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# ON THE DECOMPOSABLE NUMERICAL RANGE OF OPERATORS

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Abstract. Let $V$ be an $n$-dimensional complex inner product space. Suppose $H$ is a subgroup of the symmetric group of degree $m$, and $\chi : H \to \mathbb{C}$ is an irreducible character (not necessarily linear). Denote by $V_\chi(H)$ the symmetry class of tensors associated with $H$ and $\chi$. Let $K(T) \in \mathrm{End}(V_\chi(H))$ be the operator induced by $T \in \mathrm{End}(V)$. The decomposable numerical range $W_\chi(T)$ of $T$ is a subset of the classical numerical range $W(K(T))$ of $K(T)$ defined as:

$$W_\chi(T) = \{(K(T)x^*, x^*) : x^* \text{ is a decomposable unit tensor}\}.$$

In this paper, we study the interplay between the geometric properties of $W_\chi(T)$ and the algebraic properties of $T$. In fact, we extend some of the results of [C. K. Li and A. Zaharia, Decomposable numerical range on orthonormal decomposable tensors, *Linear Algebra Appl.* 308 (2000), no, 1-3, 139–152] and [C. K. Li and A. Zaharia, Induced operators on symmetry classes of tensors, *Trans. Amer. Math. Soc.* 354 (2002), no. 2, 807–836], to non-linear irreducible characters.

**Keywords:** Symmetry class of tensors, decomposable numerical range, induced operator.
**MSC(2010):** Primary: 20C15; Secondary: 15A69, 15A60.

## 1. Introduction

In this paper, we study the decomposable numerical range of an operator $T$, which is defined by the induced operator $K(T)$ acting on symmetry classes of tensors. The decomposable numerical range $W_\chi(T)$ of

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$T$ is the set of complex numbers of the form $(K(T)x^*, x^*)$ with $x^*$ ranging over all decomposable unit vectors in $V_\chi(H)$, where $V_\chi(H)$ is the symmetry class of tensors associated with $H$ and $\chi$. This concept has been studied extensively because of its applications in many branches of pure and applied mathematics (see e.g., [1, 2, 3, 7, 8]).

In Section 2, we give a review of the symmetry classes of tensors; see [9, 10] for general background. In Section 3, we study some properties of the induced operator. Section 4 is devoted to the study of the relations between geometric properties of $W_\chi(T)$ and algebraic properties of $T$.

## 2. Symmetry classes of tensors

Let $V$ be an $n$-dimensional inner product space over $\mathbb{C}$ and $H$ be a subgroup of the full symmetric group $S_m$. Let $\bigotimes^m V$ be the tensor product of $m$ copies of $V$ and for any $\sigma \in H$, define the *permutation operator*

$$P(\sigma) : \bigotimes^m V \to \bigotimes^m V$$

by

$$P(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}.$$

Suppose $\chi$ is a complex irreducible character of $H$ and define the *symmetrizer*,

$$T(\chi, H) = \frac{\chi(1)}{|H|} \sum_{\sigma \in H} \chi(\sigma) P(\sigma),$$

where $|H|$ denotes the order of $H$. The range of $T(\chi, H)$,

$$V_\chi(H) := T(\chi, H)(\bigotimes^m V)$$

is called the *symmetry class of tensors associated with $H$ and $\chi$*. The elements in $V_\chi(H)$ of the form

$$T(\chi, H)(v_1 \otimes v_2 \otimes \cdots \otimes v_m)$$

are called *decomposable symmetrized tensors* and are denoted by $v_1 * v_2 * \cdots * v_m$ (or briefly $v^*$). The inner product on $V$ induces an inner product on $V_\chi(H)$ such that

$$(u^*,\ v^*) = \frac{\chi(1)}{|H|}\sum_{\sigma\in H}\chi(\sigma)\prod_{i=1}^{m}(u_i, v_{\sigma(i)}).$$

Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of $V$ and suppose $\Gamma_{m,n}$ is the set of all $m$-tuples of integers $\alpha = (\alpha(1),\ldots,\alpha(m))$ with $1 \leq \alpha(i) \leq n$. For $\alpha = (\alpha(1),\ldots,\alpha(m)) \in \Gamma_{m,n}$, we use the notation $e^*_\alpha$ for the decomposable symmetrized tensor $e_{\alpha(1)} * \cdots * e_{\alpha(m)}$. It is clear that $V_\chi(H)$ is generated by all $e^*_\alpha$, $\alpha \in \Gamma_{m,n}$. We define an action of $H$ on $\Gamma_{m,n}$ by

$$\alpha\sigma = (\alpha(\sigma(1)),\ldots,\alpha(\sigma(m))),$$

for any $\sigma \in H$ and $\alpha \in \Gamma_{m,n}$. Given two elements $\alpha, \beta \in \Gamma_{m,n}$, we say that $\alpha \sim \beta$ if and only if $\alpha$ and $\beta$ lie in the same orbit. Suppose $\Delta$ is the set of minimum elements of orbits of this action with respect to the lexicographic order and let $H_\alpha$ denote the stabilizer subgroup of $\alpha$ (see [9]). Define

$$\Omega = \{\alpha \in \Gamma_{m,n} :\ [\chi,\ 1_{H_\alpha}] \neq 0\},$$

where [ , ] denotes the inner product of characters (see [5]). It is well known that $e^*_\alpha \neq 0$, if and only if $\alpha \in \Omega$ (see for example [10]). Suppose $\bar{\Delta} = \Delta \cap \Omega$. For any $\alpha \in \bar{\Delta}$, we have the subspace

$$V^*_\alpha = \langle\ e^*_{\alpha\sigma}\ :\ \sigma \in H\ \rangle,$$

where $\langle$set of vectors$\rangle$ denotes the subspace generated by a given set of vectors.
It is proved [10] that

$$(e^*_\alpha,\ e^*_\beta) = \begin{cases} 0 & \text{if } \alpha \nsim \beta, \\ \frac{\chi(1)}{|H|}\sum_{\sigma\in H_\alpha}\chi(\sigma) & \text{if } \alpha = \beta, \end{cases}$$

and thus

$$\|e^*_\alpha\|^2 = \frac{\chi(1)}{|H|}\sum_{\sigma\in H_\alpha}\chi(\sigma).$$

Hence we have the orthogonal decomposition [10]

$$V_\chi(H) = \dot{\sum_{\alpha\in\bar{\Delta}}} V^*_\alpha.$$

It is also proved that

$$s_\alpha := \dim V_\alpha^* = \chi(1)[\chi,\ 1_{H_\alpha}],$$

and in particular, if $\chi$ is linear, then $s_\alpha = 1$. So the set

$$\{e_\alpha^*\ :\ \alpha \in \bar{\Delta}\}$$

is a basis of $V_\chi(H)$. In the general case, let $\alpha \in \bar{\Delta}$ and suppose

$$e_{\alpha\sigma_1}^*,\ e_{\alpha\sigma_2}^*,\ \ldots,\ e_{\alpha\sigma_t}^*$$

is a basis of $V_\alpha^*$. Let

$$A_\alpha = \{\alpha\sigma_1, \alpha\sigma_2, \ldots, \alpha\sigma_t\}.$$

Then we define $\hat{\Delta} = \bigcup_{\alpha\in\bar{\Delta}} A_\alpha$. It is clear that

$$\bar{\Delta} \subseteq \hat{\Delta} \subseteq \Omega,$$

and the set

$$\{e_\alpha^*\ :\ \ \alpha \in \hat{\Delta}\}$$

is a basis of $V_\chi(H)$. But in general, this basis may be non-orthogonal (see [6]). Let $m_j(\alpha)$ be the number of occurrences of $j$ in the sequence $\alpha \in \hat{\Delta}$.

## 3. **Some properties of induced operators**

Let $V$ be an $n$-dimensional inner product space over $\mathbb{C}$. Let $H$ be a subgroup of the full symmetric group $S_m$ and suppose $\chi$ is a complex irreducible character of $H$. For any $T \in End(V)$, there is a unique induced operator $K(T)$ acting on $V_\chi(H)$ satisfying

$$K(T)v_1 * \cdots * v_m = Tv_1 * \cdots * Tv_m.$$

Indeed $V_\chi(H)$ is an invariant subspace of $\bigotimes^m T$ and $K(T)$ is the restriction $\bigotimes^m T$ to $V_\chi(H)$ (see [10, p. 185]). Clearly $K(\xi T) = \xi^m K(T),\ \xi \in \mathbb{C}$.

If $H = S_m$ and $\chi \equiv 1$ is the principal character of $H$, then $V_\chi(H)$ is the $m$th completely symmetric space over $V$ and $K(T)$ is the $m$th induced power of $T$, usually denoted by $P_m(T)$. If $H = S_m$ and $\chi$ is the alternating character, that is $\chi(\sigma) = sgn(\sigma)$, then $V_\chi(H)$ is the $m$th exterior space over $V$ and $K(T)$ is the $m$th compound of $T$, usually denoted by $C_m(T)$. If $H = \{1\}$, where 1 is the identity in $S_m$ ($\chi \equiv 1$ is the only irreducible character of $H$), then $V_\chi(H) = \bigotimes^m V$ and $K(T) = \bigotimes^m T$ is the $m$th tensor power of $T$.

The following contains some properties of the induced operator.

**Proposition 3.1.** [10]. *Let $S$, $T$ be linear operators on $V$ and assume $\bar{\Delta} \neq \emptyset$.*

- **(a):** $K(I_V) = I_{V_\chi(H)}$.
- **(b):** $K(ST) = K(S)K(T)$.
- **(c):** *$T$ is invertible if and only if $K(T)$ is . Moreover, we have $K(T^{-1}) = K(T)^{-1}$.*
- **(d):** $K(T^*) = K(T)^*$.
- **(e):** *If $T$ is normal, unitary, positive (semi)-definite, Hermitian or skew-Hermitian (when $m$ is odd), then so is $K(T)$.*
- **(f):** *If $T$ has eigenvalues $\lambda_1, \dots, \lambda_n$, then for any $\sigma \in S_n$, $K(T)$ has eigenvalues $\prod_{j=1}^{n} \lambda_{\sigma(j)}^{m_j(\alpha)}$, $\alpha \in \hat{\Delta}$.*
- **(g):** *If rank $(T) = r$, then rank $(K(T)) = |\Gamma_{m,r} \bigcap \hat{\Delta}|$.*

We shall use $\mu(\bar{\Delta})$ to denote the smallest integer $r$ such that $\Gamma_{m,\ r} \cap \bar{\Delta} \neq \emptyset$. Similarly we can define $\mu(\hat{\Delta})$, but it is clear that $\mu(\bar{\Delta}) = \mu(\hat{\Delta})$. By Proposition 3.1, if rank $(T) = r$, then rank $(K(T)) = |\Gamma_{m,\ r} \cap \mu(\hat{\Delta})|$. Thus an operator $T$ on $V$ satisfies $K(T) = 0$ if and only if rank$(T) < \mu(\bar{\Delta})$.

We say that $\chi$ is of *determinant type* if every element $\alpha \in \bar{\Delta}$ satisfies

$$m_1(\alpha) = \cdots = m_n(\alpha) = \frac{m}{n},$$

with $\mu(\bar{\Delta}) > 1$. Furthermore, we say that $\chi$ is of *special type* if every element $\alpha \in \Gamma_{m,\ r} \cap \bar{\Delta}$ satisfies

$$m_1(\alpha) = \cdots = m_r(\alpha),$$

with $r = \mu(\bar{\Delta}) > 1$.
Otherwise, we say that $\chi$ is of *general type*. Notice that the determinant type is a particular case of the special type. One can find examples of different types of characters in [6].

**Theorem 3.2.** *Suppose $\chi$ is not of the determinant type, and $T \in End(V)$. Suppose $\eta \in \mathbb{C}$ with $|\eta| = 1$. Then $\eta K(T)$ is positive definite if and only if there exists $\xi \in \mathbb{C}$ with $\xi^m = \eta$ such that $\xi T$ is positive definite.*

*Proof.* The necessity part is clear. The converse is proved in [6, Theorem 5.3]. □

**Theorem 3.3.** [6]. *Suppose $\chi$ is not of the determinant type. Let $T \in End(V)$. Then $K(T)$ is unitary or a nonzero scalar operator if and only if $T$ has the corresponding property.*

**Theorem 3.4.** *Let $r = \mu(\bar{\Delta})$ and $T \in End(V)$ with $rank(T) \geq r$. Let $\eta \in \mathbb{C}$ with $|\eta| = 1$. Then $\eta K(T)$ is (i) Hermitian, (ii) positive semi-definite if and only if one of the following holds:*

(a) *There exists $\xi \in \mathbb{C}$ such that $\xi T$ has the corresponding property, where $\xi^m = \pm\eta$ for case (i) and $\xi^m = \eta$ for case (ii) .*

(b) *$\chi$ is of the special type, and $T = T_1 \oplus 0$, where $T_1$ is an invertible operator acting on an $r$-dimensional subspace $V_1$ of $V$, and $\eta\ (det\, T_1)^{m/r}$ is real or positive according to case (i) or (ii).*

*Proof.* The necessary part is proved in [6, Theorem 5.5]. Now we prove the converse. If (a) holds, then the assertion is clear. Suppose (b) holds. Then $V = V_1 \oplus V_1^{\perp}$ such that $V_1$ is invariant under both $T$ and $T^*$; $T_1$ is the restriction of $T$ on $V_1$, and the restriction of $T$ on $V_1^{\perp}$ is the zero operator. Thus

$$\bigotimes^m V = \bigotimes^m V_1 \bigoplus (\bigotimes^m V_1)^{\perp}.$$

By applying $T(\chi, H)$ to the both sides of the equation above, we get

$$V_{\chi}(H) = V_{1\chi}(H) \bigoplus V_{1\chi}(H)^{\perp}.$$

It is easy to see that $V_{1\chi}(H)$ is invariant under both $K(T)$ and $K(T)^*$; $K(T_1)$ is the restriction of $K(T)$ on $V_{1\chi}(H)$.

Now suppose $u \in V_{1\chi}(H)^{\perp}$. Then $u = T(\chi, H)(z)$, where $z \in (\bigotimes^m V_1)^{\perp}$, because $V_{1\chi}(H)^{\perp} = T(\chi, H)((\bigotimes^m V_1)^{\perp})$.
Since

$$(\otimes^m V_1)^{\perp} = V_1^{\perp} \otimes V_1 \otimes \cdots \otimes V_1 + V_1 \otimes V_1^{\perp} \otimes \cdots \otimes V_1 + \cdots + V_1 \otimes \cdots \otimes V_1 \otimes V_1^{\perp},$$

so $z = \sum w_1 \otimes \cdots \otimes w_m$, where at least one of the $w_i$ belongs to $V_1^{\perp}$. Hence

$$\begin{aligned} K(T)u &= K(T)T(\chi, H)z \\ &= K(T)(\sum w_1 * \cdots * w_m) \\ &= \sum Tw_1 * \cdots * Tw_m \\ &= 0, \quad (V_1 \subseteq \text{Ker T}). \end{aligned}$$

This implies that $K(T) = K(T_1) \oplus 0$.
Since $\chi$ is of the special type, so by Theorem 4.1 in [6], $K(T)$ is a multiple of an orthogonal projection $P_T$ on $V_\chi(H)$, i.e., $K(T) = \xi P_T$. Indeed $P_T$ is the natural projection from $V_\chi(H)$ onto $V_{1\chi}(H)$. Thus the restriction $P_T$ on $V_{1\chi}(H)$ is the identity. Since $K(T_1)$ is the restriction $K(T)$ on to $V_{1\chi}(H)$, so $K(T_1) = \xi\, I$. Now we show that $\xi = (\text{ det}T_1)^{\frac{m}{r}}$. Suppose $T_1$ has eigenvalues $\lambda_1, \ldots, \lambda_r$. Then

$$\prod_{j=1}^{r} \lambda_j^{m_j(\alpha)}, \quad \alpha \in \Gamma_{m,\, r} \cap \hat{\Delta}$$

are the eigenvalues of $K(T_1)$. Since $\chi$ is of the special type, so by Theorem 4.1 in [6], $m_1(\alpha) = \cdots = m_r(\alpha) = \frac{m}{r}$ for every $\alpha \in \Gamma_{m,\, r} \cap \hat{\Delta}$. Hence

$$\prod_{j=1}^{r} \lambda_j^{m_j(\alpha)} = \prod_{j=1}^{r} \lambda_j^{\frac{m}{r}} = \left( \prod_{j=1}^{r} \lambda_j \right)^{\frac{m}{r}} = (\det\, T_1)^{\frac{m}{r}},$$

is the only eigenvalue of $T_1$. Thus $\xi = (\det\, T_1)^{\frac{m}{r}}$. Therefore $\eta\, K(T) = \eta(\det\, T_1)^{\frac{m}{r}} P_T$. Since $P_T$ is an orthogonal projection and by assumption $\eta\, (\det\, T_1)^{m/r}$ is real or positive, so the result follows from the fact that every orthogonal projection is a positive semi-definite operator. □

## 4. Decomposable numerical range

Let $V$ be a finite dimensional inner product space over $\mathbb{C}$ and $T \in End(V)$. Let $H$ be a subgroup of the full symmetric group $S_m$ and let $\chi$ be a complex irreducible character of $H$. The decomposable numerical range of $T$ is defined by

$$W_\chi(T) = \{(K(T)x^*, x^*) : x^* \text{ is a decomposable unit tensor in } V_\chi(H)\}.$$

Evidently, when $m = 1$, $W_\chi(T)$ reduces to the classical numerical range $W(T)$. Since the set of decomposable unit tensors is usually a proper subset of the unit vectors in $V_\chi(H)$, we have

$$W_\chi(T) \subseteq W(K(T)),$$

and the inclusion is often strict.
The following lemma can be easily proved by using elementary properties of induced operators.

**Lemma 4.1.** *Let $T \in End(V)$. Then*

(a) *For any $\mu \in \mathbb{C}$, $W_\chi(\mu T) = \mu^m W_\chi(T)$.*

b) *The set $W_\chi(T)$ is invariant under unitary similarities, i.e., $W_\chi(T) = W_\chi(U^*TU)$ for any unitary operator $U$.*

There is an interesting interplay between the geometric properties of $W(T)$ and the algebraic properties of the operator $T$. For example, we have the following result (see [4]).

**Proposition 4.2.** *Let $T \in End(V)$.*

(a) *$W(T) = \{\lambda\}$ if and only if $T = \lambda I$.*

(b) *$W(T) \subseteq \mathbb{R}$ if and only if $T$ is Hermitian.*

(c) *$W(T) \subseteq (0, \infty)$ if and only if $T$ is positive definite.*

(d) *$W(T)$ has no interior point if and only if $T$ is a normal operator with eigenvalues lying on a straight line.*

In [7], Proposition 4.2 was generalized to the decomposable numerical range, when the irreducible character $\chi$ is linear. In this paper, it is generalized to arbitrary irreducible characters. The following theorem is a generalization of Theorem 3.5 in [8], to non-linear irreducible characters of $G$. We need the following result of Robinson [11].

**Proposition 4.3.** *A linear operator $L$ on $V_\chi(H)$ satisfies $(Lv^*,\ v^*) = 0$ for all decomposable tensors $v^* = v_1 * \cdots * v_m \in V_\chi(H)$ if and only if $L = 0$.*

**Theorem 4.4.** *Let $r = \mu(\bar{\Delta})$ and $T \in End(V)$ with $rank(T) \geq r$. Let $\eta \in \mathbb{C}$ with $|\eta| = 1$. Then $W_\chi(T)$ is a subset of* *(i)* $\eta\mathbb{R}$ *or (ii)* $\eta[0,\infty)$ *if and only if one of the following conditions holds.*

(a) *There exists* $\xi \in \mathbb{C}$ *such that* $\xi T$ *is (i) Hermitian or (ii) positive semi-definite, where* $\xi^m = \pm\bar{\eta}$ *for case (i) and* $\xi^m = \bar{\eta}$ *for case (ii).*

(b) $\chi$ *is of the special type and* $T = T_1 \oplus 0$*, where* $T_1$ *is an operator acting on an* $r$*-dimensional subspace* $V_1$ *of* $V$ *and* $\bar{\eta}(\det T_1)^{\frac{m}{r}}$ *is (i) real or (ii) nonnegative according to case (i), or (ii).*

*Proof.* If $\chi$ and $T$ satisfy (a) or (b), then $\bar{\eta}K(T)$ is $(i)$ Hermitian $(ii)$ positive semi-definite by Theorem 3.4. So $W_\chi(T)$ is a subset of $\eta\mathbb{R}$ or $\eta[0,\infty)$, by Proposition 4.2.
Conversely, suppose $W_\chi(T) \subseteq \bar{\eta}\mathbb{R}$ or $\eta[0,\infty)$. Then

$$\begin{aligned}\Big((\eta K(T)^* - \bar{\eta}K(T))v^*, v^*\Big) &= \eta(K(T)^*v^*, v^*) - \bar{\eta}(K(T)v^*, v^*)\\ &= \eta(v^*, K(T)v^*) - \bar{\eta}(K(T)v^*, v^*)\\ &= \eta\overline{(K(T)v^*, v^*)} - \bar{\eta}(K(T)v^*, v^*)\\ &= 0,\end{aligned}$$

for all unit decomposable tensors $v^* \in V_\chi(H)$. By Proposition 4.3, $\eta K(T)^* - \bar{\eta}K(T) = 0$, i.e., $\bar{\eta}K(T)$ is Hermitian.
Applying Theorem 3.4, we see that $T$ satisfies (a) or (b). □

**Corollary 4.5.** *Let* $\chi$ *be not of the determinant type. Let* $T \in End(V)$*. Then* $T$ *is (i) Hermitian, (ii) positive semi- definite if and only if* $W_\chi(T + \eta I) \subseteq S$ *for all* $\eta \in S$ *with (i)* $S = \mathbb{R}$*,(ii)* $S = [0,\infty)$*.*

*Proof.* If $T$ is $(i)$ Hermitian, $(ii)$ positive semi-definite, then for any $\eta \in S$ with (i) $S = \mathbb{R}$, (ii) $S = [0,\infty)$, $K(T + \eta I)$ has the corresponding property. So the result follows from Proposition 4.2.

Conversely, suppose that $W_\chi(T + \eta I) \subseteq \mathbb{R}$ for all $\eta \in \mathbb{R}$. If $\eta \in \mathbb{R}$ is not an eigenvalue of $T$, then $T + \eta I$ is an invertible operator. Thus $\text{rank}(T + \eta I) \geq r = \mu(\bar{\Delta})$. So, $T + \eta I$ satisfies (a) or (b) of Theorem 4.4. If (b) holds, then $\chi$ is of the special type, and $T + \eta I = T_1 \oplus 0$, where $T_1$ is an invertible operator acting on an $r$-dimensional subspace. Since $T + \eta I$ is invertible, so $T + \eta I = T_1$, i.e., $r = n$, and $\chi$ is of the determinant type, a contradiction. Thus (a) holds and there exists $\xi \in \mathbb{C}$ with $\xi^m = \pm 1$ such that $\xi(T + \eta I)$ is Hermitian.
Since the number of the $m$th roots of unity is finite, there exist two

distinct real numbers $\eta_1$ and $\eta_2$ such that $\xi(T+\eta_1 I)$ and $\xi(T+\eta_2 I)$ are Hermitian, where $\xi$ is an $m$th root of unity, i.e.,

$$\begin{aligned}\bar{\xi}(T^* + \eta_1 I) &= \xi(T+\eta_1 I),\\ \bar{\xi}(T^* + \eta_2 I) &= \xi(T+\eta_2 I).\end{aligned}$$

Thus $\bar{\xi}(\eta_1 - \eta_2)I = \xi(\eta_1 - \eta_2)I$, and we get $\xi = \bar{\xi}$. Hence $T = T^*$ and the assertion holds.

Now, suppose that $W_\chi(T+\eta I) \subseteq [0,\infty)$ for all $\eta \in [0,\infty)$. Similar to the proof of case $(i)$, where $S = \mathbb{R}$, we conclude that $T$ is a Hermitian operator. To prove that $T$ is positive semi-definite, we show that the eigenvalues of $T$ are all nonnegative (see [10, Corollary 2.35]).
Let $\lambda$ be an eigenvalue of $T$ and $v$ be the corresponding eigenvector. Since the number of the $m$th roots of unity is finite, hence similar to the proof of case $(i)$, there exist a complex number $\xi$ with $\xi^m = 1$ and a decreasing sequence $\{\eta_n\}$ of real positive numbers such that $\eta_1 > -\lambda$, $\eta_n \to 0$ and for any $n \in \mathbb{N}$, $\xi(T+\eta_n I)$ is positive semi-definite. This implies that $\xi(\lambda+\eta_n) \geq 0$ for every $n \in \mathbb{N}$. Since $\eta_1 > -\lambda$, so we must have $\xi = 1$ and hence $\lambda + \eta_n \geq 0$ for all $n > 1$. Using the fact that $\eta_n \to 0$ completes the proof. $\square$

**Theorem 4.6.** *Suppose $\chi$ is not of the determinant type. Let $T \in End(V)$ and $\eta \in \mathbb{C}$ with $|\eta| = 1$.*

(a) *$W_\chi(T) \subseteq \eta(0,\infty)$ if and only if there exists $\xi \in \mathbb{C}$ with $\xi^m = \bar{\eta}$ such that $\xi T$ is positive definite.*

(b) *$W_\chi(T)$ is a singleton if and only if $T$ is a scalar operator.*

*Proof.*
(a) Suppose $W_\chi(T) \subseteq \eta(0,\infty)$. Then $\bar{\eta}\, W_\chi(T) \subseteq (0,\infty)$. So $(\bar{\eta}K(T)x^*, x^*) > 0$ for all unit decomposable tensors $x^*$. Therefore $\bar{\eta}K(T)$ is positive definite. By Theorem 3.2, we see that there exists $\xi \in \mathbb{C}$ with $\xi^m = \bar{\eta}$ such that $\xi\, T$ is positive definite.
Conversely, assume that there exists $\xi \in \mathbb{C}$ with $\xi^m = \bar{\eta}$ such that $\xi T$ is positive definite. Then $\bar{\eta}K(T) = \xi^m K(T) = K(\xi T)$ is positive definite. So $W(\bar{\eta}K(T)) \subseteq (0,\infty)$, by Proposition 4.2. Therefore $W_\chi(T) \subseteq W(K(T)) \subseteq \eta(0,\infty)$.

(b) If $T = \xi I$ for some $\xi \in \mathbb{C}$, then $K(T) = \xi^m I$. So $W_\chi(T) = \{\xi^m\}$ by definition. Conversely, let $W_\chi(T) = \{\xi\}$ for some $\xi \in \mathbb{C}$. Then

$(K(T)x^*, x^*) = \xi$ for all unit decomposable tensors $x^* \in V_\chi(H)$. So $((K(T) - \xi I)x^*, x^*) = 0$. Therefore, by Proposition 4.3, $K(T) = \xi I$. Now, we see that $T$ is a scalar operator from Theorem 3.3. □

By Theorem 4.4, we know that if $\chi$ is not of the determinant type, then $W_\chi(T)$ is a subset of a line passing through the origin if and only if $\xi T$ is Hermitian for some $\xi \in \mathbb{C}$. In the following theorem, we determine the conditions under which $W_\chi(T)$ is a subset of a line not passing through the origin.

**Theorem 4.7.** *Suppose $\chi$ is not of the determinant type, and $T \in End(V)$ is non-scalar. Then $W_\chi(T)$ is a subset of a line not passing through the origin if and only if for every $\alpha \in \hat{\Delta}$,*

$$(m_1(\alpha), \ldots, m_n(\alpha)) \text{ is a permutation of } (k, \ldots, k, \underbrace{k+1, \ldots, k+1}_{t}),$$

*where $m = nk+t$ with $1 \leq t < n$, and $T$ is an invertible normal operator such that one of the following holds:*

- **(a):** *$t = 1$ and the eigenvalues of $T$ lie on a line not passing through the origin;*
- **(b):** *$1 < t < n-1$ and one of the eigenvalues of $T$ has multiplicity $n-1$;*
- **(c):** *$t = n-1$ and the eigenvalues of $T^{-1}$ lie on a line not passing through the origin.*

*Proof.* Similar to the case of linear characters (see Theorem 3.8 in [8]). □

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