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# BILATERAL COMPOSITION OPERATORS ON VECTOR-VALUED HARDY SPACES

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Abstract. Let $T$ be a bounded operator on the Banach space $X$ and $\varphi$ be an analytic self-map of the unit disk $\mathbb{D}$. We investigate some operator theoretic properties of bilateral composition operator $C_{\varphi,T} : f \to T \circ f \circ \varphi$ on the vector-valued Hardy space $H^p(X)$ for $1 \leq p \leq +\infty$. Compactness and weak compactness of $C_{\varphi,T}$ on $H^p(X)$ are characterized and when $p = 2$, a concrete formula for its adjoint is given.
**Keywords:** Vector-valued Hardy space, composition operator, linear fractional map, weak compactness.
**MSC(2010):** Primary: 47B33; Secondary: 47B48.

## 1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk of the complex plane $\mathbb{C}$ and $X = (X; \|.\|)$ be any complex Banach space. A function $f : \mathbb{D} \to X$ is called analytic if it is weakly analytic, i.e., if $x^* \circ f$ is analytic for all functionals $x^* \in X^*$, where $X^*$ is the dual space of $X$. For every analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$, the (right) composition operator $C_\varphi$ is defined by $C_\varphi(f) = f \circ \varphi$ where $f : \mathbb{D} \to X$ is an analytic function. Basic properties of such operators have been studied on many classical Banach spaces such as Hardy spaces [3, 8], Bergman and Bloch spaces, and BMOA [1, 2, 8, 10]. Recently, these studies have been extended by considering weak compactness of composition operators on spaces of analytic $X$-valued functions, where $X$ is an arbitrary complex Banach space [4, 5, 6].

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The purpose of this paper is to initiate the study of special types of composition operators on some Banach spaces of analytic vector-valued functions. For every analytic function $f : \mathbb{D} \to X$ and a bounded operator $T$ on $X$, the composition $T \circ f$ defines an analytic function from $\mathbb{D}$ into $X$. This introduces two new operators; the left composition operator $C_T$ by $C_T(f) = T \circ f$ and the bilateral composition operator $C_{\varphi,T}$ by

$$C_{\varphi,T}(f) = T \circ f \circ \varphi$$

which is a composition of the (right) composition operator $C_\varphi$ and the left composition operator $C_T$. Note that when $X$ is a complex plane, $C_T = \lambda I$ and $C_{\varphi,T} = \lambda C_\varphi$ for some complex scalar $\lambda$, where $I$ is the identity operator.

Since Hardy spaces are closely related to other function spaces of our interest, it will be useful to investigate some fundamental properties related to these operators on them.

**Definition 1.1.** *Let $X$ be a normed space. The vector-valued Hardy space $H^p(X)$ for $1 \leq p < +\infty$, is the class of analytic functions $f : \mathbb{D} \to X$ satisfying*

$$\|f\|_{H^p(X)} := \lim_{r\to 1}\left(\int_{t\in\mathbb{T}} \|f(rt)\|_X^p dm(t)\right)^{\frac{1}{p}} < +\infty,$$

*where $m$ is the Lebesgue measure on the unite circle $\mathbb{T} = \partial\mathbb{D}$ normalized so that $m(\mathbb{T}) = 1$. The space $H^\infty(X)$ is defined as the vector space of bounded analytic functions $f : \mathbb{D} \to X$, with the norm*

$$\|f\|_{H^\infty(X)} := \sup_{z\in\mathbb{D}} \|f(z)\|_X < +\infty.$$

*For $1 \leq p \leq +\infty$, $H^p(X)$ becomes a Banach space with norm $\|.\|_p$. When $X$ is a Hilbert space, $H^2(X)$ becomes a Hilbert space with the following inner product:*

$$\langle f, g\rangle_{H^2(X)} = \lim_{r\to 1}\left(\int_{t\in\mathbb{T}} \langle f(rt), g(rt)\rangle\right) dm.$$

*In particular, when $X = \mathbb{C}$ the space $H^p(\mathbb{C})$, is exactly the classical Hardy space $H^p$.*

Basic structures of vector-valued Hardy spaces will be studied in Section 2.

Some of the main properties of composition operators on such spaces have been considered by several authors. The boundedness of $C_\varphi$ on $H^p(X)$ for $1 \leq p \leq +\infty$ was proved by Saksman and Tylli [4]. In fact, it follows from the Littlewood subordination theorem in a well known manner, a suitable change of variables and subharmonicity of the function $z \to \|f(z)\|_X$ (see [4], Theorem 3.1). Applying the reproducing kernels argument, the Hilbert adjoint of composition operator $C_\varphi$ on general Hilbert spaces of analytic functions is computed by Martin and Vukotic, [7]. For an infinite dimensional Banach space $X$, it is shown that $C_\varphi$ is never compact on $H^p(X)$ for $1 < p \leq +\infty$, however it is weakly compact on $H^1(X)$ if and only if $X$ is a reflexive Banach space and the Nevanlinna counting function $N_\varphi$ satisfies the Shapiro condition [4].

Here boundedness, compactness and weak compactness of the operator $C_{\varphi,T}$ on the vector-valued Hardy spaces $H^p(X)$ with $1 \leq p \leq +\infty$, are obtained. In Section 3, we show that $C_{\varphi,T}$ is bounded on $H^p(X)$ for $1 \leq p \leq +\infty$, and moreover

$$\|T\|\|C_\varphi\|_{H^p} \leq \|C_{\varphi,T}\|_{H^p(X)} \leq \|C_\varphi\|_{H^p(X)}\|T\|.$$

We give a formula in terms of Bochner integral for the Hilbert adjoint of $C_{\varphi,T}$ on $H^2(X)$ where $X$ is a Hilbert space. Compactness and weak compactness of $C_{\varphi,T}$ will be considered in Section 4. In fact, we show $C_{\varphi,T}$ is compact on $H^p(X)$ if and only if $T$ is compact and either $T = 0$ or $C_\varphi$ is compact on $H^p$. A similar result will be obtained for weak compactness of $C_{\varphi,T}$ on $H^1(X)$ and $H^\infty(X)$.

## 2. **Vector-valued Hardy space**

In this preliminary section we consider some well known basic properties of vector-valued Hardy spaces (see [9]). Throughout this section $\varphi$ is an analytic self-map of $\mathbb{D}$, $X$ is a Banach space and $T$ is a bounded linear operator on $X$.

The first observation characterizes the structure of $H^p(X)$ in some cases.

**Proposition 2.1.** *Let $X$ be a normed space then*

(1) *If dim*$X < +\infty$, *then* $H^p(X)$ *for* $1 \leq p < +\infty$ *is isometrically isomorphic to* $\bigoplus_{k=1}^{n} H^p$ *for some integer* $n \geq 1$.

(2) *If* $X$ *is a separable Hilbert space then* $H^2(X)$ *is isometrically isomorphic to* $(\bigoplus_{i\in\mathbb{N}} H^2)_{\ell^2}$.

*Proof.* (1) If $\dim(X) < +\infty$, without loss of generality suppose that $X = \mathbb{C}^n$ for some integer $n \geq 1$ with the $p$-norm topology. Then every $X$-valued analytic map $f : \mathbb{D} \to X$ in $H^p(X)$ can be written as $f = (f_1, f_2, \ldots, f_n)$ where $f_i \in H^p$ for $1 \leq i \leq n$. Now the operator $\Lambda(f) = \{f_i\}_{1\leq i\leq n}$ defines an isometrically isomorphism between $H^p(X)$ and $\bigoplus_{k=1}^{n} H^p$.

(2) Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis for the Hilbert space $X$. For any $X$-valued analytic function $f \in H^2(X)$ and vector $x \in X$, the analytic function $x \otimes f$ by $(x \otimes f)(z) = \langle f(z), x\rangle$ belongs to $H^2$. Put $\mathcal{H} = (\bigoplus_{i\in\mathbb{N}} H^2)_{\ell^2}$ and define $\Lambda : H^2(X) \to \mathcal{H}$ by $\Lambda(f) = \{e_i \otimes f\}_{i\in\mathbb{N}}$. Then the Lebesgue Dominated Converges Theorem with an application of Fubini Theorem implies that

$$\begin{aligned}
\|\Lambda(f)\|_{\mathcal{H}}^2 &= \Big\|\{e_i \otimes f\}_{i\in\mathbb{N}}\Big\|_{\mathcal{H}} \\
&= \sum_{i=1}^{+\infty} \big|e_i \otimes f\big|_{H^2}^2 \\
&= \sum_{i=1}^{+\infty} \lim_{r\to 1}\left(\int_{t\in\mathbb{T}} \big|(e_i \otimes f)(rt)\big|^2\right) dm(t) \\
&= \lim_{r\to 1}\left(\int_{t\in\mathbb{T}}\left(\sum_{i=1}^{+\infty} \big| < f(rt), e_i > \big|^2 dm(t)\right)\right) \\
&= \lim_{r\to 1}\left(\int_{t\in\mathbb{T}} \big|f(rt)\big|^2 dm(t)\right) = \|f\|_{H^2(X)}^2.
\end{aligned}$$

It remains to prove $\Lambda$ is a surjection. Let $g = \{f_i\}_i \in \mathcal{H}$. Then the Parseval identity

$$f(t) = \sum_{i=1}^{+\infty} f_i(t)e_i$$

defines an analytic function in $H^2(X)$ and $\Lambda(f) = g$. Now the proof is completed. $\square$

**Corollary 2.2.** *Let* $\{e_i : i \in \mathbb{N}\}$ *be an orthonormal basis for the Hilbert space* $X$ *and* $e(i,j)(z) = z^j e_i$ *for* $i \in \mathbb{N}$ *and* $j \in \mathbb{N} \cup \{0\}$. *Then*

(1) $\mathcal{B} := \Big\{e(i,j) : (i,j) \in \mathbb{N} \times \mathbb{N} \cup \{0\}\Big\}$ *is an orthonormal basis for* $H^2(X)$.

(2) *For any* $i \in \mathbb{N}$ *and* $w \in \mathbb{D}$, *the mapping*

$$K_{i,w}(z) = e_i \sum_{n=0}^{+\infty} \bar{w}^n z^n = \frac{e_i}{1 - \bar{w}z}$$

*belongs to* $H^2(X)$, $\|K_{i,w}\|^2 = \sum_{n=0}^{+\infty} |w|^{2n} = \frac{1}{1-|w|^2}$ *and for any* $f \in H^2(X)$,

$$\langle f, K_{i,w}\rangle_{H^2(X)} = (e_i \otimes f)(w).$$

*Proof.* (1) For $i \in \mathbb{N}$ and $j \in \mathbb{N} \cup \{0\}$ define $E_{i,j}(\alpha) = 0$ if $\alpha \neq i$ and $E_{i,j}(\alpha) = z^j$ if $\alpha = i$. Then $E_{i,j} \in \mathcal{H}$, $e(i,j) = z^j e_i = \Lambda^{-1} E_{i,j} \in H^2(X)$ and since

$$\Big\{E_{i,j} : (i,j) \in \mathbb{N} \times \mathbb{N} \cup \{0\}\Big\}$$

is an orthonormal basis for $\mathcal{H}$, so is $\mathcal{B}$ for $H^2(X)$.

(2) Let $k_w(z) := \sum_{n=0}^{+\infty} \overline{w}^n z^n$. For $i \in \mathbb{N}$, define $K_i(\alpha) = 0$ if $\alpha \neq i$ and $K_i(\alpha) = k_w$ if $\alpha = i$. Then $K_i \in \mathcal{H}$ and we can easily see that $K_{i,w} = \Lambda^{-1} K_i$. On the other hand,

$$\begin{aligned} < f, K_{i,w} >_{H^2(X)} &= < \Lambda^{-1}(\{e_i \otimes f\}_{i\in\mathbb{N}}), \Lambda^{-1} K_i >_{H^2(X)} \\ &= < \{e_i \otimes f\}_{i\in\mathbb{N}}, K_i >_{\mathcal{H}} \\ &= < e_i \otimes f, k_w >_{H^2} = (e_i \otimes f)(w) \end{aligned}$$

for every $i \in \mathbb{N}$ and $w \in \mathbb{D}$ and this completes the proof. □

## 3. Norm and adjoint

In this section the norm of $C_{\varphi,T}$ on vector-valued Hardy space $H^p(X)$ for $1 \leq p \leq +\infty$, is estimated and then its Hilbert adjoint on $H^2(X)$, where $X$ is a Hilbert space, is computed.

**Theorem 3.1.** *Let* $X$ *be a Banach space and* $\varphi$ *be an analytic self-map of* $\mathbb{D}$. *Then the operator* $C_{\varphi,T}$ *is bounded on* $H^p(X)$ *for* $1 \leq p \leq +\infty$.

*Moreover,*

$$\|T\|\|C_\varphi\|_{H^p} \leq \|C_{\varphi,T}\|_{H^p(X)} \leq \|T\|\|C_\varphi\|_{H^p(X)} \tag{3.1}$$

*and in particular,* $\|C_T\|_{H^p(X)} = \|T\|$.

*Proof.* First note that by proposition 1 in [4], the composition operator $C_\varphi$ is bounded on $H^p(X)$ for $1 \leq p \leq +\infty$. Let $1 \leq p < +\infty$, then for $f \in H^p(X)$ and $z \in \mathbb{D}$, $\|T(f(z))\|_X \leq \|T\|\|f(z)\|_X$ and so

$$\begin{aligned} \|C_{\varphi,T}(f)\|_{H^p(X)} &= \|T \circ f \circ \varphi\|_{H^p(X)} \\ &= \sup_{0<r<1} \left( \int_{t \in \mathbb{T}} \|T(f(\varphi(rt)))\|_X^p dm(t) \right)^{\frac{1}{p}} \\ &\leq \|T\|\|C_\varphi(f)\|_{H^p(X)} = \|T\|\|C_\varphi\|_{H^p(X)}\|f\|_{H^p(X)}. \end{aligned}$$

Hence

$$\|C_{\varphi,T}\|_{H^p(X)} \leq \|T\|\|C_\varphi\|_{H^p(X)}.$$

and so $C_{\varphi,T}$ is bounded on $H^p(X)$. On the other hand, for non-zero elements $f \in H^p$ and $x \in X$, consider the analytic vector-valued function $x * f : \mathbb{D} \to X$ by $(x * f)(z) = xf(z)$ for any $z \in \mathbb{D}$. Then $x * f \in H^p(X)$,

$$\|x * f\|_{H^p(X)} = \|x\|\|f\|_{H^p},$$

and

$$\begin{aligned} \|C_{\varphi,T}(x * f)\|^p &= \sup_{0<r<1} \left( \int_{t \in \mathbb{T}} \|T(x * f)(\varphi(rt))\|^p dm(t) \right) \\ &= \|Tx\|^p \sup_{0<r<1} \left( \int_{t \in \mathbb{T}} |(f \circ \varphi(rt))|^p dm(t) \right) \\ &= \|Tx\|^p \|C_\varphi(f)\|_{H^p}^p. \end{aligned}$$

Hence

$$\begin{aligned} \|Tx\|^p \|C_\varphi(f)\|_{H^p}^p &= \|C_{\varphi,T}(x * f)\|^p \\ &\leq \|C_{\varphi,T}\|^p \|x * f\|_{H^p(X)}^p = \|C_{\varphi,T}\|^p \|x\|^p \|f\|_{H^p}^p. \end{aligned}$$

and so

$$\frac{\|Tx\|}{\|x\|} \frac{\|C_\varphi(f)\|_{H^p}}{\|f\|_{H^p}} \leq \|C_{\varphi,T}\|.$$

Taking the supremum of the left hand side, we get

$$\|T\|\|C_\varphi\|_{H^p} \leq \|C_{\varphi,T}\|.$$

The proof for $H^\infty(X)$ is entirely similar. Letting $\varphi(z) = z$ in (3.1), we see that $\|C_T\|_{H^p(X)} = \|T\|$ and the proof is completed. □

Let $\varphi$ be any analytic self-map of the unite disc. Then for the scalar-valued composition operator $C_\varphi$ an adjoint formula exists and has interesting applications [7]. In fact, let $\mathcal{H}$ be any Hilbert space of analytic functions in $\mathbb{D}$ on which all point-evaluations are bounded. Assume that $C_\varphi$ is well defined and bounded on $\mathcal{H}$. Then $\mathcal{H}$ possesses a family of reproducing kernels $\{K_w : w \in \mathbb{D}\} \subseteq \mathcal{H}$; that is, $K_w(f) = f(w)$ for every $f \in \mathcal{H}$. Then for each $w \in \mathbb{D}$ and all $f \in \mathcal{H}$ an adjoint formula for $C_\varphi$ is:

$$C^*_\varphi(f)(w) =< f, K_w \circ f > .$$

In particular, for the Hardy space $H^2$,

$$C^*_\varphi f(w) = \lim_{r\to 1} \int_{t\in\mathbb{T}} \frac{f(rt)}{1 - w\bar{\varphi}(rt)} dm(t) \quad (f \in H^2).$$

Here we obtain a similar formula for the adjoint of $C_{\varphi,T}$ containing a formula for the adjoint of $C_\varphi$ on $H^2(X)$.

**Theorem 3.2.** *Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and let $T$ be a bounded linear operator on a separable Hilbert space $X$. Then the Hilbert adjoint of $C_{\varphi,T}$ on Hilbert space $H^2(X)$ is verified by the following formula:*

$$< C^*_{\varphi,T} f(w), x >_X = \lim_{r\to 1} \int_{t\in\mathbb{D}} \frac{(Tx \otimes f)(rt)}{1 - w\bar{\varphi}(rt)} dm(t) = C^*_\varphi(Tx \otimes f)(w) \tag{3.2}$$

*where $C^*_\varphi$ is the adjoint of $C_\varphi$ on the usual Hardy space $H^2$.*

*Proof.* Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis for $X$. Fix $i \in \mathbb{N}$ and $w \in \mathbb{D}$, then by Corollary 2.2 we have

$$\begin{aligned} \left\langle C^*_{\varphi,T} f(w), e_i \right\rangle_X &= \left\langle C^*_{\varphi,T} f, K_{i,w} \right\rangle_{H^2(X)} \\ &= \left\langle f, C_{\varphi,T} K_{i,w} \right\rangle_{H^2(X)} \\ &= \left\langle f, T \circ K_{i,w} \circ \varphi \right\rangle_{H^2(X)}. \end{aligned}$$

Hence

$$\begin{aligned}
\left\langle C_{\varphi,T}^{*} f(w), e_i \right\rangle_X &= \lim_{r\to 1} \int_{t\in\mathbb{T}} \left\langle f(rt), T(K_{i,w}(\varphi(rt))) \right\rangle_X dm(t) \\
&= \lim_{r\to 1} \int_{t\in\mathbb{T}} \left\langle T^{*} f(rt), K_{i,w}(\varphi(rt)) \right\rangle_X dm(t) \\
&= \lim_{r\to 1} \int_{t\in\mathbb{T}} \left\langle T^{*} f(rt), \frac{e_i}{1-\bar{w}\varphi(rt)} \right\rangle_X dm(t) \\
&= \lim_{r\to 1} \int_{t\in\mathbb{T}} \frac{\left\langle f(rt), Te_i \right\rangle_X}{1-w\bar{\varphi}(rt)} dm(t) \\
&= \lim_{r\to 1} \int_{t\in\mathbb{T}} \frac{(Te_i \otimes f)(rt)}{1-w\bar{\varphi}(rt)} dm(t) \\
&= C_{\varphi}^{*}(T(e_i) \otimes f)(w)
\end{aligned}$$

Now by linearity and continuity, the relation (3.2) follows. □

**Corollary 3.3.** *Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and let $T$ be a bounded operator on a separable Hilbert space $X$. Then the Hilbert adjoint of $C_\varphi$ on Hilbert space $H^2(X)$ is given by*

$$C_{\varphi}^{*} f(w) = \lim_{r\to 1} \int_{t\in\mathbb{T}} \frac{f(rt)}{1-w\bar{\varphi}(rt)} dm(t) \quad (f \in H^2(X)),$$

*where the integral is the well known Bochner integral of the vector-valued function*

$$t \to \frac{f(rt)}{1-w\bar{\varphi}(rt)}.$$

*In particular, $C_T^{*} = C_{T^{*}}$, where $T^{*}$ is the Hilbert adjoint of $T$.*

## 4. **Compact and weak compactness**

In this section we characterize the compact (weak compact) bilateral composition operators on $H^p(X)$ for $1 \le p \le +\infty$. Recall that a bounded linear operator $T$ on a Banach space $X$ is compact (respectively weakly compact) if $T(B_X)$ is a compact (respectively weakly compact) subset of $X$, where $B_X = \{x \in X : \|x\| \le 1\}$ is the closed unit ball of $X$. Compactness properties of composition operators have been quite intensively studied in connection with various function spaces. It is easy to check if $C_\varphi$ is compact on $H^p(X)$ for $1 \le p < +\infty$, then $X$ must be finite dimensional and if it is weakly compact on $H^p(X)$, then $X$ must be reflexive. More precisely:

**Theorem 4.1.** *For the analytic self-map $\varphi$ on $\mathbb{D}$, the following statements hold:*

(1) *$C_\varphi$ is compact on $H^p(X)$ for $1 \leq p < \infty$ if and only if $X$ is finite dimensional and $C_\varphi$ is compact on $H^p$.*

(2) *$C_\varphi$ is weakly compact on $H^1(X)$ if and only if $X$ is reflexive and*

$$N_\varphi(w) = o(1 - |w|) \text{ as } |w| \to 1^-.$$

(3) *$C_\varphi$ is weakly compact on $H^\infty(X)$ if and only if $X$ is reflexive and $\|\varphi\|_\infty < 1$.*

*Proof.* Assume that $C_\varphi$ is compact on $H^p(X)$. Then $X$ can be viewed as a closed subspace of $H^p(X)$ via the mapping $x \to f_x$, where $f_x(t) = x$ for all $t \in \mathbb{D}$. Then $C_\varphi(f_x) = f_x$ for every $x \in X$, i.e., it fixes the constant functions and hence a copy of $X$ into itself. This means that the identity operator on $X$ is compact and hence $X$ must be of finite dimension. Hence, by Proposition 2.1, $H^p(X)$ is isometrically isomorphic to $\bigoplus_{k=1}^n H^p$ for some integer $n \geq 1$ and so $C_\varphi$ is compact on $H^p$. The converse follows similarly with Proposition 2.1. To prove (2) and (3), see part (ii) of Theorem 3, and Theorem 6 in [4]. $\square$

For any non-zero element $x \in X$ and any analytic function $f : \mathbb{D} \to \mathbb{C}$, the analytic vector-valued function $x * f : \mathbb{D} \to X$ is defined by $(x*f)(z) = xf(z)$. It is easy to see that $f \in H^p$ if and only if $f \in H^p(X)$ and

$$\|x * f\|_{H^p(X)} = \|x\|\|f\|_{H^p}.$$

We denote by $x * H^p$, the set all of vector-valued function $x * f$ where $f \in H^p$. In fact, $x * H^p$ is a closed subspace of $H^p(X)$, isometrically isometric to $H^p$.

Before stating the main result of this section, the following lemma is needed.

**Lemma 4.2.** *Suppose $T : X \to X$ and $S : Y \to Y$ are bounded linear operators of Banach spaces, and $\Lambda_i : X \to Y$ is a bounded linear operator for $i = 1, 2$ such that $\Lambda_1 T = S\Lambda_2$. If $\Lambda_1$ is an isometry and $S$ is compact (weakly compact), then $T$ is also compact (weakly compact).*

*Proof.* Let $Y_i := \Lambda_i(X)$, i=1,2. Since $\Lambda_i$ is an isometry, it is one to one and $Y_i$ is closed. Applying the inverse mapping theorem, there exists a bounded linear operator $\Lambda_i^{-1} : Y_i \to X$ such that $\Lambda_i^{-1}\Lambda_i = I$, where $I : X \to X$ is the identity operator and $i = 1, 2$. Since $\Lambda_1 T = S\Lambda_2$,

$$S(Y_2) = S\Lambda_2(X) = \Lambda_1 T(X) \subseteq \Lambda_1(X) = Y_1,$$

and $S : Y \to Y$ is compact, $S : Y_2 \to Y_1$ and consequently $T = \Lambda_1^{-1} S\Lambda_2$ is also compact. The proof for the case of weakly compact is similar. □

**Theorem 4.3.** *Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $T$ be a bounded linear operator on $X$, then*

(1) *$C_{\varphi,T}$ is compact on $H^p(X)$ for $1 \le p < \infty$, if and only if $T$ is compact on $X$ and either $T = 0$ or $C_\varphi$ is compact on $H^p$.*

(2) *If $T \neq 0$, then $C_{\varphi,T} : H^1(X) \to H^1(X)$ is weakly compact if and only if $T : X \to X$ is weakly compact and $\varphi$ satisfies the Shapiro condition $N_\varphi(w) = o(1 - |w|)$ as $|w| \to 1^-$.*

(3) *If $T \neq 0$, then $C_{\varphi,T} : H^\infty(X) \to H^\infty(X)$ is weakly compact if and only if $T : X \to X$ is weakly compact and $\|\varphi\|_\infty < 1$.*

*Proof.* (1) Assume that $C_{\varphi,T}$ is compact on $H^p(X)$ for $1 \le p < \infty$. Define $\Lambda : X \to H^p(X)$ by $\Lambda(x) = f_x$ where $f_x(t) = x$ for every $t \in \mathbb{D}$. Then $\Lambda$ is an isometry and the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\Lambda} & H^p(X) \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle C_{\varphi,T}} \\ X & \xrightarrow{\Lambda} & H^p(X) \end{array}$$

that is $\Lambda T = C_{\varphi,T}\Lambda$. Since $\Lambda$ is an isometry and $C_{\varphi,T}$ is compact on $H^p(X)$, by the previous lemma, $T$ is compact on $X$. Now if $T \neq 0$ and $Ta \neq 0$ for some unit vector $a \in X$, then considering bounded linear operators

$$\Lambda_a : H^p \to H^p(X) \text{ and } \Lambda_{Ta} : H^p \to H^p(X)$$

by $\Lambda_a(f) = a * f$ and $\Lambda_{Ta}(f) = (Ta) * f$, we see that the following diagram is also commutative:

$$\begin{array}{ccc} H^p & \xrightarrow{\Lambda_a} & H^p(X) \\ \Big\downarrow C_\varphi & & \Big\downarrow C_{\varphi,T} \\ H^p & \xrightarrow{\Lambda_{Ta}} & H^p(X) \end{array}$$

that is $\Lambda_{Ta}C_\varphi = C_{\varphi,T}\Lambda_a$. Since $\Lambda_{Ta}$ is an isometry and $C_{\varphi,T}$ is compact on $H^p(X)$, by the previous lemma, $C_\varphi$ is compact on $H^p$.

For the converse, first suppose that $T$ is of finite rank type and $\hat{X} = TX$. We will use the notation $\hat{C}_\varphi$ instead of $C_\varphi$ when it acts on $H^p(\hat{X})$. Since $\hat{X}$ is of finite dimensional and $C_\varphi$ is compact on $H^p$, by Theorem 4.1, part (1), $\hat{C}_\varphi$ is also compact on $H^p(\hat{X})$. Applying the following commutative diagram

$$\begin{array}{ccc} H^p(X) & \xrightarrow{C_T} & H^p(\hat{X}) \\ \Big\downarrow C_\varphi & & \Big\downarrow \hat{C}_\varphi \\ H^p(X) & \xrightarrow{C_T} & H^p(\hat{X}) \end{array}$$

we see that $C_{\varphi,T} = C_T C_\varphi = \hat{C}_\varphi C_T$ and so $C_{\varphi,T}$ is compact. Now suppose that $T$ is compact. Then there is an isometric embedding $J : X \to \ell^\infty(B_{X^*})$, where $\ell^\infty(B_{X^*})$ has the approximation property, so that $\|JT - T_n\| \to 0$ as $n \to +\infty$ for a suitable sequence of finite rank operators $T_n : X \to \ell^\infty(B_{X^*})$. Here $B_{X^*}$ is the closed unit ball of $X^*$. By the following estimate

$$\|C_{\varphi,T_n} - C_{\varphi,JT}\|_{H^p(X)} \leq \|C_\varphi\|_{H^p(X)}\|T_n - JT\|_X$$

we observe that $\|C_{\varphi,T_n} - C_{\varphi,JT}\|_{H^p(X)} \to 0$. On the other hand, each of the operators $C_{\varphi,T_n}$ is compact, implying that $C_{\varphi,JT}$ is compact. Since the left composition operator $C_J$ mapping $f \to J \circ f$ is an isometric embedding from $H^p(X)$ to $H^p(\ell^\infty(B_{X^*}))$, one can consider the left inverse operator $(C_J)^{-1}$ of $C_J$. Considering the relation

$$C_{\varphi,JT} = C_\varphi C_T C_J = C_J C_\varphi C_T = C_J C_{\varphi,T},$$

we observe that $C_{\varphi,T} = (C_J)^{-1}C_{\varphi,JT}$ whence $C_{\varphi,T}$ is also compact and the proof of part (1) is completed.

(2) Let $C_{\varphi,T}$ be weakly compact on $H^1(X)$. Then the same arguments as in the proof of part (1) together with Lemma 4.2 imply that $T$ is weakly compact on $X$ and $C_\varphi$ is weakly compact on $H^1$, respectively. Now by Theorem 3(ii) in [4], $\varphi$ must satisfy the Shapiro's condition.

Suppose conversely that $T : X \to X$ is weakly compact and $\varphi$ satisfies Shapiro's condition. By the DFJP-factorization there is a reflexive Banach space $Z$ and operators $U : X \to Z$ and $V : Z \to X$ such that $T = UV$. It follows from Theorem 3 of [4] that the operator $\hat{C}_\varphi : H^1(Z) \to H^1(Z)$ and hence the operator $C_{\varphi,U} = C_U C_\varphi = \hat{C}_\varphi C_U$ is weakly compact. Therefore, $C_{\varphi,T} = C_V C_{\varphi,U}$ is also weakly compact on $H^1(X)$ and the proof of part (2) is completed.

(3) this follows by using the same argument as in the proof of part (2), but employing Theorem 6 instead of Theorem 3 in [4]. □

When $\varphi(z) = z$, the composition operator $C_\varphi$ is the identity operator which is not compact on $H^p(X)$ for $1 \leq p \leq +\infty$, while it is weakly compact on $H^1(X)$ and $H^\infty(X)$. This is led to the following corollary:

**Corollary 4.4.** *For any bounded operator $T$ on $X$,*

*(1) $C_T$ is compact on $H^p(X)$ for $1 \leq p \leq \infty$, if and only if $T = 0$.*

*(2) If $T \neq 0$, then $C_T$ is weakly compact on $H^1(X)$ (or $H^\infty(X)$) if and only if $T : X \to X$ is weakly compact.*

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