Title:
On the decomposable numerical range of operators

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ABSTRACT. Let $V$ be an $n$-dimensional complex inner product space. Suppose $H$ is a subgroup of the symmetric group of degree $m$, and $\chi : H \to \mathbb{C}$ is an irreducible character (not necessarily linear). Denote by $V_\chi(H)$ the symmetry class of tensors associated with $H$ and $\chi$. Let $K(T) \in \text{End}(V_\chi(H))$ be the operator induced by $T \in \text{End}(V)$. The decomposable numerical range $W_\chi(T)$ of $T$ is a subset of the classical numerical range $W(K(T))$ of $K(T)$ defined as:

$$W_\chi(T) = \{(K(T)x^*, x^*) : x^* \text{ is a decomposable unit tensor}\}.$$

In this paper, we study the interplay between the geometric properties of $W_\chi(T)$ and the algebraic properties of $T$. In fact, we extend some of the results of [C. K. Li and A. Zaharia, Decomposable numerical range on orthonormal decomposable tensors, Linear Algebra Appl. 308 (2000), no. 1-3, 139–152] and [C. K. Li and A. Zaharia, Induced operators on symmetry classes of tensors, Trans. Amer. Math. Soc. 354 (2002), no. 2, 807–836], to non-linear irreducible characters.

Keywords: Symmetry class of tensors, decomposable numerical range, induced operator.

1. Introduction

In this paper, we study the decomposable numerical range of an operator $T$, which is defined by the induced operator $K(T)$ acting on symmetry classes of tensors. The decomposable numerical range $W_\chi(T)$ of
$T$ is the set of complex numbers of the form $(K(T)x^*, x^*)$ with $x^*$ ranging over all decomposable unit vectors in $V_\chi(H)$, where $V_\chi(H)$ is the symmetry class of tensors associated with $H$ and $\chi$. This concept has been studied extensively because of its applications in many branches of pure and applied mathematics (see e.g., [1, 2, 3, 7, 8]).

In Section 2, we give a review of the symmetry classes of tensors; see [9, 10] for general background. In Section 3, we study some properties of the induced operator. Section 4 is devoted to the study of the relations between geometric properties of $W_\chi(T)$ and algebraic properties of $T$.

2. Symmetry classes of tensors

Let $V$ be an $n$-dimensional inner product space over $\mathbb{C}$ and $H$ be a subgroup of the full symmetric group $S_m$. Let $\otimes^m V$ be the tensor product of $m$ copies of $V$ and for any $\sigma \in H$, define the permutation operator

$$P(\sigma) : \otimes^m V \to \otimes^m V$$

by

$$P(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}.$$

Suppose $\chi$ is a complex irreducible character of $H$ and define the symmetrizer,

$$T(\chi, H) = \frac{\chi(1)}{|H|} \sum_{\sigma \in H} \chi(\sigma) P(\sigma),$$

where $|H|$ denotes the order of $H$. The range of $T(\chi, H)$,

$$V_\chi(H) := T(\chi, H)(\otimes^m V)$$

is called the symmetry class of tensors associated with $H$ and $\chi$. The elements in $V_\chi(H)$ of the form

$$T(\chi, H)(v_1 \otimes v_2 \otimes \cdots \otimes v_m)$$

are called decomposable symmetrized tensors and are denoted by $v_1 * v_2 * \cdots * v_m$ (or briefly $v^*$). The inner product on $V$ induces an
inner product on $V_{\chi}(H)$ such that

$$(u^*, v^*) = \frac{\chi(1)}{|H|} \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{m} (u_{\sigma(i)}, v_{\sigma(i)}).$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $V$ and suppose $\Gamma_{m,n}$ is the set of all $m$-tuples of integers $\alpha = (\alpha(1), \ldots, \alpha(m))$ with $1 \leq \alpha(i) \leq n$. For $\alpha = (\alpha(1), \ldots, \alpha(m)) \in \Gamma_{m,n}$, we use the notation $e_\alpha^*$ for the decomposable symmetrized tensor $e_{\alpha(1)} \cdots e_{\alpha(m)}$. It is clear that $V_{\chi}(H)$ is generated by all $e_\alpha^*$, $\alpha \in \Gamma_{m,n}$. We define an action of $H$ on $\Gamma_{m,n}$ by

$$\alpha \sigma = (\alpha(\sigma(1)), \ldots, \alpha(\sigma(m))),$$

for any $\sigma \in H$ and $\alpha \in \Gamma_{m,n}$. Given two elements $\alpha, \beta \in \Gamma_{m,n}$, we say that $\alpha \sim \beta$ if and only if $\alpha$ and $\beta$ lie in the same orbit. Suppose $\Delta$ is the set of minimum elements of orbits of this action with respect to the lexicographic order and let $H_\alpha$ denote the stabilizer subgroup of $\alpha$ (see [9]). Define

$$\Omega = \{\alpha \in \Gamma_{m,n} : [\chi, 1_{H_\alpha}] \neq 0\},$$

where $[\cdot, \cdot]$ denotes the inner product of characters (see [5]). It is well known that $e_\alpha^* \neq 0$, if and only if $\alpha \in \Omega$ (see for example [10]). Suppose $\Delta = \Delta \cap \Omega$. For any $\alpha \in \Delta$, we have the subspace

$$V^*_\alpha = \langle e_{\alpha \sigma}^* : \sigma \in H \rangle,$$

where $\langle \text{set of vectors} \rangle$ denotes the subspace generated by a given set of vectors.

It is proved [10] that

$$(e_\alpha^*, e_\beta^*) = \begin{cases} 0 & \text{if } \alpha \sim \beta, \\ \frac{\chi(1)}{|H|} \sum_{\sigma \in H_\alpha} \chi(\sigma) & \text{if } \alpha = \beta, \end{cases}$$

and thus

$$\|e_\alpha^*\|^2 = \frac{\chi(1)}{|H|} \sum_{\sigma \in H_\alpha} \chi(\sigma).$$

Hence we have the orthogonal decomposition [10]

$$V_{\chi}(H) = \sum_{\alpha \in \Delta} V^*_\alpha.$$
It is also proved that
\[ s_\alpha := \dim V_\alpha^* = \chi(1)[\chi, 1_H], \]
and in particular, if \( \chi \) is linear, then \( s_\alpha = 1 \). So the set
\[ \{ e_\alpha^* : \alpha \in \tilde{\Delta} \} \]
is a basis of \( V_\chi(H) \). In the general case, let \( \alpha \in \tilde{\Delta} \) and suppose
\[ e_{\alpha\sigma_1}^*, e_{\alpha\sigma_2}^*, \ldots, e_{\alpha\sigma_t}^* \]
is a basis of \( V_\alpha^* \). Let
\[ A_\alpha = \{ \alpha\sigma_1, \alpha\sigma_2, \ldots, \alpha\sigma_t \}. \]
Then we define \( \tilde{\Delta} = \bigcup_{\alpha \in \Delta} A_\alpha \). It is clear that
\[ \Delta \subseteq \tilde{\Delta} \subseteq \Omega, \]
and the set
\[ \{ e_\alpha^* : \alpha \in \tilde{\Delta} \} \]
is a basis of \( V_\chi(H) \). But in general, this basis may be non-orthogonal (see [6]). Let \( m_j(\alpha) \) be the number of occurrences of \( j \) in the sequence \( \alpha \in \tilde{\Delta} \).

3. Some properties of induced operators

Let \( V \) be an \( n \)-dimensional inner product space over \( \mathbb{C} \). Let \( H \) be a subgroup of the full symmetric group \( S_m \) and suppose \( \chi \) is a complex irreducible character of \( H \). For any \( T \in \text{End}(V) \), there is a unique induced operator \( K(T) \) acting on \( V_\chi(H) \) satisfying
\[ K(T)v_1 \ast \cdots \ast v_m = Tv_1 \ast \cdots \ast Tv_m. \]
Indeed \( V_\chi(H) \) is an invariant subspace of \( \bigotimes^m T \) and \( K(T) \) is the restriction \( \bigotimes^m T \) to \( V_\chi(H) \) (see [10, p. 185]). Clearly \( K(\xi T) = \xi^m K(T) \), \( \xi \in \mathbb{C} \).

If \( H = S_m \) and \( \chi \equiv 1 \) is the principal character of \( H \), then \( V_\chi(H) \) is the \( m \)th completely symmetric space over \( V \) and \( K(T) \) is the \( m \)th induced power of \( T \), usually denoted by \( P_m(T) \). If \( H = S_m \) and \( \chi \) is the alternating character, that is \( \chi(\sigma) = \text{sgn}(\sigma) \), then \( V_\chi(H) \) is the \( m \)th exterior space over \( V \) and \( K(T) \) is the \( m \)th compound of \( T \), usually denoted by \( C_m(T) \). If \( H = \{1\} \), where 1 is the identity in \( S_m \) (\( \chi \equiv 1 \) is the only irreducible character of \( H \)), then \( V_\chi(H) = \bigotimes^m V \) and \( K(T) = \bigotimes^m T \) is
Proposition 3.1. [10]. Let $S$, $T$ be linear operators on $V$ and assume \( \Delta \neq \emptyset \).

(a): \( K(I_V) = I_{V^N} \).
(b): \( K(ST) = K(S)K(T) \).
(c): $T$ is invertible if and only if $K(T)$ is. Moreover, we have $K(T^{-1}) = K(T)^{-1}$.
(d): \( K(T^*) = K(T)^* \).
(e): If $T$ is normal, unitary, positive (semi)-definite, Hermitian or skew-Hermitian (when $m$ is odd), then so is $K(T)$.
(f): If $T$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then for any $\sigma \in S_n$, $K(T)$ has eigenvalues $\prod_{j=1}^n \lambda_{\sigma(j)}^{m_j(\alpha)}$, $\alpha \in \tilde{\Delta}$.
(g): If rank($T$) = $r$, then rank($K(T)$) = $|\Gamma_{m, r} \cap \tilde{\Delta}|$.

We shall use $\mu(\tilde{\Delta})$ to denote the smallest integer $r$ such that $\Gamma_{m, r} \cap \tilde{\Delta} \neq \emptyset$. Similarly we can define $\mu(\Delta)$, but it is clear that $\mu(\Delta) = \mu(\tilde{\Delta})$. By Proposition 3.1, if rank($T$) = $r$, then rank($K(T)$) = $|\Gamma_{m, r} \cap \mu(\Delta)|$. Thus an operator $T$ on $V$ satisfies $K(T) = 0$ if and only if rank($T$) < $\mu(\Delta)$.

We say that $\chi$ is of determinant type if every element $\alpha \in \tilde{\Delta}$ satisfies

$$m_1(\alpha) = \cdots = m_n(\alpha) = \frac{m}{n},$$

with $\mu(\tilde{\Delta}) > 1$. Furthermore, we say that $\chi$ is of special type if every element $\alpha \in \Gamma_{m, r} \cap \tilde{\Delta}$ satisfies

$$m_1(\alpha) = \cdots = m_r(\alpha),$$

with $r = \mu(\tilde{\Delta}) > 1$.

Otherwise, we say that $\chi$ is of general type. Notice that the determinant type is a particular case of the special type. One can find examples of different types of characters in [6].

Theorem 3.2. Suppose $\chi$ is not of the determinant type, and $T \in \text{End}(V)$. Suppose $\eta \in \mathbb{C}$ with $|\eta| = 1$. Then $\eta K(T)$ is positive definite if and only if there exists $\xi \in \mathbb{C}$ with $\xi^m = \eta$ such that $\xi T$ is positive definite.
Proof. The necessity part is clear. The converse is proved in [6, Theorem 5.3].

Theorem 3.3. [6]. Suppose $\chi$ is not of the determinant type. Let $T \in \text{End}(V)$. Then $K(T)$ is unitary or a nonzero scalar operator if and only if $T$ has the corresponding property.

Theorem 3.4. Let $r = \mu(\Delta)$ and $T \in \text{End}(V)$ with rank$(T) \geq r$. Let $\eta \in \mathbb{C}$ with $|\eta| = 1$. Then $\eta K(T)$ is (i) Hermitian, (ii) positive semi-definite if and only if one of the following holds:

(a) There exists $\xi \in \mathbb{C}$ such that $\xi T$ has the corresponding property, where $\xi^m = \pm \eta$ for case (i) and $\xi^m = \eta$ for case (ii).

(b) $\chi$ is of the special type, and $T = T_1 \oplus 0$, where $T_1$ is an invertible operator acting on an $r$-dimensional subspace $V_1$ of $V$, and $\eta(\det T_1)^{m/r}$ is real or positive according to case (i) or (ii).

Proof. The necessary part is proved in [6, Theorem 5.5]. Now we prove the converse. If (a) holds, then the assertion is clear. Suppose (b) holds. Then $V = V_1 \oplus V_1^\perp$ such that $V_1$ is invariant under both $T$ and $T^*$; $T_1$ is the restriction of $T$ on $V_1$, and the restriction of $T$ on $V_1^\perp$ is the zero operator. Thus

$$\bigotimes^m V = \bigotimes^m V_1 \bigoplus \bigotimes^m (V_1^\perp).$$

By applying $T(\chi, H)$ to the both sides of the equation above, we get

$$V_\chi(H) = V_1 \chi(H) \bigoplus V_1(\chi(H))^\perp.$$

It is easy to see that $V_1 \chi(H)$ is invariant under both $K(T)$ and $K(T)^*$; $K(T_1)$ is the restriction of $K(T)$ on $V_1 \chi(H)$.

Now suppose $u \in V_1 \chi(H)^\perp$. Then $u = T(\chi, H)(z)$, where $z \in (\bigotimes^m V_1)^\perp$, because $V_1 \chi(H)^\perp = T(\chi, H)((\bigotimes^m V_1)^\perp)$. Since

$$(\bigotimes^m V_1)^\perp = V_1^\perp \otimes V_1 \otimes \cdots \otimes V_1 + V_1 \otimes V_1^\perp \otimes \cdots \otimes V_1 + \cdots + V_1 \otimes \cdots \otimes V_1 \otimes V_1^\perp,$$
so \( z = \sum w_1 \otimes \cdots \otimes w_m \), where at least one of the \( w_i \) belongs to \( V_1^\perp \). Hence

\[
K(T)u = K(T)T(\chi, H)z = K(T)(\sum w_1 \cdots \cdot w_m) = \sum Tw_1 \cdots \cdot Tw_m = 0, \quad (V_1 \subseteq \text{Ker } T).
\]

This implies that \( K(T) = K(T_1) \oplus 0 \).

Since \( \chi \) is of the special type, so by Theorem 4.1 in [6], \( K(T) \) is a multiple of an orthogonal projection \( P_T \) on \( V_\chi(H) \), i.e., \( K(T) = \xi P_T \). Indeed \( P_T \) is the natural projection from \( V_\chi(H) \) onto \( V_1 \chi(H) \). Thus the restriction \( P_T \) on \( V_1 \chi(H) \) is the identity. Since \( K(T_1) \) is the restriction \( K(T) \) on to \( V_1 \chi(H) \), so \( K(T_1) = \xi I \). Now we show that \( \xi = (\det T_1)^\frac{m}{r} \). Suppose \( T_1 \) has eigenvalues \( \lambda_1, \ldots, \lambda_r \). Then

\[
\prod_{j=1}^r \lambda_j^{m_j(\alpha)}, \quad \alpha \in \Gamma_m, r \cap \Delta
\]

are the eigenvalues of \( K(T_1) \). Since \( \chi \) is of the special type, so by Theorem 4.1 in [6], \( m_1(\alpha) = \cdots = m_r(\alpha) = \frac{m}{r} \) for every \( \alpha \in \Gamma_m, r \cap \Delta \). Hence

\[
\prod_{j=1}^r \lambda_j^{m_j(\alpha)} = \prod_{j=1}^r \lambda_j^{m} = \left( \prod_{j=1}^r \lambda_j \right)^{\frac{m}{r}} = (\det T_1)^{\frac{m}{r}},
\]

is the only eigenvalue of \( T_1 \). Thus \( \xi = (\det T_1)^\frac{m}{r} \). Therefore \( \eta K(T) = \eta(\det T_1)^\frac{m}{r} P_T \). Since \( P_T \) is an orthogonal projection and by assumption \( \eta (\det T_1)^{m/r} \) is real or positive, so the result follows from the fact that every orthogonal projection is a positive semi-definite operator. \( \square \)

4. Decomposable numerical range

Let \( V \) be a finite dimensional inner product space over \( \mathbb{C} \) and \( T \in \text{End}(V) \). Let \( H \) be a subgroup of the full symmetric group \( S_m \) and let \( \chi \) be a complex irreducible character of \( H \). The decomposable numerical range of \( T \) is defined by

\[
W_\chi(T) = \{ (K(T)x^*, x^*) : x^* \text{ is a decomposable unit tensor in } V_\chi(H) \}.
\]
Evidently, when $m = 1$, $W_\chi(T)$ reduces to the classical numerical range $W(T)$. Since the set of decomposable unit tensors is usually a proper subset of the unit vectors in $V_\chi(H)$, we have

$$W_\chi(T) \subseteq W(K(T)),$$

and the inclusion is often strict.

The following lemma can be easily proved by using elementary properties of induced operators.

**Lemma 4.1.** Let $T \in \text{End}(V)$. Then

(a) For any $\mu \in \mathbb{C}$, $W_\chi(\mu T) = \mu^m W_\chi(T)$.

(b) The set $W_\chi(T)$ is invariant under unitary similarities, i.e., $W_\chi(T) = W_\chi(U^*TU)$ for any unitary operator $U$.

There is an interesting interplay between the geometric properties of $W(T)$ and the algebraic properties of the operator $T$. For example, we have the following result (see [4]).

**Proposition 4.2.** Let $T \in \text{End}(V)$.

(a) $W(T) = \{\lambda\}$ if and only if $T = \lambda I$.

(b) $W(T) \subseteq \mathbb{R}$ if and only if $T$ is Hermitian.

(c) $W(T) \subseteq (0, \infty)$ if and only if $T$ is positive definite.

(d) $W(T)$ has no interior point if and only if $T$ is a normal operator with eigenvalues lying on a straight line.

In [7], Proposition 4.2 was generalized to the decomposable numerical range, when the irreducible character $\chi$ is linear. In this paper, it is generalized to arbitrary irreducible characters. The following theorem is a generalization of Theorem 3.5 in [8], to non-linear irreducible characters of $G$. We need the following result of Robinson [11].

**Proposition 4.3.** A linear operator $L$ on $V_\chi(H)$ satisfies $(Lv^*, v^*) = 0$ for all decomposable tensors $v^* = v_1 \ast \cdots \ast v_m \in V_\chi(H)$ if and only if $L = 0$.

**Theorem 4.4.** Let $r = \mu(\Delta)$ and $T \in \text{End}(V)$ with $\text{rank}(T) \geq r$. Let $\eta \in \mathbb{C}$ with $|\eta| = 1$. Then $W_\chi(T)$ is a subset of
(i) $\eta\mathbb{R}$ or (ii) $\eta[0,\infty)$ if and only if one of the following conditions holds.

(a) There exists $\xi \in \mathbb{C}$ such that $\xi T$ is (i) Hermitian or (ii) positive semi-definite, where $\xi^m = \pm \eta$ for case (i) and $\xi^m = \bar{\eta}$ for case (ii).

(b) $\chi$ is of the special type and $T = T_1 \oplus 0$, where $T_1$ is an operator acting on an $r$-dimensional subspace $V_1$ of $V$ and $\eta(|\det T_1|)^{\frac{1}{m}}$ is (i) real or (ii) nonnegative according to case (i), or (ii).

Proof. If $\chi$ and $T$ satisfy (a) or (b), then $\eta K(T)$ is (i) Hermitian or (ii) positive semi-definite by Theorem 3.4. So $W_\chi(T)$ is a subset of $\eta\mathbb{R}$ or $\eta[0,\infty)$, by Proposition 4.2.

Conversely, suppose $W_\chi(T) \subseteq \eta\mathbb{R}$ or $\eta[0,\infty)$. Then

$$
\left( (\eta K(T)^* - \bar{\eta} K(T)) v^*, v^* \right) = \eta (K(T)^* v^*, v^*) - \bar{\eta} (K(T) v^*, v^*) = 0,
$$

for all unit decomposable tensors $v^* \in V_\chi(H)$. By Proposition 4.3, $\eta K(T)^* - \bar{\eta} K(T) = 0$, i.e., $\eta K(T)$ is Hermitian.

Applying Theorem 3.4, we see that $T$ satisfies (a) or (b). $\square$

**Corollary 4.5.** Let $\chi$ be not of the determinant type. Let $T \in \text{End}(V)$. Then $T$ is (i) Hermitian, (ii) positive semi-definite if and only if $W_\chi(T + \eta I) \subseteq S$ for all $\eta \in S$ with (i) $S = \mathbb{R}$, (ii) $S = [0,\infty)$.

Proof. If $T$ is (i) Hermitian, (ii) positive semi-definite, then for any $\eta \in S$ with (i) $S = \mathbb{R}$, (ii) $S = [0,\infty)$, $K(T + \eta I)$ has the corresponding property. So the result follows from Proposition 4.2.

Conversely, suppose that $W_\chi(T + \eta I) \subseteq \mathbb{R}$ for all $\eta \in \mathbb{R}$. If $\eta \in \mathbb{R}$ is not an eigenvalue of $T$, then $T + \eta I$ is an invertible operator. Thus $\text{rank}(T + \eta I) \geq r = \mu(\Delta)$. So, $T + \eta I$ satisfies (a) or (b) of Theorem 4.4. If (b) holds, then $\chi$ is of the special type, and $T + \eta I = T_1 \oplus 0$, where $T_1$ is an invertible operator acting on an $r$-dimensional subspace. Since $T + \eta I$ is invertible, so $T + \eta I = T_1$, i.e., $r = n$, and $\chi$ is of the determinant type, a contradiction. Thus (a) holds and there exists $\xi \in \mathbb{C}$ with $\xi^m = \pm 1$ such that $\xi(T + \eta I)$ is Hermitian.

Since the number of the $m$th roots of unity is finite, there exist two
distinct real numbers \( \eta_1 \) and \( \eta_2 \) such that \( \xi(T + \eta_1 I) \) and \( \xi(T + \eta_2 I) \) are Hermitian, where \( \xi \) is an \( m \)th root of unity, i.e.,
\[
\tilde{\xi}(T^* + \eta_1 I) = \xi(T + \eta_1 I), \\
\tilde{\xi}(T^* + \eta_2 I) = \xi(T + \eta_2 I).
\]
Thus \( \tilde{\xi}(\eta_1 - \eta_2)I = \xi(\eta_1 - \eta_2)I \), and we get \( \xi = \tilde{\xi} \). Hence \( T = T^* \) and the assertion holds.

Now, suppose that \( W_\chi(T + \eta I) \subseteq [0, \infty) \) for all \( \eta \in [0, \infty) \). Similar to the proof of case (i), where \( S = \mathbb{R} \), we conclude that \( T \) is a Hermitian operator. To prove that \( T \) is positive semi-definite, we show that the eigenvalues of \( T \) are all nonnegative (see [10, Corollary 2.35]).

Let \( \lambda \) be an eigenvalue of \( T \) and \( v \) be the corresponding eigenvector. Since the number of the \( m \)th roots of unity is finite, hence similar to the proof of case (i), there exist a complex number \( \xi \) with \( \xi^m = 1 \) and a decreasing sequence \( \{\eta_n\} \) of real positive numbers such that \( \eta_1 > -\lambda, \eta_n \to 0 \) and for any \( n \in \mathbb{N} \), \( \xi(T + \eta_n I) \) is positive semi-definite. This implies that \( \xi(\lambda + \eta_n) \geq 0 \) for every \( n \in \mathbb{N} \). Since \( \eta_1 > -\lambda \), so we must have \( \xi = 1 \) and hence \( \lambda + \eta_n \geq 0 \) for all \( n > 1 \). Using the fact that \( \eta_n \to 0 \) completes the proof. \( \square \)

**Theorem 4.6.** Suppose \( \chi \) is not of the determinant type. Let \( T \in \text{End}(V) \) and \( \eta \in \mathbb{C} \) with \( |\eta| = 1 \).

(a) \( W_\chi(T) \subseteq \eta(0, \infty) \) if and only if there exists \( \xi \in \mathbb{C} \) with \( \xi^m = \bar{\eta} \) such that \( \xi T \) is positive definite.

(b) \( W_\chi(T) \) is a singleton if and only if \( T \) is a scalar operator.

**Proof.**

(a) Suppose \( W_\chi(T) \subseteq \eta(0, \infty) \). Then \( \bar{\eta} W_\chi(T) \subseteq (0, \infty) \). So \( (\bar{\eta}K(T)x^*, x^*) > 0 \) for all unit decomposable tensors \( x^* \). Therefore \( \bar{\eta}K(T) \) is positive definite. By Theorem 3.2, we see that there exists \( \xi \in \mathbb{C} \) with \( \xi^m = \bar{\eta} \) such that \( \xi T \) is positive definite.

Conversely, assume that there exists \( \xi \in \mathbb{C} \) with \( \xi^m = \bar{\eta} \) such that \( \xi T \) is positive definite. Then \( \bar{\eta}K(T) = \xi^mK(T) = K(\xi T) \) is positive definite. So \( W(\bar{\eta}K(T)) \subseteq (0, \infty) \), by Proposition 4.2. Therefore \( W_\chi(T) \subseteq W(K(T)) \subseteq \eta(0, \infty) \).

(b) If \( T = \xi I \) for some \( \xi \in \mathbb{C} \), then \( K(T) = \xi^m I \). So \( W_\chi(T) = \{\xi^m\} \) by definition. Conversely, let \( W_\chi(T) = \{\xi\} \) for some \( \xi \in \mathbb{C} \). Then
$(K(T)x^*,x^*) = \xi$ for all unit decomposable tensors $x^* \in V_\chi(H)$. So $((K(T) - \xi I)x^*,x^*) = 0$. Therefore, by Proposition 4.3, $K(T) = \xi I$.

Now, we see that $T$ is a scalar operator from Theorem 3.3.

By Theorem 4.4, we know that if $\chi$ is not of the determinant type, then $W_\chi(T)$ is a subset of a line passing through the origin if and only if $\xi T$ is Hermitian for some $\xi \in \mathbb{C}$. In the following theorem, we determine the conditions under which $W_\chi(T)$ is a subset of a line not passing through the origin.

**Theorem 4.7.** Suppose $\chi$ is not of the determinant type, and $T \in \text{End}(V)$ is non-scalar. Then $W_\chi(T)$ is a subset of a line not passing through the origin if and only if for every $\alpha \in \hat{\Delta}$,

$$(m_1(\alpha), \ldots, m_n(\alpha))$$

is a permutation of $(k, \ldots, k, k+1, \ldots, k+1)$,

where $m = nk + t$ with $1 \leq t < n$, and $T$ is an invertible normal operator such that one of the following holds:

(a): $t = 1$ and the eigenvalues of $T$ lie on a line not passing through the origin;

(b): $1 < t < n - 1$ and one of the eigenvalues of $T$ has multiplicity $n - 1$;

(c): $t = n - 1$ and the eigenvalues of $T^{-1}$ lie on a line not passing through the origin.

**Proof.** Similar to the case of linear characters (see Theorem 3.8 in [8]).

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