ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 40 (2014), No. 2, pp. 339–355

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 2, pp. 339–355
Online ISSN: 1735-8515

PERIODICALLY CORRELATED AND MULTIVARIATE SYMMETRIC STABLE PROCESSES RELATED TO PERIODIC AND CYCLIC FLOWS

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(Communicated by Hamid Pezeshk)

Abstract. In this work we introduce and study discrete time periodically correlated stable processes and multivariate stationary stable processes related to periodic and cyclic flows. Our study involves producing a spectral representation and a spectral identification for such processes. We show that the third component of a periodically correlated stable process has a component related to a periodic-cyclic flow.
Keywords: Periodically correlated stable processes, multivariate stationary stable pro- cesses, fows, periodic and cyclic fows, cocycles.
MSC(2010): Primary: 60G51; Secondary: 60G52, 60G57.

1. Introduction

The spectral characterization for stationary stable processes became important and challenging, due to the observation made by Cambanis and Soltani [1] that no nontrivial stationary stable process is moving average and harmonizable as well. Rosinski [4] elegantly connected the Hardin [2] spectral form derivations to the theory of flows, and came with a solution, namely, a univariate stationary stable process $X = \{X_t, t \in \mathbb{T}\}$ (where $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$) can be decomposed in distribution as the sum of three independent stationary stable processes with somewhat known

Article electronically published on April 30, 2014.
Received: 13 February 2011, Accepted: 17 February 2013.
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spectral structures. Indeed

$$X \stackrel{d}{=} X^{(1)} + X^{(2)} + X^{(3)}, \tag{1.1}$$

where $X^{(1)} = \{X_t^1, t \in \mathbb{T}\}$ is a mixed moving average, $X^{(2)} = \{X_t^2, t \in \mathbb{T}\}$ is harmonizable and $X^{(3)} = \{X_t^3, t \in \mathbb{T}\}$ is of the third kind (the latest is the Rosinski terminology). Unlike $X^{(3)}$ the spectral structures of $X^{(1)}$ and $X^{(2)}$ were already well known. No further details, except that it possesses no moving average or nor harmonizable components, were given for $X^{(3)}$, by Rosinski [4]. The processes of the third kind were scrutinized by Pipiras and Taqqu [3]. They provided a refinement of the decomposition (1.1), namely

$$X \stackrel{d}{=} X^{(1)} + X^{(2)} + X^{(3)1} + X^{(3)2}, \tag{1.2}$$

where independent stationary stable processes $X^{(3)1}$ and $X^{(3)2}$ are cyclic and purely conservative, respectively. Soltani and Parvardeh [8] established a decomposition of type (1.1) for discrete time univariate periodically correlated, and also multivariate stationary, stable processes. In this work we produce (1.2) for such processes. Naturally, similar to Pipiras and Taqqu [3], we consider periodically correlated, as well as multivariate stationary stable processes that are generated by periodic flows. Periodic flows $\{\phi_t, t \in \mathbb{T}\}$ were extensively studied and characterized in Soltani and Parvardeh [8]. In this article we deal with a periodic flow for which $\mathbb{T} = \{nT, n \in \mathbb{Z}\}$, where T is a fixed positive integer. We have noticed that this change of time index does not affect the course of Pipiras and Taqqu [3] derivations on periodic flows. Thus we only bring the main results of Sections 2 and 3 of Pipiras and Taqqu [3] to the context of this work without giving the proofs. We also assume that readers are familiar with the notions and notations given in Soltani and Parvardeh [8] and Pipiras and Taqqu [3], and do not redefine them here. We mostly concentrate on representations of discrete time periodically correlated processes generated by periodic flows. To avoid reproduction, we only bring proofs that are different from those in Pipiras and Taqqu [3]. This give a resonable size to this article, and brings to light the effects of the correlation periodicity of the processes that are generated by periodic flows. The following are furnished in this article. (i): structural decomposition for PC SαS processes generated by periodic flows, Theorem 2.5. (ii): Specification for the periodic component set, say C_P of a PC SαS process, and showing that the harmonizable component set, C_F, of such a process, as identified in Soltani and Parvardeh [8] is indeed included in C_P. Also proving that on the component set $C_L = C_P - C_F$ the resulting process is cyclic, Theorem 3.1. (iii) Including a section of examples of PC SαS processes that are either of the component of the structural decomposition discussed in this article.

2. Period and cyclic flows

Let $\{X_n\}_{n\in Z}$ be a discrete time periodically correlated symmetric $\alpha-$stable process (PC $S\alpha S$, for in short) that has an integral representation

$$\{X_n\}_{n\in\mathbb{Z}} \stackrel{d}{=} \left\{\int_S f_n(s)dM(s)\right\}_{n\in\mathbb{Z}}, \tag{2.1}$$

where $\stackrel{d}{=}$ stands for equality in the sense of the finite-dimensional distributions. Here $(S,\mathcal{B},\mu)$ is a standard Lebesgue space, $\{f_n\}_{n\in\mathbb{Z}} \subset L^\alpha(S,\mathcal{B},\mu)$ is a collection of real valued (complex valued) deterministic functions and M is, respectively, either a real-valued or a complex valued rotationally invariant $S\alpha S$ random measure on $(S,\mathcal{B})$ with control measure μ. Recall that $\{X_n\}_{n\in\mathbb{Z}}$ is periodically correlated (or PC in short) if

$$\{X_n\}_{n\in\mathbb{Z}} \stackrel{d}{=} \{X_{n+mT}\}_{n\in\mathbb{Z}},$$

for any $m \in \mathbb{Z}$ and some positive integer T. The smallest T is the period. Clearly the process is stationary when $T = 1$. For the spectral representation and more on stable processes see Samorodnitsky and Taqqu [7].

A PC stable process is connected to a multivariate stationary stable process. It easily follows that $\{X_n\}_{n\in\mathbb{Z}}$ is a PC $S\alpha S$ process with a period T if and only if the T-variate process

$$\{\mathbf{Y}_n = (X_{nT}, X_{nT+1}, \cdots, X_{nT+T-1}),\ n \in \mathbb{Z}\}$$

is $S\alpha S$ stationary. Recall that every $n \in \mathbb{Z}$ can be uniquely expressed as $n = [[n]] + r_n$, where $[[n]]$ is a multiple of T, $0 \le r_n \le T-1$ and $r_n = r_{n+T}$ for all $n \in \mathbb{Z}$. We also let define $G = \{kT, k \in \mathbb{Z}\}$.

Definition 2.1. *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a PC $S\alpha S$ process with the spectral representation (2.1). Then we say X is generated by a nonsingular measurable flow $\left\{\phi_{[[n]]}\right\}_{n\in\mathbb{Z}}$ on $(S,\mathcal{B},\mu)$ if for all $n \in \mathbb{Z}$*

$$f_n = a_{[[n]]}\left\{\frac{d\mu\circ\phi_{[[n]]}}{d\mu}\right\}^{1/\alpha} V_n\circ\phi_{[[n]]} \quad a.e.(\mu), \tag{2.2}$$

where $V_r = f_r$ *for* $r = 0, ..., T-1$ *and* $V_{n+T} = V_n$ *for any* $n \in \mathbb{Z}$ *, and* $\left\{a_{[[n]]}\right\}_{n\in\mathbb{Z}}$ *is a cocycle for the flow* $\left\{\phi_{[[n]]}\right\}_{n\in\mathbb{Z}}$*, and*

$$\operatorname{supp}\{f_n, n \in \mathbb{Z}\} = S \quad a.e.(\mu). \tag{2.3}$$

If a set S satisfies (2.3), then we refer to it as *the spectral support* of the process.

For a spectral representation $\{f_n\}_{n\in\mathbb{Z}}$ satisfying (2.2) and (2.3), we use the notation $f_n = [\phi_{[[n]]}, a_{[[n]]}, V_n]$.

Let $\left\{\phi_{[[n]]}\right\}_{n\in\mathbb{Z}}$ be a flow on a standard Lebesgue space $(S, \mathcal{B}, \mu)$ and

$$\begin{aligned} P &= \left\{s : \exists[[p]] = [[p(s)]] \in G - \{0\} : \phi_{[[p]]}(s) = s\right\} \\ F &= \left\{s : \phi_{[[n]]}(s) = s \text{ for all } n \in \mathbb{Z}\right\} \\ L &= P - F \end{aligned}$$

be the periodic, fixed and cyclic points of the flow $\left\{\phi_{[[n]]}\right\}_{n\in\mathbb{Z}}$, respectively. Note that $P, F, L \in \mathcal{B}$.

Definition 2.2. *A flow* $\left\{\phi_{[[n]]}\right\}_{n\in\mathbb{Z}}$ *on* $(S, \mathcal{B}, \mu)$ *is periodic if* $S = P$ *a.e.*(μ)*, is identity if* $S = F$ *a.e.*(μ)*, and is cyclic if* $S = L$ *a.e.*(μ)

We can also use the following alternative definition of a cyclic flow, which is equivalent to Definition 2.2 (Theorem 2.4).

Definition 2.3. *A flow* $\left\{\phi_{[[n]]}\right\}_{n\in\mathbb{Z}}$ *on* $(S, \mathcal{B}, \mu)$ *is cyclic if it is null isomorphic (mod 0) to the flow*

$$\widetilde{\phi}_{[[n]]}(y, [[v]]) = (y, \{[[v]] + [[n]]\}_{[[q(y)]]})$$

on $(Y \times ([0, [[q(\cdot)]]) \cap G), \mathcal{Y} \times \mathcal{B}([0, [[q(\cdot)]]) \cap G), \nu \times \lambda)$*, where* $q(y) \in T\mathbb{Z}_+$ *is some measurable function, where* $\mathbb{Z}_+ = \{2, 3, \ldots\}$.

For the definition of the null isomorphic (mod 0) see [3].

Theorem 2.4. *Definitions 2.2 and 2.3 of cyclic flows are equivalent.*

Proof. If the flow $\left\{\phi_{[[n]]}\right\}_{n\in\mathbb{Z}}$ is cyclic in the sense of Definition 2.3, then every point in the space $Y \times ([0, [[q(\cdot)]]) \cap G)$ is cyclic, because if $p = [[q(y)]] \neq 0$, then

$$\begin{aligned} \widetilde{\phi}_{[[q(y)]]}(y, [[v]]) &= (y, \{[[v]] + [[q(y)]]\}_{[[q(y)]]}) \\ &= (y, \{[[v]]\}_{[[q(y)]]}) \\ &= (y, [[v]]) \end{aligned}$$

and hence $S = L$ $a.e.(\mu)$, by using definition of null isomorphic. Now suppose that $\left\{\phi_{[[n]]}\right\}_{n\in Z}$ is cyclic in the sense of Definition 2.2, i.e., $S = L$ $a.e.(\mu)$. Then the result follows by reasonings very similar to the those given in the proof of Theorem 2.1 in Pipiras and Taqqu [3] with the function

$$g(s) = [[q(s)]] = \min\{[[m]] : [[m]] \in T\mathbb{Z}_+, \phi_{[[m]]}(s) = s\}.$$

□

Now we state a representation of PC stable processes generated by periodic flows. For definitions of PC harmonizable and PC trivial $S\alpha S$ processes see Soltani and Parvardeh [8].

Theorem 2.5. *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a PC $S\alpha S$ process generated by a periodic flow in the sense of Definition 2.1. Then X can be decomposed, in distribution, into the sum of two independent PC stable processes. The first process is a harmonizable processes (or trivial process) and the second process can be represented as*

(2.4)

$$\int_Y \sum_{m:mT<[[q(y)]]} b(y)^{[mT+[[n]]]_{[[q(y)]]}} g_n(y, \{mT+[[n]]\}_{[[q(y)]]}) M(dy, mT),$$

here $(Y, \mathcal{Y}, \nu)$ is a standard Lebesgue space, $q(y) \in \{T, 2T, 3T, \dots\}$,

$$b(y) \in \begin{cases} \{z : |z| = 1\} & \textit{(complex-valued case)}, \\ \{-1, 1\} & \textit{(real-valued case)}, \end{cases}$$

and $g_n \in L^\alpha(Y \times ([0, [[q(\cdot)]]) \cap G), \nu \times \lambda)$ is T-periodic $g_{n+T} = g_n$, and M is a $S\alpha S$ random measure on $Y \times ([0, [[q(\cdot)]]) \cap G)$ with control measure $v \times \lambda$.

Note that a PC $S\alpha S$ process generated by a cyclic flow also have a representation (2.4), (see Remark 3.1 in [3]).

Definition 2.6. *A PC $S\alpha S$ process $X = \{X_n\}_{n\in\mathbb{Z}}$ is said to be PC periodic process if it is decomposed into the sum of a harmonizable process (or trivial process) and a process given by (2.4), such that these two processes are independent.*

If $\{X_n\}_{n\in\mathbb{Z}}$ is a PC periodic process, then it does not necessarily imply that an underlying generating flow of the process is periodic. The following can be the case.

Lemma 2.7. *A PC harmonizable process (or trivial process) can be represented as in (2.4).*

Proof. Suppose that $\{X_n\}_{n\in\mathbb{Z}}$ is a PC stable process such that

$$\begin{aligned} X_n &= \int_S \sum_{m=0}^{1} \left(e^{i2Tk(y)}\right)^{[mT+[[n]]]_{2T}} \left(e^{ik(y)}\right)^{\{mT+[[n]]\}_{2T}} V_n(y)M(dy,mT) \\ &= \int_S \sum_{m=0}^{1} \left(e^{ik(y)}\right)^{2T[mT+[[n]]]_{2T}+\{mT+[[n]]\}_{2T}} V_n(y)M(dy,mT) \\ &= \int_S \sum_{m=0}^{1} \left(e^{ik(y)}\right)^{mT+[[n]]} V_n(y)M(dy,mT), \quad n\in\mathbb{Z}, \end{aligned}$$

where M is a complex-valued rotationally invariant $S\alpha S$ random measure with control measure $\mu\times\lambda$ where λ is the counting measure on $G=\{[[n]], n\in\mathbb{Z}\}$, $k: S\to[0,\frac{2\pi}{T})$ and $V_n(y)$ are measurable functions such that $V_n = V_{n+T}$. Then the process $\{X_n\}_{n\in\mathbb{Z}}$ has a representation (2.4) with $Y=S$, $v(dy)=\mu(dy)$, $b(y)=e^{i2Tk(y)}$, $q(y)=2T$, and $g_n(y,u)=e^{izu}V_n(y)$. Note that since $\left(e^{ik(y)}\right)^{mT+[[n]]} = e^{ik(y)mT}e^{ik(y)[[n]]}$, $|e^{ik(y)mT}|=1$, hence $e^{ik(y)mT}$ does not depend on n. Therefore

$$\begin{aligned} \{X_n\}_{n\in\mathbb{Z}} &\stackrel{d}{=} \left\{\int_S \sum_{m=0}^{1} e^{ik(y)[[n]]} V_n(y)M(dy,mT)\right\}_{n\in\mathbb{Z}} \\ &\stackrel{d}{=} \left\{2^{1/\alpha}\int_S e^{ik(y)[[n]]}V_n(y)N(dy)\right\}_{n\in\mathbb{Z}}, \end{aligned}$$

where N is a complex-valued rotationally invariant measure with control measure $\mu(dy)$. Hence the process $\{X_n\}_{n\in\mathbb{Z}}$ is also harmonizable. In the case that the process is trivial, take $Y=\{1,2\}$, $b(z)=1$, and $g_n(y,mT)=a(y)^{mT}V_n(y)$ with $a(1)=1$, $a(2)=-1$ and $q(y)=2T$.

Then (2.4) becomes

$$\begin{aligned}
&\int_{\{1,2\}} \sum_{m=0}^{1} a(y)^{\{mT+[[n]]\}_{2T}} V_n(y) M(dy, mT) \\
&= \int_{\{1,2\}} \sum_{m=0}^{1} a(y)^{mT+[[n]]} V_n(y) M(dy, mT) \\
&\stackrel{d}{=} \int_{\{1,2\}} a(y)^{[[n]]} V_n(y) N(dy) \\
&= \int_{\{1\}} V_n(z) N(dy) + (-1)^{[[n]]} \int_{\{2\}} V_n(y) N(dy),
\end{aligned}$$

giving the result. The proof is complete. □

Corollary 2.8. *A PC periodic process can also be represented by (2.4).*

The situation is clear if (2.1) is the minimal spectral representation. This is illustrated in the following theorem.

Theorem 2.9. *If the spectral representation (2.1) of a PC periodic stable process X is minimal, then X is generated by a unique flow in the sense of Definition 2.1. This flow is periodic for (2.4) and harmonizable (or trivial) process, identity for harmonizable (or trivial process), and is cyclic for processes given by (2.4).*

Now we define PC cyclic processes.

Definition 2.10. *A PC periodic stable process is said to be PC cyclic if in distribution it can not written as a sum of two independent PC stable processes for which one is harmonizable (or trivial).*

It is interesting and challenging to impose conditions on f_n so that the corresponding PC stable process given by (2.1) becomes periodic or cyclic. We provide such conditions below, which indeed are among major contributions of this article, as is not easy to use Theorem 2.9. Indeed it might be hard to determine whether a representation $\{f_n\}_{n\in\mathbb{Z}}$ is minimal. We know that harmonizable processes (or trivial processes in the real-valued case) can be identified through the harmonizable (or trivial) component set (see Soltani and Parvardeh [8] and the following definition).

Definition 2.11. *Let $X = \{X_n\}_{n\in Z}$ be a PC $S\alpha S$ process with the spectral representation (2.1).*

(i) A periodic component for X is defined as

$$C_P = \{s \in S : \exists m = m(s) : f_{n+Tm}(s) = a(m,s)f_n(s) \text{ for all } n \text{ and for some } a(m,s) \neq 0\}.$$

(ii) A harmonizable (or trivial) component set is defined as

$$C_F = \bigcup_{r=0}^{T-1} C_F^r \cap \operatorname{supp}(f_r),$$

where

$$C_F^r = \{s: \ f_r(s)f_{n+mT}(s) = f_n(s)f_{mT+r}(s), \text{for each } n, m \in Z\},$$

and $r = 0, \ldots, T-1$. *(iii) A cyclic component set is defined as*

$$C_L = C_P - C_F.$$

Note that $C_P, C_F, C_L \in \mathcal{B}$ and are invariant under the flow $\{\phi_{[[n]]}\}_{n\in\mathbb{Z}}$. Also we can easily see that $C_F \subset C_P$ *a.e.*(μ) and $C_P \subset C_L$ *a.e.*(μ). The following theorem characterizes PC periodic (cyclic respectively) processes.

Theorem 2.12. *Let* $X = \{X_n\}_{n\in\mathbb{Z}}$ *be a PC SαS process given by (2.1) with* $\operatorname{supp}\{f_n : n \in \mathbb{Z}\} = S$ *a.e.*(μ).
(i) The process X is PC periodic (PC cyclic respectively) if and only if

$$C_P = S \quad a.e.(\mu). \ \ (C_L = S \quad a.e.(\mu). \ \ \text{respectively})$$

(ii) We have a.e.(μ)

$$C_F = \{s \in S : f_{n+T}(s) = a(s)f_n(s) \text{ for all } n \in \mathbb{Z}, \quad \text{for some } a(s) \neq 0\}$$

$$C_L = \{s \in S : \exists m = m(s) \in \mathbb{Z} - \{0\} : f_{n+Tm}(s) = a(m,s)f_n(s) \text{ for all } n \in \mathbb{Z}, \text{for some} \quad a(m,s) \neq 0\}$$
$$\bigcap \{s \in S : f_{n+T}(s) \neq a(s)f_n(s) \quad \text{for all} \quad n \in \mathbb{Z}, \text{ for all } a(s) \neq 0\}.$$

3. Further decompositions

In this section, we give a refinement for the decomposition (1.1) for PC $S\alpha S$ process. Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a PC $S\alpha S$ process generated by a nonsingular flow $\{\phi_{[[n]]}\}_{n\in\mathbb{Z}}$ defined on a standard Lebesgue space (S, μ) with $\operatorname{supp}\{f_n, n \in \mathbb{Z}\} = S$, *a.e.*$(\mu)$. Also let $S = C \cup D$ be the Hopf decomposition for $\{\phi_{[[n]]}\}_{n\in\mathbb{Z}}$, i.e., D and C are (*a.e.*(μ)) the dissipative and the conservative parts of the flow $\{\phi_{[[n]]}\}_{n\in\mathbb{Z}}$. Then

$$X \stackrel{d}{=} X^D + X^C,$$

where

$$X_n^D = X_n^{(1)} = \int\limits_D f_n(s)M(ds), \quad X_n^C = \int\limits_C f_n(s)M(ds).$$

If the representation of $\{X_n\}_{n\in\mathbb{Z}}$ is minimal, then $C_F = F\ a.e.(\mu)$, where F is the set of the fixed points of the generating flow. To obtain (1.1) for PC $S\alpha S$ process, we decomposed X^C as the following (note that $C_F \subset C\ a.e(\mu)$):

$$X_n^{(2)} = \int\limits_{C_F} f_n(s)M(ds), \quad X_n^{(3)} = \int\limits_{C-C_F} f_n(s)M(ds).$$

For further decomposition, we use the facts that $C_P \subset C\ a.e.(\mu)$ and $C_F \subset C_P\ a.e.(\mu)$, and write

$$X^{(3)} \overset{d}{=} X^{(3)1} + X^{(3)2}, \tag{3.1}$$

$$X_n^{(3)1} = \int_{C_L} f_n(s)M(ds), \quad X_n^{(3)2} = \int_{C-C_P} f_n(s)M(ds).$$

Theorem 3.1. *The decomposition (3.1) is unique in distribution. Moreover, the process $X^{(3)1}$ is a PC cyclic process and the process $X^{(3)2}$ is a $S\alpha S$ PC process generated by a conservative flow without a periodic component.*

In the case that a minimal representation is given, we obtain the following result.

Proposition 3.2. *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be PC $S\alpha S$ process with minimal representation $f_n = [\phi_{[[n]]}, a_{[[n]]}, V_n]$ in the sense of Definition 2.1. Then*

$$C_L = L\ \ a.e.(\mu) \quad \text{and} \quad C_P = P \quad a.e.(\mu),$$

and X is a PC periodic (cyclic, respectively) process if and only if

$$S = P\ \ a.e.(\mu) \quad (S = L\ \ a.e.(\mu)\ \text{respectively.}),$$

where P and L are the periodic and the cyclic points of the generating flow, respectively. The latter is also true if and only if the generating flow is periodic (cyclic respectively).

In the following we give a unique decomposition of a PC $S\alpha S$ process into four independent components.

Theorem 3.3. *Every PC $S\alpha S$ process $\{X_n\}_{n\in\mathbb{Z}}$ admits a unique in distribution decomposition*

$$X \stackrel{d}{=} X^{(1)} + X^{(2)} + X^{(3)1} + X^{(3)2}, \tag{3.2}$$

into four mutually independent PC $S\alpha S$ processes: (i) $X^{(1)}$ is a mixed moving average PC $S\alpha S$ process given by (3.5) in [1]; (ii) $X^{(2)}$, in the complex valued case, is a harmonizable $S\alpha S$ process and is trivial process in the real case; (iii) $X^{(3)1}$ is a $S\alpha S$ PC cyclic process in the sense of Definition 2.10.; and (iv) $X^{(3)2}$ is a PC $S\alpha S$ process generated by a conservative flow without a periodic component.

Now we deal with a T-variate stationary $S\alpha S$ process $\{\mathbf{Y}_n = (Y_n^0, \ldots, Y_n^{T-1})\}_{n\in\mathbb{Z}}$, whose components are, in general, complex stable. For more insights to these processes, we refer readers to Soltani and Parvardeh [8] and Samorodnitski and Taqqu [7]. We will provide a decomposition for discrete time multivariate stationary $S\alpha S$ processes (DTMS($S\alpha S$)P in short) using the decomposition that was established for PC stable processes in (3.2). A PC stable process $\{X_n\}_{n\in\mathbb{Z}}$ with period T can be formed from a DTMS($S\alpha S$)P $\{\mathbf{Y}_n\}_{n\in\mathbb{Z}}$, and vise versa, through

$$Y_n^j = X_{nT+j}, \;\; j = 0, \ldots, T-1, \; n \in \mathbb{Z}. \tag{3.3}$$

A similar relation to (3.3) also relates the spectral representations $\{\mathbf{f}_n\}_{n\in\mathbb{Z}}$ and $\{f_n\}_{n\in\mathbb{Z}}$; namely

$$\{X_n\}_{n\in\mathbb{Z}} \stackrel{d}{=} \left\{\int_S f_n(s)dM(s)\right\}_{n\in\mathbb{Z}},$$

if and only if

$$\{\mathbf{Y}_n\}_{n\in\mathbb{Z}} \stackrel{d}{=} \left\{\int_S \mathbf{f}_n(s)dM(s)\right\}_{n\in\mathbb{Z}},$$

where

$$\mathbf{f}_n = (f_n^0, \ldots, f_n^{T-1}),$$

$$f_n^j = f_{nT+j}, \;\; j = 0, \ldots, T-1, \; n \in \mathbb{Z}. \tag{3.4}$$

By using (3.4), we rewrite basic terms which are needed for the decomposition. Following Definition 2.1, we say that (see [8]) a DTMS($S\alpha S$)P $\{\mathbf{Y}_n\}_{n\in\mathbb{Z}}$ is generated by a nonsingular measurable flow $\{\phi_n\}_{n\in\mathbb{Z}}$ on $(S, \mathcal{B}, \mu)$ if, for all $n \in \mathbb{Z}$,

$$\mathbf{f}_n = a_n \left\{\frac{d\mu \circ \phi_n}{d\mu}\right\}^{1/\alpha} \mathbf{f}_0 \circ \phi_n \quad a.e.(\mu),$$

where $\{a_n\}_{n\in\mathbb{Z}}$ is a cocycle for $\{\phi_n\}_{n\in\mathbb{Z}}$, $\mathbf{f}_0=(f_0^0,\ldots,f_0^{T-1})=(V_0,\ldots,V_{T-1})$, where the later coordinates are as in (2.3) such that

$$\operatorname{supp}\left\{f_0^j\circ\phi_n,\ \ r=0,\ldots,T-1,\ \ n\in\mathbb{Z}\right\}=S.$$

Let us define the corresponding subsets C_F^r and C_F, as follows:

$$C_F^r=\{s:\ f_0^r(s)\ \mathbf{f}_{n+m}(s)=f_n^r(s)\mathbf{f}_m(s),\ \ \text{for each}\ \ n,m\in\mathbb{Z}\}$$

for $r=0,\ldots,T-1$, and

$$C_F=\bigcup_{r=0}^{T-1}C_F^r\cap\operatorname{supp}(f_0^r),$$

$$\begin{aligned}
C_P &= \{s\in S:\exists m=m(s):\mathbf{f}_{n+m}(s)=a(m,s)\mathbf{f}_n(s)\\
&\qquad \text{for all } n \text{ and for some } a(m,s)\neq 0\},\\
C_F &= \{s\in S:\mathbf{f}_{n+1}(s)=a(s)\mathbf{f}_n(s)\ \text{ for all } n\in\mathbb{Z},\ \text{for some } a(s)\neq 0\},\\
C_L &= \{s\in S:\exists m=m(s)\in\mathbb{Z}-\{0\}:\mathbf{f}_{n+m}(s)=a(m,s)\mathbf{f}_n(s)\\
&\qquad \text{for all } n\in\mathbb{Z},\ \ \text{for some}\ \ a(m,s)\neq 0\}\\
&\qquad \bigcap\{s\in S:\mathbf{f}_{n+1}(s)\neq a(s)\mathbf{f}_n(s)\ \text{ for all } n\in\mathbb{Z},\ \text{for all } a(s)\neq 0\}.
\end{aligned}$$

respectively. The definitions of T-variate $S\alpha S$ periodic and cyclic stationary processes is similar to the Definitions 2.6 and 2.10 for which the term “PC” is changed to ”T-variate stationary”, and (2.4) for this T-variate stationary $S\alpha S$ process is state as

$$\int_Y\sum_{0\le m<q(y)}b(y)^{[m+n]_{q(y)}}\mathbf{g}(y,\{m+n\}_{q(y)})M(dy,m),$$

where $g(y)\in\{2,3,\ldots\}$ and $\mathbf{g}=(g_1,\ldots,g_{T-1})$ satisfies

$$\int_Y\sum_{0\le m<q(y)}|g_j(y,\{m+n\}_{q(y)})|^{\alpha}\nu(dy),\ \ j=0,\ldots,T-1,\ \ n\in\mathbb{Z}.$$

For definitions of T-variate stationary $S\alpha S$ mixed moving average and T-variate harmonizable $S\alpha S$ processes see Soltani and Parvardeh [8], Theorem 5.1.

Theorem 3.4. *Suppose $\{\mathbf{Y}_n=(Y_n^0,\ldots,Y_n^{T-1})\}_{n\in\mathbb{Z}}$ is a DTMS($S\alpha S$)P. Then $\{\mathbf{Y}_n\}_{n\in\mathbb{Z}}$ can be uniquely decomposed, in distribution, into sum of four mutually independent stationary $S\alpha S$ processes,*

$$\mathbf{Y}_n=\mathbf{Y}_n(1)+\mathbf{Y}_n(2)+\mathbf{Y}_n(3)+\mathbf{Y}_n(4),$$

where $\mathbf{Y}_n(1) = (Y_n^0(1), \ldots, Y_n^{T-1}(1))$ *is a* T*-variate stationary* $S\alpha S$ *mixed moving average process,* $\mathbf{Y}_n(2) = (Y_n^0(2), \ldots, Y_n^{T-1}(2))$ *is a* T*-variate harmonizable* $S\alpha S$ *process,* $\mathbf{Y}_n(3)$ *is a* T*-variate stationary cyclic process, and* $\mathbf{Y}_n(4)$ *is a* T*-variate stationary* $S\alpha S$ *process generated by a conservative flow without a periodic component.*

Proof. Define $X_n = Y^j_{[[n]]/T}$, $n = [[n]] + j$. Then $\{X_n\}_{n\in Z}$ is a PC $S\alpha S$ process. Apply Theorem 3.3 to obtain

$$\mathbf{Y}_n = \mathbf{Y}_n(1) + \mathbf{Y}_n(2) + \mathbf{Y}_n(3) + \mathbf{Y}_n(4),$$

where

$$Y_n^j(q) = X^{(q)}_{nT+j},\ q = 1,2,3,4,\ \ j = 0,\ldots,T-1,\ n \in Z.$$

□

4. **Examples**

In this section we provide some examples of PC $S\alpha S$ processes that are of the third and the fourth kind in decomposition (3.2).

Example 4.1. Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary $S\alpha S$ process with spectral representation $\{g_n\}_{n\in\mathbb{Z}}$, that is

$$\{X_n\}_{n\in\mathbb{Z}} \stackrel{d}{=} \left\{\int_S g_n(s)dM(s)\right\}_{n\in\mathbb{Z}}$$

where $(S,\mathcal{B},\mu)$ is a standard Lebesgue space, $\{g_n\}_{n\in\mathbb{Z}} \subset L^\alpha(S,\mathcal{B},\mu)$ is a collection of real valued (complex valued) and M is, respectively, either a real-valued or a complex valued rotationally invariant $S\alpha S$ random measure on $(S,\mathcal{B})$ with control measure μ. The stationarity of $\{X_n\}_{n\in\mathbb{Z}}$ is equivalent to

$$\int_S |\sum_{j=1}^k \theta_j g_{n_j+m}(s)|^\alpha \mu(ds) = \int_S |\sum_{j=1}^k \theta_j g_{n_j}(s)|^\alpha \mu(ds), \tag{4.1}$$

for every $n_1,\ldots,n_k, m \in \mathbb{Z}$, $\theta_1,\ldots,\theta_k \in \mathbb{R}$ ($\mathbb{C}$ in complex valued case) and $k \in \mathbb{N}$. Suppose that $\{b_n\}_{n\in\mathbb{Z}}$ be sequence of integers such that $b_n = b_{n+T}$ for all $n \in \mathbb{Z}$ such that $T > 2$, $b_0 < \ldots < b_{T-1}$ and $b_1 - b_0 \neq b_2 - b_1 \neq \ldots \neq b_{T-1} - b_{T-2}$. Then the process

$$\{Y_n\}_{n\in\mathbb{Z}} \stackrel{d}{=} \left\{\int_S f_n(s)dM(s)\right\}_{n\in\mathbb{Z}},$$

where $f_n = g_{n+b_n}$, $n \in \mathbb{Z}$, is a PC $S\alpha S$ process with period T. Indeed with a use of (4.1), for every $n_1, \ldots, n_k, m \in \mathbb{Z}$, $\theta_1, \ldots, \theta_k \in \mathbb{R}$ ($\mathbb{C}$ in the complex valued case) and $k \in \mathbb{N}$ we have

$$\begin{aligned}
\int_S |\sum_{j=1}^k \theta_j f_{n_j+mT}(s)|^\alpha \mu(ds) &= \int_S |\sum_{j=1}^k \theta_j g_{n_j+mT+b_{n_j+mT}}(s)|^\alpha \mu(ds) \\
&= \int_S |\sum_{j=1}^k \theta_j g_{n_j+b_{n_j}+mT}(s)|^\alpha \mu(ds) \\
&= \int_S |\sum_{j=1}^k \theta_j g_{n_j+b_{n_j}}(s)|^\alpha \mu(ds) \\
&= \int_S |\sum_{j=1}^k \theta_j f_{n_j}(s)|^\alpha \mu(ds).
\end{aligned}$$

Now if

$$g_n = a_n \left\{ \frac{d\mu \circ \phi_n}{d\mu} \right\}^{1/\alpha} g_0 \circ \phi_n, \quad a.e.(\mu)$$

where $\{\phi_n\}_{n\in\mathbb{Z}}$ is a nonsingular flow on $(S, \mathcal{B}, \mu)$, $\{a_n\}_{n\in\mathbb{Z}}$ is a cocycle for the flow $\{\phi_n\}_{n\in\mathbb{Z}}$ and $g_0 \in L^\alpha(S, \mathcal{B}, \mu)$, then $\text{supp}\{g_n, n \in \mathbb{Z}\} = S$ $a.e.(\mu)$ (that is $\{X_n\}_{n\in\mathbb{Z}}$ is generated by the nonsingular flow $\{\phi_{[[n]]}\}_{n\in\mathbb{Z}}$). We note that for $a.e.(\mu)$,

$$\begin{aligned}
f_n &= g_{n+b_n} = a_{n+b_n} \left\{ \frac{d\mu \circ \phi_{n+b_n}}{d\mu} \right\}^{1/\alpha} g_0 \circ \phi_{n+b_n} \\
&= a_{[[n]]+r_n+b_n} \left\{ \frac{d\mu \circ \phi_{[[n]]+r_n+b_n}}{d\mu} \right\}^{1/\alpha} g_0 \circ \phi_{[[n]]+r_n+b_n} \\
&= a_{[[n]]} a_{r_n+b_n} \circ \phi_{[[n]]} \left\{ \frac{d\mu \circ \phi_{r_n+b_n}}{d\mu} \circ \phi_{[[n]]} \right\}^{1/\alpha} \\
&\quad \times \left\{ \frac{d\mu \circ \phi_{[[n]]}}{d\mu} \right\}^{1/\alpha} g_0 \circ \phi_{r_n+b_n} \circ \phi_{[[n]]} \\
&= a_{[[n]]} \left\{ \frac{d\mu \circ \phi_{[[n]]}}{d\mu} \right\}^{1/\alpha} V_n \circ \phi_{[[n]]},
\end{aligned}$$

where

$$V_n = a_{r_n+b_n} \left\{ \frac{d\mu \circ \phi_{r_n+b_n}}{d\mu} \right\}^{1/\alpha} g_0 \circ \phi_{r_n+b_n}.$$

Clearly every $V_n \in L^\alpha(S, \mathcal{B}, \mu)$. Also $V_{n+T} = V_n$, since $b_{n+T} = b_n$ and $r_{n+T} = r_n$. Therefore $\{Y_n\}_{n\in\mathbb{Z}}$ is generated by nonsingular flow $\{\phi_{[[n]]}\}_{n\in\mathbb{Z}}$. But not necessarily $\text{supp}\{f_n, n \in \mathbb{Z}\} = S \quad a.e.(\mu)$. Note that $C_P^g \subset C_P^f$, where C_P^g is a periodic component set for the stationary process $\{X_n\}_{n\in\mathbb{Z}}$ and C_P^f is a periodic component set for the PC process $\{Y_n\}_{n\in\mathbb{Z}}$. Indeed if $s \in C_P^g$, then (by Definition 4.1 Pipiras and Taqqu [3])

$$\exists m = m(s) \in \mathbb{Z} - \{0\} \text{ such that } g_{n+m}(s) = a(m,s)g_n(s) \text{ for all } n \in \mathbb{Z}, \\ \text{for some } a(m,s) \neq 0.$$

Therefore

$$g_{n+mT}(s) = (a(m,s))^T g_n(s) \ \forall n \in \mathbb{Z}.$$

Then for all $n \in \mathbb{Z}$,

$$\begin{aligned} f_{n+mT}(s) &= g_{n+mT+b_{n+mT}}(s) = g_{n+b_n+mT}(s) \\ &= (a(m,s))^T g_{n+b_n}(s) = (a(m,s))^T f_n(s), \end{aligned}$$

that is $s \in C_P^f$. Similarly $C_F^g \subset C_F^f$, where C_F^g is a harmonizable component set for the stationary process $\{X_n\}_{n\in\mathbb{Z}}$ and C_P^f is a harmonizable component set for the PC process $\{Y_n\}_{n\in\mathbb{Z}}$. Note that not necessarily $C^g \subset C^f$, where C stands for the conservative part but we have $D^g \subset D^f$, where D stands for the dissipative part (for definition of C_P^g, C_F^g, C^g and D^g see Pipiras and Taqqu [3]). Therefore if $\{X_n\}_{n\in\mathbb{Z}}$ is stationary $S\alpha S$ mixed moving average, harmonizable process or periodic, then $\{Y_n\}_{n\in\mathbb{Z}}$ will be PC $S\alpha S$ mixed moving average, harmonizable or periodic. Note that if $\{X_n\}_{n\in\mathbb{Z}}$ is a stationary $S\alpha S$ process of the third (or forth) kind, then $\{Y_n\}_{n\in\mathbb{Z}}$ is not necessarily a PC $S\alpha S$ process of third and forth kind (for examples of third and forth stationary $S\alpha S$ process see Rosinski and Samorodnitsky [6], Rosinski [5] and Pipiras and Taqqu [3]).

Example 4.2. Let $\{g_n\}_{n\in\mathbb{Z}}$ be the spectral kernel of the stationary $S\alpha S$ process $\{X_n\}_{n\in\mathbb{Z}}$ in Example 4.1, and let $b_0, ..., b_{T-1}$ be nonzero distinct real (complex) numbers, $T > 2$ and $b_n = b_{n+T}$ for all $n \in \mathbb{Z}$. Consider the process

$$\{Z_n\}_{n\in\mathbb{Z}} = \left\{ \int_S h_n dM \right\}_{n\in\mathbb{Z}},$$

where $h_n(s) = g_{[[n]]/T} b_{r_n}$ for $n \in \mathbb{Z}$. Then for all $n_1, \ldots, n_k, m \in \mathbb{Z}$, $\theta_1, \ldots, \theta_k \in \mathbb{R}$ ($\mathbb{C}$ in the complex valued case) and $k \in \mathbb{N}$, we have

$$\begin{aligned}
\int_S |\sum_{j=1}^k \theta_j h_{n_j+mT}(s)|^\alpha \mu(ds) &= \int_S |\sum_{j=1}^k \theta_j g_{[[n_j+mT]]/T}(s) b_{r_{n_j+mT}}|^\alpha \mu(ds) \\
&= \int_S |\sum_{j=1}^k \theta_j b_{r_{n_j}} g_{[[n_j]]/T+m}(s)|^\alpha \mu(ds) \\
&= \int_S |\sum_{j=1}^k \theta_j b_{r_{n_j}} g_{[[n_j]]/T}(s)|^\alpha \mu(ds) \\
&= \int_S |\sum_{j=1}^k \theta_j h_{n_j}(s)|^\alpha \mu(ds),
\end{aligned}$$

that is the process $\{Z_n\}_{n\in\mathbb{Z}}$ is PC $S\alpha S$. Note that also here if the process $\{X_n\}_{n\in\mathbb{Z}}$ is generated by the nonsingular flow $\{\phi_n\}_{n\in\mathbb{Z}}$, then the process $\{Y_n\}_{n\in\mathbb{Z}}$ will be generated by the nonsingular flow $\{\phi_{[[n]]}\}_{n\in\mathbb{Z}}$. Also $C_P^g = C_P^h$ and $C_F^g = C_F^h$. Indeed if $s \in C_P^g$, then

$$\exists m = m(s) \in \mathbb{Z} - \{0\} \text{ such that } g_{n+m}(s) = a(m,s) g_n(s) \ \forall n \in \mathbb{Z}, \quad \text{for some } a(m,s) \neq 0.$$

Therefore for all $n \in \mathbb{Z}$

$$\begin{aligned}
h_{n+mT}(s) &= g_{[[n+mT]]/T}(s) b_{r_{n+mT}} = g_{[[n]]/T+m}(s) b_{r_n} \\
&= (a(m,s)) g_{[[n]]/T}(s) b_{r_n} = (a(m,s)) h_n(s),
\end{aligned}$$

that is $s \in C_P^h$. Hence $C_P^g \subset C_P^h$. Also if $s \in C_P^h$, then

$$\exists m = m(s) \in \mathbb{Z} - \{0\} \text{ such that } h_{n+mT}(s) = b(m,s) h_n(s) \ \forall n \in \mathbb{Z}, \quad \text{for some } b(m,s) \neq 0.$$

Therefore

$$g_{[[n+mT]]/T}(s) b_{r_{n+mT}} = b(m,s) g_{[[n]]/T}(s) b_{r_n} \text{ for all } n \in \mathbb{Z},$$

or

$$g_{[[n]]/T+m}(s) = b(m,s) g_{[[n]]/T}(s) \quad \forall n \in \mathbb{Z}.$$

For $n = kT$,

$$g_{k+m}(s) = b(m,s)g_k(s) \ \forall k \in \mathbb{Z},$$

that is $s \in C_P^g$. Hence $C_P^h \subset C_P^g$. Similarly $C_F^g = C_F^h$, $D^g = D^h$ and $C^g = C^h$. Therefore if the stationary process $\{X_n\}_{n\in\mathbb{Z}}$ is $S\alpha S$ mixed moving average, harmonizable, of third or of forth kind, then correspondingly, the PC $S\alpha S$ process $\{Z_n\}_{n\in\mathbb{Z}}$ will be mixed moving average, harmonizable, of third or of forth kind.

In the following example we were motivated by Pipiras and Taqqu [3] Example 6.1.

Example 4.3. Let $\{Y_n\}_{n\in\mathbb{Z}}$ be a stationary process which is defined on the canonical coordinate space $(S, \mathcal{B}, \mu) = (\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \mu)$ (or $(\mathbb{C}^{\mathbb{Z}}, \mathcal{B}(\mathbb{C}^{\mathbb{Z}}), \mu)$ in the complex case) and satisfies $\mu(|Y_n| < c) < 1$ for all $c > 0$, $E_\mu|Y_n|^\alpha < \infty$. Let θ be the shift transformation on S, that is

$$\theta(..., s_{-1}, s_0, s_1, ...) = (..., s_0, s_1, s_2, ...)$$

We know that θ is measure preserving and conservative. Let $V_n \in L^\alpha(S, \mathcal{B}, \mu)$ such that $V_r(s) = s_r$, $r = 0, ..., T-1$, $V_n = V_{n+T}$. Consider the PC $S\alpha S$ process

$$X_n = \int_S f_n(s) M(ds),$$

where $f_n = V_n \circ \phi_{[[n]]}$, $\phi_{[[n]]} = \theta^{[[n]]}$ and M is, respectively, either a real-valued or a complex- valued rotationally invariant $S\alpha S$ random measure on $(S, \mathcal{B})$ with control measure μ. The process $\{X_n\}_{n\in\mathbb{Z}}$ is generated by the flow $\{\phi_{[[n]]}\}_{n\in\mathbb{Z}}$ in the sense of Definition 2.1. The process $\{X_n\}_{n\in\mathbb{Z}}$ is of type $X^{(3)2}$. For this fact we should show that $\mu(C_P) = 0$ or $C_P = \emptyset$ a.e-P. By Definition 2.11, we have

$$C_P = \{s \in S : \ \exists m = m(s) : f_{n+Tm}(s) = a(m,s) f_n(s) \text{ for all } n \in \mathbb{Z} \text{ and for some } a(m,s) \neq 0\}$$

If $s \in C_P$, then

$$\exists m = m(s) \text{ such that } V_0 \circ \phi_{kT+mT}(s) = a(m,s) V_0 \circ \phi_{kT}(s) \text{ for all } k \in \mathbb{Z},$$

or

$$V_0 \circ \phi_{kT+mT}(s) = a(m,s) V_0 \circ \phi_{kT}(s) \text{ for all } k \in \mathbb{Z},$$

and then

$$s_{kT+mT} = a(m,s) s_{kT} \text{ for all } k \in \mathbb{Z}.$$

Following Example 6.1 given by Pipiras and Taqqu [3], and the stationarity of the process $\{Y_{nT}\}_{n\in Z}$ we have $\mu(C_P) = 0$, (the integral in their example becomes a sum here).

Acknowledgments

The authors thank the referee for valuable suggestions which improved the presentation of the paper.

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