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OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE PREINVEK

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ABSTRACT. In this paper, making use of a new identity, we establish new inequalities of Ostrowski type for the class of preinvex functions and gave some midpoint type inequalities.

Keywords: Ostrowski type inequalities, preinvex function, condition c.

MSC(2010): Primary: 26D10; Secondary: 26D15, 26A51.

1. Introduction and preliminaries

Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the interior of I , and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then the following inequality holds:

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right].$$

This result is well known in the literature as the Ostrowski's inequality [10, p. 469]. For recent results and generalizations concerning Ostrowski's inequality see [1], [5] and the references therein.

Definition 1.1. *The function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the following inequality holds:*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

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for all $x, y \in [a, b]$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The following theorem contains Hadamard type inequality for M-Lipschitzian functions. (see [3]).

Theorem 1.2. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an M-Lipschitzian mapping on I , and $a, b \in I$ with $a < b$. Then we have the inequality:*

$$(1.2) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq M \frac{(b-a)}{4}.$$

In [6], and in [7] U.S. Kirmaci proved the following theorems.

Theorem 1.3. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If the mapping $|f'|$ is convex on $[a, b]$, then we have*

$$(1.3) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{8} (|f'(a)| + |f'(b)|)$$

Theorem 1.4. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then we have*

$$(1.4) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} \\ \times \left\{ \left(3|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(3|f'(b)|^{\frac{p}{p-1}} + |f'(a)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \right\}.$$

Theorem 1.5. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then we have*

$$(1.5) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} (|f'(a)| + |f'(b)|).$$

Theorem 1.6. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the mapping $|f'|^p$ is convex on $[a, b]$, then

$$(1.6) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left(\frac{3^{1-\frac{1}{p}}}{8}\right) (b-a) (|f'(a)| + |f'(b)|)$$

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [4]. Weir and Mond [13] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. Pini [12] introduced the concept of prequasi-invex function as a generalization of invex functions. Later, Mohan and Neogy [9] obtained some properties of generalized preinvex functions. Noor [11] established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions.

The aim of this paper is to establish some Ostrowski type inequalities for functions whose derivatives in absolute value are preinvex. Now we recall some notions in invexity analysis which will be used through the paper (see [2, 8, 14] and references therein)

Let $f : A \rightarrow \mathbb{R}$ and $\eta : A \times A \rightarrow \mathbb{R}$, where A is a nonempty set in \mathbb{R}^n , be continuous functions.

Definition 1.7. The set $A \subset \mathbb{R}^n$ is said to be invex with respect to $\eta(\cdot, \cdot)$, if for every $x, y \in A$ and $t \in [0, 1]$,

$$x + t\eta(y, x) \in A.$$

The invex set A is also called a η -connected set.

It is obvious that every convex set is invex with respect to $\eta(y, x) = y - x$, but there exist invex sets which are not convex [2].

Definition 1.8. The function f on the invex set A is said to be preinvex with respect to η if

$$f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in A, \quad t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

Note that every convex function is a preinvex function, but the converse is not true [8]. For example $f(x) = -|x|$, $x \in \mathbb{R}$, is not a convex

function, but it is a preinvex function with respect to

$$\eta(x, y) = \begin{cases} x - y, & \text{if } xy \geq 0 \\ y - x, & \text{if } xy < 0 \end{cases} .$$

We also need the following assumption regarding the function η which is due to Mohan and Neogy [9]:

Condition C: Let $A \subset \mathbb{R}^n$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$. For any $x, y \in A$ and any $t \in [0, 1]$,

$$\begin{aligned} \eta(y, y + t\eta(x, y)) &= -t\eta(x, y) \\ \eta(x, y + t\eta(x, y)) &= (1 - t)\eta(x, y). \end{aligned}$$

Note that for any $x, y \in A$ and any $t_1, t_2 \in [0, 1]$ from condition C, we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).$$

There are many vector functions that satisfy the condition C [8], besides the trivial case $\eta(x, y) = x - y$. For example let $A = \mathbb{R} \setminus \{0\}$ and

$$\eta(x, y) = \begin{cases} x - y, & \text{if } x > 0, y > 0 \\ x - y, & \text{if } x < 0, y < 0 \\ -y, & \text{otherwise.} \end{cases}$$

Then A is an invex set and η satisfies the condition C.

2. Main results

Lemma 2.1. *Let $A \subset \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a + \eta(b, a)]$, then the following equality holds:*

$$\begin{aligned} & f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \\ &= \eta(b, a) \left(\int_0^{\frac{x-a}{\eta(b, a)}} t f'(a + t\eta(b, a)) dt + \int_{\frac{x-a}{\eta(b, a)}}^1 (t - 1) f'(a + t\eta(b, a)) dt \right) \end{aligned}$$

for all $x \in [a, a + \eta(b, a)]$.

Since $A \subset \mathbb{R}$ is an invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$, for all $t \in [0, 1]$ we have $a + t\eta(b, a) \in A$. A simple proof of the equality can be given by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

Theorem 2.2. *Let $A \subset \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function and $|f'|$ is preinvex function on A . If f' is integrable on $[a, a + \eta(b, a)]$, then the following inequality holds:*

$$(2.1) \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \leq \frac{\eta(b, a)}{6} \\ \times \left\{ \left[3 \left(\frac{x-a}{\eta(b, a)} \right)^2 - 2 \left(\frac{x-a}{\eta(b, a)} \right)^3 + 2 \left(\frac{a+\eta(b, a)-x}{\eta(b, a)} \right)^3 \right] |f'(a)| \right. \\ \left. + \left[1 - 3 \left(\frac{x-a}{\eta(b, a)} \right)^2 + 4 \left(\frac{x-a}{\eta(b, a)} \right)^3 \right] |f'(b)| \right\}$$

for all $x \in [a, a + \eta(b, a)]$. The constant $\frac{1}{6}$ is best possible in the sense that it cannot be replaced by a smaller value.

Proof. By Lemma 2.1 and since $|f'|$ is preinvex, we have

$$\left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \\ \leq \eta(b, a) \left\{ \int_0^{\frac{x-a}{\eta(b, a)}} t |f'(a + t\eta(b, a))| dt + \int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) |f'(a + t\eta(b, a))| dt \right\} \\ \leq \eta(b, a) \left\{ \int_0^{\frac{x-a}{\eta(b, a)}} t [(1-t) |f'(a)| + t |f'(b)|] dt \right.$$

$$\begin{aligned}
 & \left. + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) [(1-t)|f'(a)| + t|f'(b)|] dt \right\} \\
 \leq & \eta(b,a) \left\{ \left[\frac{1}{2} \left(\frac{x-a}{\eta(b,a)} \right)^2 - \frac{1}{3} \left(\frac{x-a}{\eta(b,a)} \right)^3 + \frac{1}{3} \left(\frac{a+\eta(b,a)-x}{\eta(b,a)} \right)^3 \right] |f'(a)| \right. \\
 & \left. + \left[\frac{1}{6} - \frac{1}{2} \left(\frac{x-a}{\eta(b,a)} \right)^2 + \frac{2}{3} \left(\frac{x-a}{\eta(b,a)} \right)^3 \right] |f'(b)| \right\}
 \end{aligned}$$

where we have used the fact that

$$\begin{aligned}
 & \int_0^{\frac{x-a}{\eta(b,a)}} t(1-t) dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^2 dt \\
 = & \frac{1}{2} \left(\frac{x-a}{\eta(b,a)} \right)^2 - \frac{1}{3} \left(\frac{x-a}{\eta(b,a)} \right)^3 + \frac{1}{3} \left(\frac{a+\eta(b,a)-x}{\eta(b,a)} \right)^3
 \end{aligned}$$

and

$$\int_0^{\frac{x-a}{\eta(b,a)}} t^2 dt + \int_{\frac{x-a}{\eta(b,a)}}^1 t(1-t) dt = \frac{1}{6} - \frac{1}{2} \left(\frac{x-a}{\eta(b,a)} \right)^2 + \frac{2}{3} \left(\frac{x-a}{\eta(b,a)} \right)^3.$$

To prove that the constant $\frac{1}{6}$ is the best possible, let us assume that (2.1) holds with constant $K > 0$, i.e.,

$$\begin{aligned}
 & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq K \cdot \eta(b,a) \\
 & \times \left\{ \left[3 \left(\frac{x-a}{\eta(b,a)} \right)^2 - 2 \left(\frac{x-a}{\eta(b,a)} \right)^3 + 2 \left(\frac{a+\eta(b,a)-x}{\eta(b,a)} \right)^3 \right] |f'(a)| \right. \\
 & \left. + \left[1 - 3 \left(\frac{x-a}{\eta(b,a)} \right)^2 + 4 \left(\frac{x-a}{\eta(b,a)} \right)^3 \right] |f'(b)| \right\}.
 \end{aligned}$$

Let $f(x) = x$, and then set $x = a + \eta(b, a)$. We get

$$\frac{\eta(b,a)}{2} \leq 3K \cdot \eta(b,a),$$

which gives $K \geq \frac{1}{6}$ and so the proof is complete. □

Remark 2.3. Suppose that all the assumptions of Theorem 2.2 are satisfied.

(a) If we choose $\eta(b, a) = b - a$ and $x = \frac{2a + \eta(b, a)}{2}$, then we obtain the following inequality which is the same as inequality (1.3)

$$(2.2) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|).$$

(b) In (a) with $|f'(x)| \leq M$, $M > 0$, we get the following inequality which is the same as inequality (1.2)

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq M \frac{(b-a)}{4}.$$

(c) If the mapping η satisfies the condition C, then by use of the preinvexity of $|f'|$ we get

$$(2.3) \quad \begin{aligned} |f'(a + t\eta(b, a))| &= |f'(a + \eta(b, a) + (1-t)\eta(a, a + \eta(b, a)))| \\ &\leq t|f'(a + \eta(b, a))| + (1-t)|f'(a)|. \end{aligned}$$

for every $t \in [0, 1]$.

Using inequality (2.3) in the proof of Theorem 2.2, inequality (2.1) becomes the following inequality:

$$(2.4) \quad \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \leq \frac{\eta(b, a)}{6} \\ \times \left\{ \left[3 \left(\frac{x-a}{\eta(b, a)} \right)^2 - 2 \left(\frac{x-a}{\eta(b, a)} \right)^3 + 2 \left(\frac{a + \eta(b, a) - x}{\eta(b, a)} \right)^3 \right] |f'(a)| \right. \\ \left. + \left[1 - 3 \left(\frac{x-a}{\eta(b, a)} \right)^2 + 4 \left(\frac{x-a}{\eta(b, a)} \right)^3 \right] |f'(a + \eta(b, a))| \right\}$$

for all $x \in [a, a + \eta(b, a)]$. We note that by use of the preinvexity of $|f'|$ we have

$$|f'(a + \eta(b, a))| \leq |f'(b)|.$$

Therefore, inequality (2.4) is better than inequality (2.1).

Theorem 2.4. Let $A \subset \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $|f'|^q$ is preinvex function on

$[a, a + \eta(b, a)]$ for some fixed $q > 1$. If f' is integrable on $[a, a + \eta(b, a)]$ and η satisfies Condition C, then for each $x \in [a, a + \eta(b, a)]$ the following inequality holds:

$$(2.5) \quad \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\ \times \left\{ \frac{(x-a)^2}{\eta(b, a)} \left(\frac{|f'(a)|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} + \frac{(a+\eta(b, a)-x)^2}{\eta(b, a)} \left(\frac{|f'(a+\eta(b, a))|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} \right\}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We first note that if $|f'|^q$ is a preinvex function on $[a, a + \eta(b, a)]$ and the mapping η satisfies Condition C, then for every $t \in [0, 1]$,

$$(2.6) \quad |f'(a + t\eta(b, a))|^q \leq t |f'(a + \eta(b, a))|^q + (1-t) |f'(a)|^q$$

and similarly

$$(2.7) \quad |f'(a + (1-t)\eta(b, a))|^q = |f'(a + \eta(b, a) + t\eta(a, a + \eta(b, a)))|^q \\ \leq (1-t) |f'(a + \eta(b, a))|^q + t |f'(a)|^q.$$

From inequalities (2.6) and (2.7)

$$(2.8) \quad |f'(a + t\eta(b, a))|^q + |f'(a + (1-t)\eta(b, a))|^q \leq |f'(a)|^q + |f'(a + \eta(b, a))|^q.$$

Then integrating inequality (2.8) with respect to t over $[0, 1]$, we obtain

$$(2.9) \quad \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} |f'(t)|^q dt \leq \frac{|f'(a)|^q + |f'(a + \eta(b, a))|^q}{2}$$

From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}
& \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \\
& \leq \eta(b, a) \left(\int_0^{\frac{x-a}{\eta(b, a)}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b, a)}} |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\
& \quad + \eta(b, a) \left(\int_{\frac{x-a}{\eta(b, a)}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b, a)}}^1 |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\
& \leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \frac{(x-a)^2}{\eta(b, a)} \left(\frac{|f'(a)|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \frac{(a + \eta(b, a) - x)^2}{\eta(b, a)} \left(\frac{|f'(a + \eta(b, a))|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

where we use the fact that

$$\int_0^{\frac{x-a}{\eta(b, a)}} t^p dt = \frac{1}{p+1} \left(\frac{x-a}{\eta(b, a)} \right)^{p+1}, \quad \int_{\frac{x-a}{\eta(b, a)}}^1 (1-t)^p dt = \frac{1}{p+1} \left(\frac{a + \eta(b, a) - x}{\eta(b, a)} \right)^{p+1},$$

and by (2.9)

$$\begin{aligned}
& \int_0^{\frac{x-a}{\eta(b, a)}} |f'(a + t\eta(b, a))|^q dt \leq \frac{x-a}{\eta(b, a)} \left(\frac{|f'(a)|^q + |f'(x)|^q}{2} \right), \\
& \int_{\frac{x-a}{\eta(b, a)}}^1 |f'(a + t\eta(b, a))|^q dt \leq \frac{a + \eta(b, a) - x}{\eta(b, a)} \left(\frac{|f'(a + \eta(b, a))|^q + |f'(x)|^q}{2} \right).
\end{aligned}$$

□

Corollary 2.5. *Suppose that all the assumptions of Theorem 2.4 are satisfied. If we choose $|f'(x)| \leq M$, $M > 0$, for each $x \in [a, a + \eta(b, a)]$,*

then we have

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} M \left(\frac{(x-a)^2 + (a+\eta(b,a)-x)^2}{\eta(b,a)} \right).$$

Corollary 2.6. *Suppose that all the assumptions of Theorem 2.4 are satisfied. Taking $x = \frac{2a+\eta(b,a)}{2}$ in inequality (2.5) and $t = \frac{1}{2}$ in inequality (2.6) we obtain*

$$\begin{aligned} & \left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \\ & \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ \frac{\eta(b,a)}{4} \left(\frac{3|f'(a)|^q + |f'(a+\eta(b,a))|^q}{4} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \frac{\eta(b,a)}{4} \left(\frac{3|f'(a+\eta(b,a))|^q + |f'(a)|^q}{4} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.7. *In Corollary 2.6, if we take $\eta(b,a) = b - a$, then we have the following inequality which is the same as inequality (1.4).*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u)du \right| \\ & \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} \left\{ (3|f'(a)|^q + |f'(b)|^q)^{\frac{1}{q}} + (3|f'(b)|^q + |f'(a)|^q)^{\frac{1}{q}} \right\} \end{aligned}$$

Let $a_1 = 3|f'(a)|^q$, $b_1 = |f'(b)|^q$, $a_2 = 3|f'(b)|^q$, $b_2 = |f'(a)|^q$. Here $0 < \frac{1}{q} < 1$, for $q > 1$. Using the fact that

$$(2.10) \quad \sum_{k=1}^n (a_k + b_k)^s \leq \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s$$

for $(0 \leq s < 1)$, $a_1, a_2, \dots, a_n \geq 0$, $b_1, b_2, \dots, b_n \geq 0$, we obtain the following inequality which is the same as inequality (1.5).

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} (|f'(a)| + |f'(b)|)$$

Theorem 2.8. *Let $A \subset \mathbb{R}$ be an open invex subset with respect to η : $A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is*

a differentiable function and $|f'|^q$ is preinvex function on $[a, a + \eta(b, a)]$ for some fixed $q \geq 1$. If f' is integrable on $[a, a + \eta(b, a)]$, then the following inequality holds:

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \leq \eta(b, a) \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \\ & \times \left\{ \left(\frac{x-a}{\eta(b, a)} \right)^{2(1-\frac{1}{q})} \left[\frac{(x-a)^2 (3\eta(b, a) - 2x + 2a)}{6\eta^3(b, a)} |f'(a)|^q \right. \right. \\ & \quad \left. \left. + \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 |f'(b)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{a+\eta(b, a) - x}{\eta(b, a)} \right)^{2(1-\frac{1}{q})} \left[\frac{1}{3} \left(\frac{a+\eta(b, a) - x}{\eta(b, a)} \right)^3 |f'(a)|^q \right. \right. \\ & \quad \left. \left. + \left(\frac{1}{6} + \frac{(x-a)^2 (2x - 3\eta(b, a) - 2a)}{6\eta^3(b, a)} \right) |f'(b)|^q \right]^{\frac{1}{q}} \right\} \end{aligned}$$

for each $x \in [a, a + \eta(b, a)]$.

Proof. By Lemma 2.1, inequality (2.3) and using the well known power mean inequality, we have

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \\ & \leq \eta(b, a) \left(\int_0^{\frac{x-a}{\eta(b, a)}} t dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{x-a}{\eta(b, a)}} t |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\ & \quad + \eta(b, a) \left(\int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\ & \leq \eta(b, a) \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ \left(\frac{x-a}{\eta(b, a)} \right)^{2(1-\frac{1}{q})} \left[\frac{(x-a)^2 (3\eta(b, a) - 2x + 2a)}{6\eta^3(b, a)} |f'(a)|^q \right. \right. \\ & \quad \left. \left. + \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 |f'(b)|^q \right]^{\frac{1}{q}} + \left(\frac{a+\eta(b, a) - x}{\eta(b, a)} \right)^{2(1-\frac{1}{q})} \left[\frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 |f'(a)|^q \right. \right. \\ & \quad \left. \left. + \left(\frac{1}{6} + \frac{(x-a)^2 (2x - 3\eta(b, a) - 2a)}{6\eta^3(b, a)} \right) |f'(b)|^q \right]^{\frac{1}{q}} \right\} \end{aligned}$$

where we use the fact that

$$\int_0^{\frac{x-a}{\eta(b,a)}} t dt = \frac{1}{2} \left(\frac{x-a}{\eta(b,a)} \right)^2,$$

$$\begin{aligned} & \int_0^{\frac{x-a}{\eta(b,a)}} t |f'(a + t\eta(b,a))|^q dt \\ \leq & \frac{(x-a)^2 (3\eta(b,a) - 2x + 2a)}{6\eta^3(b,a)} |f'(a)|^q + \frac{1}{3} \left(\frac{x-a}{\eta(b,a)} \right)^3 |f'(b)|^q, \\ & \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) dt = \frac{1}{2} \left(\frac{a + \eta(b,a) - x}{\eta(b,a)} \right)^2, \end{aligned}$$

and

$$\begin{aligned} & \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) |f'(a + t\eta(b,a))|^q dt \\ \leq & \frac{1}{3} \left(\frac{a + \eta(b,a) - x}{\eta(b,a)} \right)^3 |f'(a)|^q + \left(\frac{1}{6} + \frac{(x-a)^2 (2x - 3\eta(b,a) - 2a)}{6\eta^3(b,a)} \right) |f'(b)|^q. \end{aligned}$$

The proof is complete. □

Corollary 2.9. *Suppose that all the assumptions of Theorem 2.8 are satisfied and let $x = \frac{2a + \eta(b,a)}{2}$. Then by inequality (2.10) we have the following inequality*

$$\left| f\left(\frac{2a + \eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a + \eta(b,a)} f(x) dx \right| \leq \left(\frac{3^{1-\frac{1}{q}}}{8}\right) \eta(b,a) (|f'(a)| + |f'(b)|).$$

Remark 2.10. *Suppose that all the assumptions of Theorem 2.8 are satisfied.*

(a) *Taking $\eta(b,a) = b - a$, in Corollary 2.9, we have the following inequality which is the same with inequality (1.6)*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left(\frac{3^{1-\frac{1}{q}}}{8}\right) (b-a) (|f'(a)| + |f'(b)|).$$

(b) *If the mapping η satisfies Condition C, then by using the inequality (2.6) for $|f'|^q$ in the proof of Theorem 2.8, we obtain*

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) du \right| \leq \eta(b, a) \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \\ & \times \left\{ \left(\frac{x-a}{\eta(b, a)} \right)^{2\left(1-\frac{1}{q}\right)} \left[\frac{(x-a)^2 (3\eta(b, a) - 2x + 2a)}{6\eta^3(b, a)} |f'(a)|^q \right. \right. \\ & + \frac{1}{3} \left(\frac{x-a}{\eta(b, a)} \right)^3 |f'(a + \eta(b, a))|^q \left. \right]^{\frac{1}{q}} + \left(\frac{a + \eta(b, a) - x}{\eta(b, a)} \right)^{2\left(1-\frac{1}{q}\right)} \left[\frac{1}{3} \left(\frac{a + \eta(b, a) - x}{\eta(b, a)} \right)^3 |f'(a)|^q \right. \\ & \left. \left. + \left(\frac{1}{6} + \frac{(x-a)^2 (2x - 3\eta(b, a) - 2a)}{6\eta^3(b, a)} \right) |f'(a + \eta(b, a))|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

REFERENCES

- [1] M. Alomari, M. Darus, S. S. Dragomir and P. Cerone, Ostrowski type inequalities for functions whose derivatives are s -convex in the second sense, *Appl. Math. Lett.* **23** (2010), no. 9, 1071–1076.
- [2] T. Antczak, Mean value in invexity analysis, *Nonlinear Analysis* **60** (2005), no. 8, 1473–1484.
- [3] S. S. Dragomir, Y. J. Cho and S. S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, *J. Math. Anal. Appl.* **245** (2000), no. 2, 489–501.
- [4] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *J. Math. Anal. Appl.* **80** (1981), no. 2, 545–550.
- [5] H. Kavurmacı, M. E. Ozdemir and M. Avcı, New Ostrowski type inequalities for m -convex functions and applications, *Hacet. J. Math. Stat.* **40** (2011), no. 2, 135–145.
- [6] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.* **147** (2004), no. 1, 137–146.
- [7] U. S. Kirmaci, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. and Comput.* **153** (2004), no. 2, 361–368.
- [8] S. K. Mishra and G. Giorgi, *Invexity and Optimization*, Springer-Verlag, Berlin, 2008.
- [9] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, *J. Math. Anal. Appl.* **189** (1995), no. 3, 901–908.
- [10] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic, Dordrecht, 1994.
- [11] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, *J. Math. Anal. Approx. Theory* **2** (2007), no. 2, 126–131.
- [12] R. Pini, Invexity and generalized convexity, *Optimization* **22** (1991), no. 4, 513–525.

- [13] T. Weir, and B. Mond, Pre-invex functions in multiple objective optimization, *J. Math. Anal. Appl.* **136** (1998), no. 1, 29–38.
- [14] X. M. Yang and D. Li, On properties of preinvex functions, *J. Math. Anal. Appl.* **256** (2001), no. 1, 229–241.

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