+2i & -2i & -2 & 1 \\ 3i & 4 & 2 & 1 \\ i & 1 & 2+i & -5 \\ 6 & 5+i & 4 & 5 \end{pmatrix}, \; E = \begin{pmatrix} 3+i & 4 & 2+i & 1 \\ -2i & -4i & -3 & 2 \\ 1 & 3+i & 4 & 6 \\ 1 & 3 & 4+2i & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} 6+6i & 19+19i & -26+4i & 23+19i \\ -23-3i & 12+20i & -37-29i & 20+12i \\ -5+7i & -7+32i & -2-i & 1+26i \\ 33+3i & 82-38i & -51+85i & 74-50i \end{pmatrix},$$

$$F = 10^2 \begin{pmatrix} -0.03+0.11i & 0.06+0.15i & 0.06+0.40i & 0.06+0.10i \\ -0.11-0.15i & -0.14-0.29i & -0.61+0.07i & 0.07+0.17i \\ -0.11-0.21i & -0.22-0.49i & -0.24-1.08i & 0.27-0.51i \\ 0.05-0.17i & 0.02-0.53i & -1.10+0.98i & 0.16+0.52i \end{pmatrix}.$$

It can be verified that the matrix equations have the reflexive solution X as

$$X = \begin{pmatrix} 1+i & 2 & 0 & 0 \\ 0 & 3-i & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2+i & 3+2i \end{pmatrix} \in \mathrm{C}_r^{4\times 4}(P),$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The matrix P can be expressed as

$$P = U \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} U^H,$$

where $U = U^H = I_4$. Therefore the matrix equations are equivalent to

$$A_1X_1B_1 + A_2X_4B_2 = C \text{ and } D_1X_1E_1 + D_2X_4E_2 = F,$$

where

$$A_1 = \begin{pmatrix} 1+i & 3 \\ -1 & -1-2i \\ i & 4 \\ 1 & 2-3i \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 2 \\ -1 & 2 \\ 5 & 3 \\ -4 & -4i \end{pmatrix},$$

$$B_1 = \begin{pmatrix} i & -1-2i & -1 & 1 \\ 2 & 1 & 2i & 1 \end{pmatrix},\ B_2 = \begin{pmatrix} 4+i & 3 & 2 & 1 \\ 1 & 4 & -5 & 4 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 2+2i & -2i \\ 3i & 4 \\ i & 1 \\ 6 & 5+i \end{pmatrix},\ D_2 = \begin{pmatrix} -2 & 1 \\ 2 & 1 \\ 2+i & -5 \\ 4 & 5 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 3+i & 4 & 2+i & 1 \\ -2i & -4i & -3 & 2 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & 3+i & 4 & 6 \\ 1 & 3 & 4+2i & 2 \end{pmatrix}.$$

By using Method 1 with the initial matrix pair $[X_1^1, X_4^1] = [0, 0]$, we obtain solutions

$$X_1^{12} = \begin{pmatrix} 1+i & 2 \\ 0 & 3-i \end{pmatrix},\ X_4^{12} = \begin{pmatrix} 0 & -2 \\ -2+i & 3+2i \end{pmatrix}.$$

The obtained results are presented in Figure 1, where

$$r_1^k = \left\| \begin{pmatrix} C - A_1X_1^kB_1 - A_2X_4^kB_2 & 0 \\ 0 & F - D_1X_1^kE_1 - D_2X_4^kE_2 \end{pmatrix} \right\|.$$

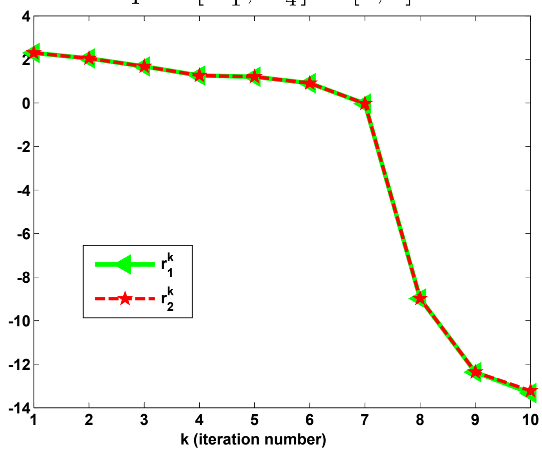
Now let sequence X^k be

$$X^k = \begin{pmatrix} X_1^k & 0 \\ 0 & X_4^k \end{pmatrix}. \tag{4.1}$$

Also in Figure 1, we display

$$r_2^k = \left\| \begin{pmatrix} C - AX^kB & 0 \\ 0 & F - DX^kE \end{pmatrix} \right\|,$$

FIGURE 1. The residuals for Example 4.1 with the initial matrix pair $[X_1^1, X_4^1] = [0, 0]$.



where X^k and X_1^k, X_4^k are obtained from (4.2) and Method 1, respectively.

Here we assume that

$$\widehat{X} = \begin{pmatrix} 2-i & 2 & 0 & 0 \\ -1+2i & -1 & 0 & 0 \\ 0 & 0 & 1+i & 1 \\ 0 & 0 & 3+i & 3+i \end{pmatrix}.$$

Also we suppose that

$$\widehat{X}_1 = \begin{pmatrix} 2-i & 2 \\ -1+2i & -1 \end{pmatrix}, \ \widehat{X}_4 = \begin{pmatrix} 1+i & 1 \\ 3+i & 3+i \end{pmatrix}.$$

By computing $C - A_1\widehat{X}_1B_1 - A_2\widehat{X}_4B_2$ and $F - D_1\widehat{X}_1E_1 - D_2\widehat{X}_4E_2$, we can obtain the solution pair $[\widetilde{X}_1, \widetilde{X}_4]$ of Problem 2.2 by finding the least norm solution pair $[Y_1, Y_4]$ of (3.20). By applying Method 1 with the initial matrix pair $[Y_1^1, Y_4^1] = [0, 0]$ for the matrix equations (3.20), we have

$$X_1^* = X_1^{11} = \begin{pmatrix} -1.0000 + 2.0000i & -0.0000 - 0.0000i \\ 1.0000 - 2.0000i & 4.0000 - 1.0000i \end{pmatrix},$$

$$X_4^* = X_4^{11} = \begin{pmatrix} -1.0000 - 1.0000i & -3.0000 + 0.0000i \\ -5.0000 + 0.0000i & -0.0000 + 1.0000i \end{pmatrix}.$$

Now we can obtain $[\widetilde{X}_1, \widetilde{X}_4]$ as follows:

$$\widetilde{X}_1 = Y_1^* + \widehat{X}_1 = \begin{pmatrix} 1.0000 + 1.0000i & 2.0000 - 0.0000i \\ 0.0000 + 0.0000i & 3.0000 - 1.0000i \end{pmatrix},$$

$$\widetilde{X}_4 = Y_4^* + \widehat{X}_4 = \begin{pmatrix} 0.0000 + 0.0000i & -2.0000 + 0.0000i \\ -2.0000 + 1.0000i & 3.0000 + 2.0000i \end{pmatrix}.$$

The obtained results are presented in Figure 2, where

$$r_1^k = \left\| \begin{pmatrix} C - A_1Y_1^kB_1 - A_2Y_4^kB_2 & 0 \\ 0 & F - D_1Y_1^kE_1 - D_2Y_4^kE_2 \end{pmatrix} \right\|,$$

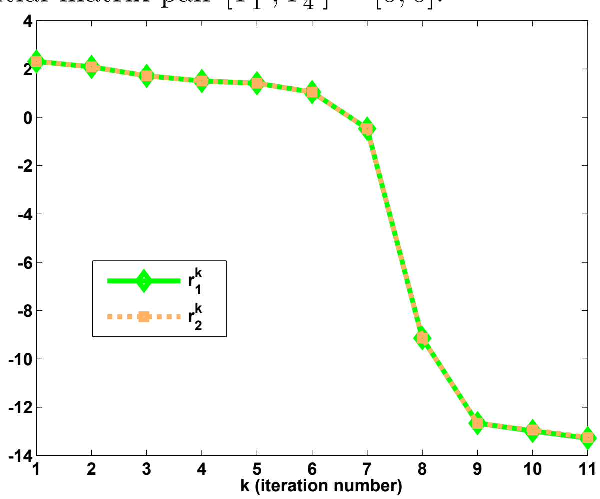
and

$$r_2^k = \left\| \begin{pmatrix} C - AY^kB & 0 \\ 0 & F - DY^kE \end{pmatrix} \right\|,$$

with

$$Y^k = \begin{pmatrix} Y_1^k & 0 \\ 0 & Y_4^k \end{pmatrix}. \tag{4.2}$$

FIGURE 2. The residuals for Example 4.1 with the initial matrix pair $[Y_1^1, Y_4^1] = [0, 0]$.



5. **Conclusions**

In this work, two iterative methods were introduced for solving the pair of matrix equations $AXB = C$ and $DXE = F$ with unknown matrix X. First two matrix equations equivalent to the matrix equations $AXB = C$ and $DXE = F$ over reflexive and anti-reflexive matrices, were given, respectively. Then for solving these matrix equations, we have introduced two iterative methods. Moreover we obtained the convergence properties of two iterative methods. Finally, a numerical example was given to show the efficiency of the presented results.

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