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WZ FACTORIZATION VIA ABAFFY-BROYDEN-SPEDICATO ALGORITHMS

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ABSTRACT. Classes of Abaffy-Broyden-Spedicato (ABS) methods have been introduced for solving linear systems of equations. The algorithms are powerful methods for developing matrix factorizations and many fundamental numerical linear algebra processes. Here, we show how to apply the ABS algorithms to devise algorithms to compute the WZ and ZW factorizations of a nonsingular matrix as well as the W^TW and Z^TZ factorizations of a symmetric positives definite matrix. We also describe the QZ and the QW factorizations, with Q orthogonal, and show how to appropriate the parameters of the ABS algorithms to compute these factorizations.
Keywords: ABS algorithms, WZ factorization, ZW factorization, W^TW factorization, Z^TZ factorization, QZ factoriation, QW factorization.
MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.

1. Introduction

ABS class of algorithms was constructed for the solution of linear systems utilizing some basic ideas such as projection and rank one update techniques [1, 3]. The ABS class later was extended to solve optimization problems [3] and systems of linear Diophantine equations (see [5, 6, 16, 17]). A scaled version of the linear ABS class was described in [3]. Reviews of ABS methods can be found in [22, 23].

A basic ABS algorithm starts with a nonsingular matrix $H_1 \in R^{n\times n}$

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(Spedicato's parameter), as a basis for the null space corresponding to the empty coefficient matrix (no equations). Given the Abaffian matrix H_i with rows generating the null space of the first $i-1$ equations, the ABS algorithm computes H_{i+1} as a null space generator of the first i equations. Consider the following linear system,

$$Ax = b, \qquad x \in R^n,\ A \in R^{n\times n},\ b \in R^n, \tag{1.1}$$

where $rank(A)$ is arbitrary. Obviously, the system (1.1) is equivalent to the following scaled system,

$$V^T Ax = V^T b, \tag{1.2}$$

where V, the scaling matrix, is an arbitrary nonsingular $n \times n$ matrix, named as the scaling parameter.

Let a_i^T be the ith row of A. A tailored scaled ABS algorithm as applied to A can be described as follows, where the output variable r gives the rank of A.

Algorithm 1. The scaled ABS (SABS) algorithm.

(1) Let $H_1 \in R^{n\times n}$ be arbitrary and nonsingular and $v_1 \in R^{n\times n}$ be an arbitrary nonzero vector. Set $i = 1$ and $r = 0$.

(2) Compute $s_i = H_i A^T v_i$.

(3) **If** $s_i = 0$, **then** set $H_{i+1} = H_i$, and **go to** (5) (the ith row is dependent on the first $i-1$ rows).

(4) If $\{s_i \neq 0\}$, then Compute $p_i = H_i^T f_i$, where $f_i \in R^n$ (Broyden's parameter), is an arbitrary vector satisfying $s_i^T f_i \neq 0$ and update H_i by

$$H_{i+1} = H_i - \frac{H_i A^T v_i q_i^T H_i}{q_i^T H_i A^T v_i}, \tag{1.3}$$

where $q_i \in R^n$ (Abaffy's parameter) is an arbitrary vector satisfying $s_i^T q_i \neq 0$. Let $r = r + 1$.

(5) **If** $i = n$, **then Stop** (columns of H_{i+1}^T generate the null space of A) **else** define $v_{i+1} \in R^n$, an arbitrary vector linearly independent of $v_1, \ldots, v_i$. Let $i = i + 1$ and **go to** (2).

The matrices H_i are generalizations of projections matrices. They probably first appeared in a book by Wedderburn [25] . They have been named *Abaffians* since the First International Conference on ABS Methods (Luoyang, China, 1991).

One important result of the ABS algorithms is the establishment of an implicit matrix factorization $V^T AP = L$, where L is a lower triangular matrix (see [3]).

Definition 1.1. *Let $A \in R^{n\times n}, P \in R^{n\times n}$ and $V \in R^{n\times n}$. The pair (P, V) is said to be A-conjugate if the matrix $L = V^T AP$ is lower triangular, and the pair (P, V) is said to be A-biconjugate if $D = V^T AP$ is nonsingular diagonal.*

Theorem 1.2. *Let $A \in R^{n\times n}$ and $V \in R^{n\times n}$. The SABS algorithms produce all of the possible A-conjugate pairs (P, V).*

Proof. See Theorem 4 in [2]. □

Corollary 1.3. *All matrix factorizations can be produced by using the SABS algorithm with proper definitions of the parameters.*

Choices of the parameters H_1, z_i and q_i determine particular methods within the class so that various matrix factorizations are derived. The implicit QR factorization via Gram-Schmidt algorithms of A is given by the choices $H_1 = I,\ q_i = f_i = a_i$ and $v_i = e_i$ [3], and the implicit LU factorization of A via Gaussian elimination techniques is given by the choices $H_1 = I$ and $q_i = f_i = v_i = e_i$ [3]. The LX factorization [24], Krylov's method [3], and Broyden's method [4] are all special cases of the ABS methods that are obtained by proper parameter settings. Furthermore, in a recent work we have shown that a specialized application of integer ABS methods leads to the Smith normal form of an integer matrix, having utility for solving linear Diophantine systems of equations [12].

Here, we show how to choose the parameters of the SABS algorithms for computing the WZ and WZ factorizations as well as the $W^T W$ and the $Z^T Z$ factorizations of a symmetric positive definite matrix. We also compute the QZ and the QW factorizations, where Q is an orthogonal matrix using the ABS algorithms.

The remainder of our work is organized as follows. In Section 2, we explain the WZ factorization. In Section 3, we discuss some characteristcs of nested submatrices of a matrix A which are used for computing new matrix factorizations. In Section 4, we present a new formulation of the existence condition for the WZ factorization of a matrix and compute the WZ and ZW factorizations using the SABS algorithm. There, we also show how to derive the W^TW and the Z^TZ factorizations of a positive definite matrix as well as the QZ and the QW factorizations. We conclude in Section 5.

2. WZ factorization

Definition 2.1. *Let s be a real number, and denote by $\lfloor s \rfloor$ ($\lceil s \rceil$), the greatest (least) integer less (bigger) than or equal to s.*

Definition 2.2. *A matrix $A = (a_{i,j}) \in R^{n\times n}$ is called a W-matrix if $a_{i,j} = 0$, for all (i,j) with $i > j$ and $i+j > n$ or with $i < j$ and $i+j \leq n$. The transpose of a W-matrix is called a Z-matrix. Thus, these matrices have the following forms:*

$$W = \begin{pmatrix} \bullet & \circ & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \circ & \bullet \end{pmatrix}, \ Z = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}, \tag{2.1}$$

where the empty bullets stand for zero and the other bullets stand for possible nonzeros.

Definition 2.3. *We say that a matrix A is factorized in the WZ form if*

$$A = WZ, \tag{2.2}$$

where the matrix W is a W-matrix and Z is a Z-matrix.

To solve a system of linear equations, the WZ factorization splitting procedure proposed in [10, 13, 20] is convenient for parallel computing. Detailed analyses of this factorization can be found in [8, 10]. The WZ factorization offers a parallel method for solving dense linear systems (1.1), where A is a square $n \times n$ matrix, and b is an n-vector [11]. A characterization for the existence of the WZ factorization is given in [14, 18]. A backward error analysis for the WZ factorization is provided in [21]. A pivoting strategy for a modified *WZ* factorization is proposed in [26].

The following theorems express the conditions for the existence of a *WZ* factorization of a nonsigular matrix (see [18]). Later in Section 4, we give a new set of conditions useful for our purposes.

Theorem 2.4. *(Factorization Theorem) Let $A \in R^{n\times n}$ be nonsingular. The matrix A has a WZ factorization if and only if for every k, $1 \le k \le s$, with $s = \lfloor n/2 \rfloor$, if n is even, and $s = \lceil n/2 \rceil$, if n is odd, the submatrix*

(2.3)

$$\Delta_k = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} & a_{1,n-k+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & \cdots & a_{k,k} & a_{k,n-k+1} & \cdots & a_{k,n} \\ a_{n-k+1,1} & \cdots & a_{n-k+1,k} & a_{n-k+1,n-k+1} & \cdots & a_{n-k+1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,k} & a_{n,n-k+1} & \cdots & a_{n,n} \end{pmatrix}_{2k,2k}$$

of A is invertible.

Proof. See Theorem 2 in [18]. □

Theorem 2.5. *If $A \in R^{n\times n}$ is nonsingular, then a WZ factorization can always be obtained by pivoting. That is, there exists a row permutation matrix Π and the factors W and Z such that*

$$\Pi A = WZ. \tag{2.4}$$

Proof. See Theorem 3 in [18]. □

Theorem 2.6. *Every symmetric positive definite matrix has a WZ factorization.*

Proof. See [18, 19]. □

When A is a symmetric positive definite matrix, it is possible to factor A in the Cholesky factorization form $A = LL^T$, for some lower triangular matrix L. A variant of the classical Cholesky factorization, called Cholesky QIF, is given by Evans [7, 9]. Existence and stability of this factorization are proved by Khazal [15].

3. Nested submatrices

It is well known that a nonsingular square matrix A has an LU factorization if and only if it is strongly nonsingular (the determinant of every leading principal submatrix is nonzero). Every sequence of nonsingular nested submatrices leads to a special matrix factorization.

In [3], the necessary and sufficient conditions for nonsingularity of the leading principal submatrices of a square matrix A using the ABS class of algorithms were given and the LU factorization was developed accordingly.

Here, we present a necessary and sufficient condition for nonsingularity of an arbitrary sequence of nested submatrices of a matrix using the ABS algorithms. Then, using the condition, we present an alternative formulation of the WZ and ZW factorizations using the ABS algorithms.

Notation: Let $A \in R^{n\times n}$. Here and subsequently $J_n = \{j_1, \ldots, j_n\}$ denotes a permutation of $\mathcal{I}_n = \{1, 2, \ldots, n\}$ and, for $k = 1, \ldots, n$, $J_k = \{j_1, \ldots, j_k\}$ denotes a subset of J_n. Let

$$A_{J_k} = (a_{i,j}),\ i, j \in J_k \tag{3.1}$$

denote a submatrix of A and

$$A_{J_1} \subset A_{J_2} \subset \ldots \subset A_{J_n} \tag{3.2}$$

be a sequence of nested submatrices of A. Then, we have the following results.

Theorem 3.1. *(Nested Submatrices) Let $A \in \mathbb{R}^{n\times n}$ and $H_1 = I$. Then, the nested submatrices A_{J_i}, $i = 1, \ldots, n$, are nonsingular if and only if $e_{j_i}^T A H_i e_{j_i} \neq 0$.*

Proof. The proof follows the lines of the proof for Theorem 5.4 in [3] replacing i by j_i. $\square$

Theorem 3.2. *Let $A \in \mathbb{R}^{n\times n}$, $H_1 = I$ and $a_{j_i}^T H_i e_{j_i} \neq 0$, for $i = 1, \ldots, n$. Then, the matrices*

$$H_{i+1} = H_i - \frac{H_i a_{j_i} e_{j_i}^T H_i}{e_{j_i}^T H_i a_{j_i}} \tag{3.3}$$

are well defined.

Proof. The proof follows the lines of the proof for Theorem 5.5 in [3] replacing i by j_i. □

Theorem 3.3. *Let the conditions of Theorem 3.2 be satisfied and H_{i+1} be defined by (3.3). Then, the following properties hold:*

(a) The jth row of H_{i+1} is zero, for $j \in J_i$.

(b) The jth column of H_{i+1} is equal to the jth column of H_1, for $j \notin J_i$.

Proof. See Theorem 5.5 in [3]. □

Now, let the conditions of Theorem 3.2 be satisfied. Then, we have

$$V^T AP = L \Rightarrow AP = V^{-T} L \Rightarrow APV^T = V^{-T} LV^T. \tag{3.4}$$

Different choices of the scale matrix V (as permutations) lead to interesting structures for $V^{-T}LV^T$, resulting in the ABS algorithms to compute new matrix factorizations. Therefore, for computing the factorization (3.4) we update H_i by

$$H_{i+1} = H_i - \frac{H_i v_i^T A q_i^T H_i}{q_i^T H_i A^T v_i} = H_i - \frac{H_i a_{j_i} q_i^T H_i}{q_i^T H_i a_{j_i}}. \tag{3.5}$$

where $q_i \in R^n$ is an arbitrary vector satisfying $a_{j_i}^T H_i q_i \neq 0$ and compute p_{j_i}, the j_ith search vector, as follows:

$$p_{j_i} = H_i^T f_i, \tag{3.6}$$

where $f_i \in R^n$ is an arbitrary vector satisfying $v_i^T A H_i f_i = a_{j_i}^T H_i f_i \neq 0$.

Next, we present some permutations J_n so that the scaled ABS algorithms produce the WZ and the ZW factorizations.

4. New matrix factorizations using ABS algorithms

Here, we compute the permutation $J_n = \{j_1, \ldots, j_n\}$ and the parameters q_i, f_i and v_i of an SABS algorithm to compute some new matrix factorizations by the class of ABS methods.

4.1. ***WZ* factorization using SABS algorithm.** We first present conditions for the existence of the WZ factorization and then compute the factorization using the ABS algorithms.

Let $J_n = \{j_1, \cdots, j_n\}$ be so that

$$j_i = \begin{cases} i & if\ i\ is\ odd, \\ \\ n-i+1 & if\ i\ is\ even. \end{cases} \tag{4.1}$$

Theorem 4.1. *Let j_k be defined by (4.1) and A_{J_k}, for $k = 1, \ldots, n$, be invertible. Then, there exists a WZ factorization for A, obtained by the ABS algorithms.*

Proof. According to Theorem 3.2, we have $e_{j_i}^T A H_i^T e_{j_i} \neq 0$, $i = 1, \ldots, n$. Now, let $q_i = e_{j_i}$, $i = 1, \ldots, n$. Then, according to Theorem 3.3, we have

$$H_{2i+1} = \begin{bmatrix} 0_{i,i} & 0 & 0 \\ R_i & I_{n-2i} & L_i \\ 0_{i,i} & 0 & 0 \end{bmatrix}, \tag{4.2}$$

with $R_i, L_i \in R^{n-2i,i}$, and

$$H_{2i} = \begin{bmatrix} 0_{i,i} & 0 & 0 \\ T_i & I_{n-2i+1} & S_i \\ 0_{i-1,i} & 0 & 0 \end{bmatrix}, \tag{4.3}$$

where $T_i \in R^{n-2i+1,i}$ and $S_i \in R^{n-2i+1,i-1}$.

Let $p_{j_i} = H_i^T e_{j_i}$. Then, P is a Z-matrix and we have

$$AP = W \Rightarrow A = WZ, \tag{4.4}$$

where $Z = P^{-1}$ is a Z-matrix. □

Theorem 4.2. *Let A be symmetric positive definite. Then, there exists a $Z^T Z$ factorization for A, obtained by the ABS algorithms.*

Proof. Consider the assumption of Theorem 4.1 and let $v_i = p_i$, for $i = 1, \ldots, n$. Then, $V^T AP$ is a diagonal matrix (see [3]), V and P are Z-matrices and we have

$$V^TAP = D \Rightarrow A = V^{-T}DP^{-1} = Z^TZ, \tag{4.5}$$

where D is a diagonal matrix, $Z^T = V^{-T}D^{1/2}$ and $Z = D^{1/2}P^{-1}$. □

4.1.1. *QZ Algorithm.*

Definition 4.3. *We say that a matrix A is factorized in the form QZ if*

$$A = QZ, \tag{4.6}$$

where the matrix Z is a Z-matrix and Q is an orthogonal matrix ($Q^TQ = I$).

Theorem 4.4. *Let $A \in R^{n\times n}$, $H_1 = I$, j_i be defined by (4.1), $q_i = z_i = a_{j_i}$ and $p_{j_i} = H_i^T a_{j_i}$, $i = 1, \cdots, n$. Then, there exists a QZ factorization, obtained by the ABS algorithms.*

Proof. According to Theorem 5.1 in [3], the p_{j_i} are orthogonal and we have

$$AP = W \Rightarrow A^T = QZ, \tag{4.7}$$

where $Z = W^T$ is a Z-matrix and $Q = P^{-T}$ is an orthogonal matrix. Thus, we admit a QZ factorization for A^T. Of course, a QZ factorization for A is easily found by applying the above process to A^T. □

4.2. ***ZW* factorization using SABS algorithm.** Here, we present an existence condition and compute the ZW factorization of a nonsingular matrix using the ABS algorithms.

Let $J_n = \{j_1, \ldots, j_n\}$ be so that

$$j_i = \begin{cases} \frac{n}{2} - i + 1 & if\ i\ is\ odd, \\ \\ \frac{n}{2} + i & if\ i\ is\ even. \end{cases} \tag{4.8}$$

Theorem 4.5. *Let j_k be defined by (4.8) and A_{J_k}, for $k = 1, \ldots, n$, be invertible. Then, there exists a WZ factorization for A, obtained by the ABS algorithms.*

Proof. According to Theorem 3.2, we have $e_{j_i}^T A H_i^T e_{j_i} \neq 0$, $i = 1, \ldots, n$. Now, let $q_i = e_{j_i}$. Then, according to Theorem 3.3, we have

$$H_{2i+1} = \begin{bmatrix} I_{\frac{n}{2}-i,\frac{n}{2}-i} & R_i & 0 \\ 0 & 0_{2i} & 0 \\ 0 & L_i & I_{\frac{n}{2}-i,\frac{n}{2}-i} \end{bmatrix}, \tag{4.9}$$

with $R_i, L_i \in R^{\frac{n}{2}-i,2i}$, and

$$H_{2i} = \begin{bmatrix} I_{\frac{n}{2}-i+1,\frac{n}{2}-i+1} & T_i & 0 \\ 0 & 0_{2i-1} & 0 \\ 0 & S_i & I_{\frac{n}{2}-i,\frac{n}{2}-i} \end{bmatrix}, \tag{4.10}$$

where $T_i \in R^{\frac{n}{2}-i,2i-1}$ and $S_i \in R^{\frac{n}{2}-i+1,2i-1}$.

Let $p_{j_i} = H_i^T e_{j_i}$. Then, P is a W-matrix and we have

$$AP = Z \Rightarrow A = ZW, \tag{4.11}$$

where $W = P^{-1}$ is a W-matrix. □

Theorem 4.6. *Let A be symmetric positive definite. Then, there exists a W^TW factorization for A, obtained by the ABS algorithms.*

Proof. Consider the assumption of Theorem 4.5 and let $v_i = p_i$, for $i = 1, \ldots, n$. Then, V^TAP is a diagonal matrix (see [3]), V and P are W-matrices and we have

$$V^TAP = D \Rightarrow A = V^{-T}DP^{-1} = W^TW, \tag{4.12}$$

where D is a diagonal matrix, $W^T = V^{-T}D^{1/2}$ and $W = D^{1/2}P^{-1}$. □

4.2.1. *QW Algorithm.*

Definition 4.7. *We say that a matrix A is factorized in the form QW if*

$$A = QW, \tag{4.13}$$

where the matrix W is a W-matrix and Q is an orthogonal matrix.

Theorem 4.8. *Let $A \in R^{n\times n}$, $H_1 = I$, j_i be defined by (4.8), $q_i = z_i = a_{j_i}$ and $p_{j_i} = H_i^T a_{j_i}$, $i = 1, \ldots, n$. Then, there exists an QW factorization, obtained by the ABS algorithms.*

Proof. According to [3, Theorem 5.1], the p_{j_i}'s are orthogonal and we have

$$AP = Z \Rightarrow A^T = QW, \tag{4.14}$$

where $W = Z^T$ is a W-matrix and $Q = P^{-T}$ is an orthogonal matrix. Thus, we admit a QW factorization for A^T. Of course, a QW factorization for A is easily found by applying the above process to A^T. □

5. Conclusions

We showed how to appropriate the parameters of ABS algorithms to construct algorithms for computing the WZ and ZW factorizations of a nonsingular matrix and the W^TW and Z^TZ factorizations of a symmetric positives definite matrix. New formulation for the existence conditions of the WZ factorization of a nonsingular matrix was given. We also derived two new factorizations, the QZ and QW, with Q orthogonal, and showed how to compute the factorizations using the ABS algorithms.

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