Title:

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IMPROVEMENTS OF TWO PRECONDITIONED AOR ITERATIVE METHODS FOR Z-MATRICES

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ABSTRACT. In this paper, we propose two preconditioned AOR iterative methods to solve systems of linear equations whose coefficient matrices are Z-matrix. These methods can be considered as improvements of two previously presented ones in the literature. Finally some numerical experiments are given to show the effectiveness of the proposed preconditioners.
Keywords: System of linear equations, preconditioner, AOR iterative method, Z-matrix.
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1. Introduction

Consider the system of linear equations

$$(1.1) \qquad\qquad Ax = b,$$

where $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is nonsingular matrix and $b \in \mathbb{R}^n$. A basic iterative method to solve Eq. (1.1) can be written as

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \ldots,$$

in which $A = M - N$, where $M, N \in \mathbb{R}^{n \times n}$ and M is nonsingular. It is well-known that this iterative method converges to the solution of Eq. (1.1) for every initial guess x_0 if and only if $\rho(M^{-1}N) < 1$, where $\rho(.)$ refers to the spectral radius of matrix. Let $a_{ii} \neq 0, \ i = 1, 2, \ldots, n$.

Therefore, without loss of generality we can assume that $a_{ii} = 1$, $i = 1, 2, \ldots, n$. In this case, we split A into

(1.2)
$$A = I - L - U,$$

where I is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular matrices, respectively. The accelerated overrelaxation (AOR) iterative method to solve Eq. (1.1) is defined by [5]

$$x^{(k+1)} = \mathcal{L}_{r,w} x^{(k)} + w(I - rL)^{-1} b,$$

in which $\mathcal{L}_{r,w} = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU]$, where w and r are real parameters and $w \neq 0$. For certain values of the parameters w and r the AOR iterative method results in the Jacobi, Gauss-Seidel and the SOR methods [5].

To improve the convergence rate of an iterative method one may apply it to the preconditioned linear system $PAx = Pb$, where the matrix P is called a preconditioner. Several preconditioners have been presented for the basic iterative methods by many authors [2-4, 6-18]. In this paper, we consider the preconditioners $P_{\alpha,\beta} = I + R_{\alpha,\beta}$ and $P^*_{\alpha,\beta} = I + C_{\alpha,\beta}$, where I is the identity matrix and

$$R_{\alpha,\beta} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ -\alpha a_{n1} + \beta & -\alpha a_{n2} + \beta & \ldots & -\alpha a_{nn-1} + \beta & 0 \end{pmatrix},$$

$$C_{\alpha,\beta} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ -\alpha a_{21} + \beta & 0 & \ldots & 0 \\ -\alpha a_{31} + \beta & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\alpha a_{n1} + \beta & 0 & \ldots & 0 \end{pmatrix},$$

in which α and β are nonnegative real numbers. In [11], Milaszewicz has applied the preconditioner $P^*_{1,0}$ to the Gauss-Seidel method. Then, Morimoto et al. have used the preconditioner $P_{1,0}$ to the Gauss-Seidel method [12]. In this paper, we apply the preconditioners $P_{\alpha,\beta}$ and $P^*_{\alpha,\beta}$ to the AOR iterative method and give some comparison theorems.

For convenience, we first present some notations, definitions and pre-liminaries which will be used in this paper. A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is said to be nonnegative and denoted by $A \geq 0$ if $a_{ij} \geq 0$ for all i and

j and A is said to be positive and denoted by $A \gg 0$ if $a_{ij} > 0$ for all i and j.

In the sequel, we recall some definitions, lemmas and theorems for later use.

Theorem 1.1. [1] *(Perron-Frobenius) If $A \geq 0$, then $\rho(A)$ is an eigenvalue of A, and corresponding to $\rho(A)$, A has a nonnegative eigenvector (Perron vector).*

Definition 1.2. A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a *Z-matrix* if $a_{ij} \leq 0$ for $i \neq j$.

Definition 1.3. A Z-matrix A is said to be an *M-matrix* if A is nonsingular and $A^{-1} \geq 0$.

Definition 1.4. Let $A \in \mathbb{R}^{n \times n}$. The representation $A = M - N$ is called a *splitting* of A if M is nonsingular. The splitting $A = M - N$ is called
(a) convergent if $\rho(M^{-1}N) < 1$;
(b) an *M-splitting* of A if M is an M-matrix and $N \geq 0$.

Theorem 1.5. [16, Lemma 1.4] *Let A be a nonnegative matrix.*
(a) If $\alpha x \leq Ax$ for some nonnegative vector $x \neq 0$, then $\alpha \leq \rho(A)$.
(b) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is positive.

Lemma 1.6. [16, Lemma 1.5] *Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*

Lemma 1.7. [16, Lemma 1.6] *Let A be a Z-matrix. Then, A is an M-matrix if and only if there is a positive vector x such that $Ax \gg 0$.*

Lemma 1.8. [15, Lemma 2.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an M-matrix. Then, there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$, $A(\epsilon) = (a_{ij}(\epsilon))$ is also an M-matrix, where*

$$a_{ij}(\epsilon) = \begin{cases} a_{ij}, & \text{if } a_{ij} \neq 0, \\ -\epsilon, & \text{if } a_{ij} = 0. \end{cases}$$

2. **Preconditioned AOR iterative method with $P_{\alpha,\beta}$**

For the sake of simplicity, let $R = R_{1,0}$, $R_\alpha = R_{\alpha,0}$, $P = I + R$ and $P_\alpha = I + R_\alpha$. We first consider the preconditioner P_α to the system

(1.1), where α is a nonnegative real number. In this case, the coefficient matrix of the preconditioned system can be written as

$$\overline{A} = (I + R_\alpha)A = I - L - U + R_\alpha - R_\alpha L - R_\alpha U = \overline{D} - \overline{L} - \overline{U},$$

where $\overline{D}$, $\overline{L}$ and $\overline{U}$ are diagonal, strictly lower triangular and strictly upper triangular matrices, respectively, i.e.,

$$(2.1) \quad \overline{D} = I - E(\alpha), \qquad \overline{L} = L - R_\alpha + R_\alpha L + F(\alpha), \qquad \overline{U} = U,$$

in which $E(\alpha)$ and $F(\alpha)$ are the diagonal and strictly lower triangular parts of $R_\alpha U$, respectively. If $\overline{D} - r\overline{L}$ is nonsingular, the iteration matrix of the AOR iterative method to solve the preconditioned system $P_\alpha A x = P_\alpha b$ is of the form

$$\overline{L}_{r,w} = (\overline{D} - r\overline{L})^{-1}[(1 - w)\overline{D} + (w - r)\overline{L} + w\overline{U}].$$

Let $\beta \geq 0$. If we apply the preconditioner $P_{\alpha,\beta}$ to the system (1.1), then the coefficient matrix of the system can be written as

$$A' = (I + R_{\alpha,\beta})A = I - L - U + R_{\alpha,\beta} - R_{\alpha,\beta}L - R_{\alpha,\beta}U = D' - L' - U',$$

where D', L' and U', are the diagonal, strictly lower triangular and strictly upper triangular matrices, respectively. Assuming $R_\beta = R_{\alpha,\beta} - R_\alpha$, it can be seen that

$$(2.2) \quad D' = \overline{D} - E(\beta), \qquad L' = \overline{L} - R_\beta + R_\beta L + F(\beta), \qquad U' = \overline{U},$$

where $E(\beta)$ and $F(\beta)$ are the diagonal and strictly lower parts of $R_\beta U$, respectively. Now, if $D' - rL'$ is nonsingular, then the iteration matrix of the AOR iterative method with the preconditioner $P_{\alpha,\beta}$ is of the form

$$L'_{r,w} = (D' - rL')^{-1}[(1 - w)D' + (w - r)L' + wU'].$$

Theorem 2.1. *Let A be an M-matrix. If*

$$(1 - \alpha)a_{nj} - \alpha \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{nk}a_{kj} + \beta(1 + \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{kj}) \leq 0,$$

for $j = 1, \ldots, n$, then A' is an M-matrix, too.

Proof. The entries a'_{ij} of $A' = (a'_{ij})$ are given by the expressions

$$a'_{ij} = \begin{cases} a_{ij}, & 1 \leq i < n - 1, \ 1 \leq j \leq n, \\ a_{nj} - \sum_{k=1}^{n-1}(\alpha a_{nk} - \beta)a_{kj}, & i = n, 1 \leq j \leq n. \end{cases}$$

Since A is a Z-matrix, we have $a'_{ij} \leq 0$ for $i \neq n$. Also, for $i = n$ we see that

$$a_{ij} = (1-\alpha)a_{nj} - \alpha \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{nk}a_{kj} + \beta(1 + \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{kj}) \leq 0.$$

Therefore, A' is a Z-matrix. By Lemma 1.7 there is a positive vector x such that $Ax \gg 0$. On the other hand, $A'x = (I + R_{\alpha,\beta})Ax \gg 0$. Again, by Lemma 1.7 we conclude that the matrix A' is an M-matrix. $\qquad \square$

Remark 2.2. *Let A be an M-matrix. If*

$$(1-\alpha)a_{nj} - \alpha \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{nk}a_{kj} \leq 0,$$

for $1 \leq j \leq n$, then $\overline{A}$ is an M-matrix, too.

Proof. It is enough to set $\beta = 0$ in Theorem 2.1. $\qquad \square$

Theorem 2.3. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z-matrix, $0 \leq r \leq w \leq 1$, $w \neq 0$ and*

$$(1-\alpha)a_{nj} - \alpha \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{nk}a_{kj} + \beta(1 + \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{kj}) \leq 0,$$

for $1 \leq j \leq n$.
(1) If $\rho(L_{r,w}) < 1$, then $\rho(L'_{r,w}) \leq \rho(L_{r,w}) < 1$.
(2) If $\rho(L_{r,w}) > 1$ and $1 - \sum_{k=1}^{n-1}(\alpha a_{nk} - \beta)a_{kn} > 0$, then $\rho(L'_{r,w}) \geq \rho(L_{r,w}) > 1$.

Proof. See Theorem 2.6 in [15]. $\qquad \square$

Remark 2.4. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z-matrix, $0 \leq r \leq w \leq 1$, $w \neq 0$ and*

$$(1-\alpha)a_{nj} - \alpha \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{nk}a_{kj} \leq 0,$$

for $1 \leq j \leq n$.
(1) If $\rho(L_{r,w}) < 1$, then $\rho(\overline{L}_{r,w}) \leq \rho(L_{r,w}) < 1$.
(2) If $\rho(L_{r,w}) > 1$ and $1 - \alpha \sum_{k=1}^{n-1} a_{nk}a_{kn} > 0$, then $\rho(\overline{L}_{r,w}) \geq \rho(L_{r,w}) > 1$.

Proof. It is enough to set $\beta = 0$ in Theorem 2.3. $\qquad \square$

In the sequel, we present a comparison theorem concerning the proposed preconditioners.

Theorem 2.5. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z-matrix, $0 \leq r \leq w \leq 1$, $w \neq 0$ and*

$$(1-\alpha)a_{nj}-\alpha \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{nk}a_{kj}+\beta(1+\sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{kj}) \leq 0, \quad (1-\alpha)a_{nj}-\alpha \sum_{\substack{k=1 \\ k \neq j}}^{n-1} a_{nk}a_{kj} \leq 0,$$

for $1 \leq j \leq n$.
(1) If $\rho(L_{r,w}) < 1$, then $\rho(L'_{r,w}) \leq \rho(\overline{L}_{r,w}) < 1$.
(2) If $\rho(L_{r,w}) > 1$ and $1 - \sum_{k=1}^{n-1}(\alpha a_{nk} - \beta)a_{kn} > 0$, then $\rho(L'_{r,w}) \geq \rho(\overline{L}_{r,w}) > 1$.

Proof. Assume that

$$A = M - N = \frac{1}{w}(I - rL) - \frac{1}{w}[(1-w)I + (w-r)L + wU],$$

$$\overline{A} = \overline{M} - \overline{N} = \frac{1}{w}(\overline{D} - r\overline{L}) - \frac{1}{w}[(1-w)\overline{D} + (w-r)\overline{L} + w\overline{U}].$$

It is easy to see that under the assumptions of the theorem $A = M - N$ is an M-splitting. We consider the following two cases.
Case I: If $\rho(L_{r,w}) < 1$, then by Lemma 1.6, A is an M-matrix, and therefore by Remark 2.2 $\overline{A}$ is an M-matrix as well. Hence, $\overline{D} > 0, \overline{L} \geq 0, \overline{U} \geq 0$ and we see that $\overline{A} = \overline{M} - \overline{N}$ is an M-splitting and as a result by Lemma 1.6 we have $\rho(\overline{L}_{r,w}) < 1$. On the other hand, we have

$$\begin{aligned}
\overline{L}_{r,w} &= (\overline{D} - r\overline{L})^{-1}[(1-w)\overline{D} + (w-r)\overline{L} + w\overline{U}] \\
&= (I - r\overline{D}^{-1}\overline{L})^{-1}[(1-w)I + (w-r)\overline{D}^{-1}\overline{L} + w\overline{D}^{-1}\overline{U}] \\
&= [\ I + (r\overline{D}^{-1}\overline{L}) + (r\overline{D}^{-1}\overline{L})^2 + \cdots\ ] \\
&\quad \times [(1-w)I + (w-r)\overline{D}^{-1}\overline{L} + w\overline{D}^{-1}\overline{U}]
\end{aligned}$$

$$(2.3) \qquad \geq [(1-w)I + (w-r)\overline{D}^{-1}\overline{L} + w\overline{D}^{-1}\overline{U}],$$

which shows that the matrix $\overline{L}_{r,w}$ is nonnegative. By Theorem 1.1 there is a nonnegative vector x such that $\overline{L}_{r,w}x = \rho(\overline{L}_{r,w})x$. For the sake of simplicity, let $\lambda = \rho(\overline{L}_{r,w})$. Then

$$[(1-w)\overline{D} + (w-r)\overline{L} + w\overline{U}]x = \lambda(\overline{D} - r\overline{L})x,$$

and

$$(2.4) \qquad wU'x = w\overline{U}x = (\lambda - 1 + w)\overline{D}x + (r - w - \lambda r)\overline{L}x.$$

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We also have

$$\lambda(D' - rL')x = \lambda(1-r)D'x + \lambda r(D' - L')x$$
$$(2.5) \qquad = \lambda(1-r)D'x + \lambda r[\overline{D} - E(\beta) - \overline{L} + R_\beta - R_\beta L - F(\beta)]x.$$

By using Eqs. (2.1), (2.2), (2.4) and (2.5), we have

$$L'_{r,w}x - \lambda x$$

$$= (D' - rL')^{-1}[(1-w)D' + (w-r)L' + wU' - \lambda(D' - rL')]x$$
$$= (D' - rL')^{-1}[(1-w)D' + (w-r)L' + (\lambda - 1 + w)\overline{D}$$
$$\qquad + (r - w - \lambda r)\overline{L} - \lambda(1-r)D' - \lambda r(\overline{D} - E(\beta) - \overline{L}$$
$$\qquad + R_\beta - R_\beta L - F(\beta))]x$$
$$= (D' - rL')^{-1}[(1 - w - \lambda + \lambda r)D' + \lambda(\overline{D} - r\overline{L}) - (1-w)\overline{D}$$
$$\qquad + (w-r)(L' - \overline{L}) - \lambda r(\overline{D} - E(\beta) - \overline{L} + R_\beta - R_\beta L - F(\beta))]x$$
$$= (D' - rL')^{-1}[(1 - w - \lambda + \lambda r)D' + \lambda(\overline{D} - r\overline{L}) - (1-w)\overline{D}$$
$$\qquad + (w-r)(D' - \overline{D} + E(\beta) - R_\beta + R_\beta L + F(\beta)) - \lambda r(\overline{D} - E(\beta)$$
$$\qquad - \overline{L} + R_\beta - R_\beta L - F(\beta))]x$$
$$= (D' - rL')^{-1}[(\lambda - 1)(1-r)(\overline{D} - D') + (w - r + \lambda r)(E(\beta) - R_\beta$$
$$\qquad + R_\beta L + F(\beta))]x$$
$$= (D' - rL')^{-1}[(\lambda - 1)(1-r)(\overline{D} - D')$$
$$\qquad + (w - r + \lambda r)(R_\beta U + R_\beta L - R_\beta)]x$$
$$= (D' - rL')^{-1}[(\lambda - 1)(1-r)(\overline{D} - D') + (r - \lambda r)R_\beta$$
$$\qquad + (\lambda r - r)(R_\beta U + R_\beta L) - wR_\beta + wR_\beta L + wR_\beta \overline{U}]x$$
$$= (D' - rL')^{-1}[(\lambda - 1)(1-r)(\overline{D} - D') + (r - \lambda r)R_\beta$$
$$\qquad + (\lambda r - r)(R_\beta U + R_\beta L) - wR_\beta + wR_\beta L$$
$$\qquad + R_\beta((\lambda - 1 + w)\overline{D} + (r - w - \lambda r)\overline{L})]x$$
$$= (D' - rL')^{-1}[(\lambda - 1)(1-r)E(\beta) - r(\lambda - 1)R_\beta + r(\lambda - 1)R_\beta U$$
$$\qquad + (\lambda - 1)R_\beta \overline{D} + wR_\beta \overline{D} - wR_\beta + (w - r + \lambda r)R_\beta(L - \overline{L})]x.$$

Since $R_\beta E(\alpha) = R_\beta F(\alpha) = R_\beta R_\alpha = 0$, $\overline{D} = I - E(\alpha)$ and $L - \overline{L} = R_\alpha - R_\alpha L - F(\alpha)$, we see that

$$(2.6) \quad L'_{r,w}x - \lambda x = (\lambda - 1)(D' - rL')^{-1}[(1-r)E(\beta) + (1-r)R_\beta + rR_\beta U].$$

Let $B = (1 - r)E(\beta) + (1 - r)R_\beta + rR_\beta U$. Obviously, $B \geq 0$. On the other hand, by Theorem 2.1 A' is also an M-matrix and therefore

$$(D' - rL')^{-1} \geq 0.$$

Now, if A is an irreducible matrix, then by Eq. (2.3) we have

$$\overline{L}_{r,w} \geq (1 - w)I + w(1 - r)\overline{D}^{-1}L + w\overline{D}^{-1}U.$$

This relation shows that if $0 \leq r < 1$ (note that $w \neq 0$), then $\overline{L}_{r,w}$ is also irreducible. Hence, by Lemma 4 in [15] we have $\lambda > 0$ and the Perron vector x is positive. Now, from Eq. (2.6) we conclude that

$$L'_{r,w}x \leq \lambda x,$$

and by using Theorem 1.5 we get

$$\rho(L'_{r,w}) \leq \lambda = \rho(\overline{L}_{r,w}).$$

If $r = 1$, then $w = r = 1$ and

$$\rho(L'_{1,1}) = \lim_{r \to 1^-} \rho(L'_{r,1}) \leq \lim_{r \to 1^-} \rho(\overline{L}_{r,1}) = \rho(\overline{L}_{1,1}) < 1.$$

On the other hand, if A is a reducible matrix, then by Lemma 1.8 for every small enough $\epsilon > 0$ the matrix $A(\epsilon)$ is an irreducible M-matrix. Hence, we have

$$\rho(L'_{r,w}) = \lim_{\epsilon \to 0^+} \rho(L'_{r,w}(\epsilon)) \leq \lim_{\epsilon \to 0^+} \rho(\overline{L}_{r,w}(\epsilon)) = \rho(\overline{L}_{r,w}) < 1.$$

This completes the first part of the theorem.
Case II: Let $\rho(L_{r,w}) > 1$. By Remark (2.4), we have $\rho(\overline{L}_{r,w}) \geq \rho(L_{r,w}) > 1$. Then by the assumption $1 - \sum_{k=1}^{n-1}(\alpha a_{nk} - \beta)a_{kn} > 0$ of the theorem we see that A' is a Z-matrix and therefore $D' - rL'$ is an M-matrix and by Eq. (2.6) and Theorem 1.5, we get

$$\rho(L'_{r,w}) \geq \rho(\overline{L}_{r,w}) > 1.$$

This proves the second part of the theorem. $\square$

3. Preconditioned AOR iterative method with $P^*_{\alpha,\beta}$

For simplicity, we assume that $C = C_{1,0}$, $C_\alpha = C_{\alpha,0}$, $P^* = I + C$ and $P^*_\alpha = I + C_\alpha$. As we mentioned Milaszewicz in [11] applied preconditioned Gauss-Seidel method in conjunction with the preconditioner P^*

to solve Eq. (1.1). We first consider the preconditioner P_α^*. In this case, $\widetilde{A} = P_\alpha^* A$ can be written as

$$\widetilde{A} = I - L - U + C_\alpha - C_\alpha U = \widetilde{D} - \widetilde{L} - \widetilde{U},$$

where $\widetilde{D}$, $\widetilde{L}$ and $\widetilde{U}$ are, respectively, diagonal, strictly lower triangular and strictly upper triangular matrices and

(3.1) $\widetilde{D} = I - E_2(\alpha), \qquad \widetilde{L} = L - C_\alpha + F_2(\alpha), \qquad \widetilde{U} = U + G_2(\alpha),$

in which $E_2(\alpha)$, $F_2(\alpha)$ and $G_2(\alpha)$ are, respectively, diagonal, strictly lower triangular and strictly upper triangular matrices such that

$$C_\alpha U = G_2(\alpha) + E_2(\alpha) + F_2(\alpha).$$

If $\widetilde{D} - r\widetilde{L}$ is nonsingular, then the iteration matrix of the AOR method with the preconditioner P_α^* can be expressed as

$$\widetilde{L}_{r,w} = (\widetilde{D} - r\widetilde{L})^{-1}[(1-w)\widetilde{D} + (w-r)\widetilde{L} + w\widetilde{U}].$$

Now, we use the preconditioner $P_{\alpha,\beta}^*$. In this case, $\widehat{A} = P_{\alpha,\beta}^* A$ can be decomposed as

$$\widehat{A} = I - L - U + C_{\alpha,\beta} - C_{\alpha,\beta} U = \widehat{D} - \widehat{L} - \widehat{U},$$

where $\widehat{D}$, $\widehat{L}$ and $\widehat{U}$ are, respectively, diagonal, strictly lower triangular and strictly upper triangular matrices. If we assume $C_\beta = C_{\alpha,\beta} - C_\alpha$, then

(3.2) $\widehat{D} = \widetilde{D} - E_2(\beta), \qquad \widehat{L} = \widetilde{L} - C_\beta + F_2(\beta), \qquad \widehat{U} = \widetilde{U} + G_2(\beta),$

in which $E_2(\beta)$, $F_2(\beta)$ and $G_2(\beta)$ are, respectively, diagonal, strictly lower triangular and strictly upper triangular matrices such that

$$C_\beta U = G_2(\beta) + E_2(\beta) + F_2(\beta).$$

If $(\widehat{D} - r\widehat{L})$ is nonsigular, then the iteration matrix of the AOR iterative method with the preconditioner $P_{\alpha,\beta}^*$ can be written as

$$\widehat{L}_{r,w} = (\widehat{D} - r\widehat{L})^{-1}[(1-w)\widehat{D} + (w-r)\widehat{L} + w\widehat{U}].$$

Theorem 3.1. *Let A be an M-matrix. If*

$$(1 - \alpha)a_{i1} + \beta \le 0,$$

for $2 \le i \le n$, then $\widehat{A}$ is an M-matrix, too.

Proof. Let A be an M-matrix and $\widehat{A} = (\widehat{a}_{ij})$. Then, we have

$$\widehat{a}_{ij} = a_{ij} + (-\alpha a_{i1} + \beta)a_{1j} = a_{ij} - \alpha a_{i1}a_{1j} + \beta a_{1j},$$

for $2 \leq i \leq n$. Since A is a Z-matrix, obviously we have $\widehat{a}_{ij} \leq 0$, for $i \neq j$ and $j \neq 1$. For $j = 1$, we see that

$$\widehat{a}_{ij} = (1 - \alpha)a_{i1} + \beta \leq 0.$$

Hence, $\widehat{A}$ is a Z-matrix. By Lemma 1.7, there exists a positive vector x such that $Ax \gg 0$. On the other hand, we have $\widehat{A}x = (I + C_{\alpha,\beta})Ax \gg 0$. Therefore, by Lemma 1.7, $\widehat{A}$ is also an M-matrix. $\qquad\square$

Remark 3.2. *Let A be an M-matrix. If*

$$(1 - \alpha)a_{i1} \leq 0,$$

for $2 \leq i \leq n$, then $\widetilde{A}$ is an M-matrix, too.

Proof. It is enough to set $\beta = 0$ in Theorem 3.1. $\qquad\square$

Theorem 3.3. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z-matrix, $0 \leq r \leq w \leq 1$, $w \neq 0$ and*

$$(1 - \alpha)a_{i1} + \beta \leq 0,$$

for $2 \leq i \leq n$.
(1) If $\rho(L_{r,w}) < 1$, then $\rho(\widehat{L}_{r,w}) \leq \rho(L_{r,w}) < 1$.
(2) If $\rho(L_{r,w}) > 1$ and $1 - \alpha a_{i1}a_{1i} + \beta a_{1i} > 0$, then $\rho(\widehat{L}_{r,w}) \geq \rho(L_{r,w}) > 1$.

Proof. See Theorem 2.6 in [15]. $\qquad\square$

Remark 3.4. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z-matrix, $0 \leq r \leq w \leq 1$, $w \neq 0$ and*

$$(1 - \alpha)a_{i1} \leq 0,$$

for $2 \leq i \leq n$.
(1) If $\rho(L_{r,w}) < 1$, then $\rho(\widetilde{L}_{r,w}) \leq \rho(L_{r,w}) < 1$.
(2) If $\rho(L_{r,w}) > 1$ and $1 - \alpha a_{i1}a_{1i} > 0$, then $\rho(\widetilde{L}_{r,w}) \geq \rho(L_{r,w}) > 1$.

Proof. It is enough to set $\beta = 0$ in Theorem 3.3. $\qquad\square$

In continuation, we present a comparison theorem concerning the proposed preconditioners.

Theorem 3.5. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z-matrix, $0 \leq r \leq w \leq 1$, $w \neq 0$ and*

$$(1 - \alpha)a_{i1} + \beta \leq 0,$$

for $2 \leq i \leq n$.
(1) If $\rho(L_{r,w}) < 1$, *then* $\rho(\widehat{L}_{r,w}) \leq \rho(\widetilde{L}_{r,w}) < 1$.
(2) If $\rho(L_{r,w}) > 1$ *and* $1 - \alpha a_{i1} a_{1i} + \beta a_{1i} > 0$, *then* $\rho(\widehat{L}_{r,w}) \geq \rho(\widetilde{L}_{r,w}) > 1$.

Proof. The proof is quite similar to that of Theorem 2.5 and is omitted here.   □

4. **Numerical examples**

In this section, to evaluate the efficiency of the preconditioners presented in this paper we compare the spectral radii of the iteration matrix of preconditioned AOR iterative methods in conjunction with the proposed preconditioners for two matrices.

Example 4.1. In this example, we consider

$$A = \begin{pmatrix} 1 & -0.1 & -0.1 & -0.1 \\ -0.5 & 1 & 0 & -0.1 \\ -0.5 & 0 & 1 & -0.4 \\ -0.7 & -0.2 & -0.4 & 1 \end{pmatrix},$$

as the coefficient matrix of the system (1.1). For $w = 0.8$, $\alpha = 0.6$, $\beta = 0.1$ and $r = 0.1, 0.2, \ldots, 0.9$, it is easy to verify that all the assumptions of Theorems 2.5 and 3.5 are satisfied. Hence, we expect that $P_{\alpha,\beta}$ (resp. $P_{\alpha,\beta}^*$) is superior to P_α (resp. P_α^*) which is itself better than the preconditioner $P = I$. To confirm this claim we report the spectral radii of the iteration matrix of the preconditioned AOR methods with the proposed preconditioners in TABLE 1. As we observe, the results confirm the claim. For more investigation, let $w = 0.7$, $\alpha = 0.3$, $\beta = 0.15$ and $r = 0.1, 0.2, \ldots, 0.9$. We report the results in TABLE 2. As we see, the results give the same conclusion as TABLE 1.

Example 4.2. Let

$$A = \begin{pmatrix} 1 & -0.5 & -0.2 & 0 & -1 \\ -0.2 & 1 & 0 & -0.2 & -0.2 \\ -0.2 & -0.2 & 1 & -0.2 & -0.3 \\ -0.4 & 0 & -0.1 & 1 & -0.1 \\ -0.3 & -0.1 & -0.1 & -0.1 & 1 \end{pmatrix}.$$

Similar to the previous example we present the spectral radii of the iteration matrix of the preconditioned AOR methods with the proposed

TABLE 1. Comparison of spectral radii of the AOR method with preconditioners $P = I, P_\alpha, P_{\alpha,\beta}, P_\alpha^*, P_{\alpha,\beta}^*$ with $w = 0.8$, $\alpha = 0.6$ and $\beta = 0.1$ for different values of r for Example 4.1.

r	$P = I$	$P = P_\alpha$	$P = P_{\alpha,\beta}$	$P = P_\alpha^*$	$P = P_{\alpha,\beta}^*$
0.1	0.6873	0.6125	0.5697	0.6375	0.6188
0.2	0.6760	0.6004	0.5579	0.6255	0.6066
0.3	0.6636	0.5871	0.5450	0.6123	0.5933
0.4	0.6498	0.5725	0.5309	0.5977	0.5787
0.5	0.6344	0.5563	0.5152	0.5815	0.5623
0.6	0.6169	0.5379	0.4976	0.5631	0.5438
0.7	0.5968	0.5167	0.4773	0.5419	0.5224
0.8	0.5731	0.4918	0.4535	0.5167	0.4971
0.9	0.5440	0.4609	0.4240	0.4856	0.4656

TABLE 2. Comparison of spectral radii of the AOR method with preconditioners $P = I, P_\alpha, P_{\alpha,\beta}, P_\alpha^*, P_{\alpha,\beta}^*$ with $w = 0.7$, $\alpha = 0.3$ and $\beta = 0.15$ for different values of r for Example 4.1.

r	$P = I$	$P = P_\alpha$	$P = P_{\alpha,\beta}$	$P = P_\alpha^*$	$P = P_{\alpha,\beta}^*$
0.1	0.7568	0.7311	0.6909	0.7393	0.7214
0.2	0.7480	0.7219	0.6818	0.7303	0.7122
0.3	0.7383	0.7119	0.6719	0.7203	0.7020
0.4	0.7276	0.7008	0.6610	0.7093	0.6909
0.5	0.7156	0.6884	0.6488	0.6970	0.6784
0.6	0.7021	0.6744	0.6352	0.6830	0.6643
0.7	0.6864	0.6583	0.6195	0.6669	0.6480
0.8	0.6679	0.6392	0.6011	0.6480	0.6288
0.9	0.6453	0.6158	0.5784	0.6246	0.6051

preconditioners in TABLES 3 and 4 for two sets of parameters. In the first set we consider $w = 0.9$, $\alpha = 0.3$, $\beta = 0.1$ and $r = 0.1, 0.2, \ldots, 0.9$ and for the second set $w = 0.9$, $\alpha = 0.4$, $\beta = 0.12$ and $r = 0.1, 0.2, \ldots, 0.9$. Both of TABLES 3 and 4 show that the preconditioner $P_{\alpha,\beta}$ (resp. $P_{\alpha,\beta}^*$) is better than P_α (resp. P_α^*) which is itself better than the preconditioner $P = I$.

TABLE 3. Comparison of spectral radii of the AOR method with preconditioners $P = I, P_\alpha, P_{\alpha,\beta}, P_\alpha^*, P_{\alpha,\beta}^*$ with $w = 0.9$, $\alpha = 0.3$ and $\beta = 0.1$ for different values of r for Example 4.1.

r	$P = I$	$P = P_\alpha$	$P = P_{\alpha,\beta}$	$P = P_\alpha^*$	$P = P_{\alpha,\beta}^*$
0.1	0.8971	0.8846	0.8523	0.8818	0.8554
0.2	0.8920	0.8789	0.8457	0.8761	0.8487
0.3	0.8864	0.8727	0.8385	0.8698	0.8414
0.4	0.8801	0.8657	0.8304	0.8627	0.8332
0.5	0.8730	0.8578	0.8213	0.8547	0.8240
0.6	0.8649	0.8488	0.8111	0.8456	0.8135
0.7	0.8556	0.8384	0.7993	0.8352	0.8015
0.8	0.8447	0.8262	0.7856	0.8230	0.7875
0.9	0.8317	0.8116	0.7694	0.8084	0.7708

TABLE 4. Comparison of spectral radii of the AOR method with preconditioners $P = I, P_\alpha, P_{\alpha,\beta}, P_\alpha^*, P_{\alpha,\beta}^*$ with $w = 0.9$, $\alpha = 0.4$ and $\beta = 0.12$ for different values of r for Example 4.1.

r	$P = I$	$P = P_\alpha$	$P = P_{\alpha,\beta}$	$P = P_\alpha^*$	$P = P_{\alpha,\beta}^*$
0.1	0.8971	0.8797	0.8342	0.8756	0.8376
0.2	0.8920	0.8738	0.8271	0.8696	0.8304
0.3	0.8864	0.8673	0.8193	0.8630	0.8224
0.4	0.8801	0.8601	0.8106	0.8556	0.8135
0.5	0.8730	0.8518	0.8009	0.8473	0.8035
0.6	0.8649	0.8425	0.7900	0.8378	0.7921
0.7	0.8556	0.8317	0.7774	0.8269	0.7791
0.8	0.8447	0.8190	0.7628	0.8142	0.7640
0.9	0.8317	0.8039	0.7456	0.7990	0.7459

5. Conclusion

We have improved two known preconditioned AOR iterative methods. Some comparison theorems have been given to show the improvement of the preconditioners. Some numerical examples have been presented to validate the presented theoretical results. Numerical results show that

the proposed preconditioners are more effective than the primary version of them.

Acknowledgments

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