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A GRAPHICAL DIFFERENCE BETWEEN THE INVERSE AND REGULAR SEMIGROUPS

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Abstract. In this paper we investigate the Green graphs for the regular and inverse semigroups by considering the Green classes of them. And by using the properties of these semigroups, we prove that all of the five Green graphs for the inverse semigroups are isomorphic complete graphs, while this doesn't hold for the regular semigroups. In other words, we prove that in a regular semigroup S two Green graph $\Gamma_{\mathcal{L}}(S)$ and $\Gamma_{\mathcal{H}}(S)$ are isomorphic, however, the other three Green graphs are non-isomorphic to them.
Keywords: Regular and inverse semigroup, green relations, green graphs.
MSC(2010): Primary: 20M05; Secondary: 05C99.

1. Introduction

The motivation of this study is that, these graphs are independent of any subset of the inverse semigroups where, the identification of inverse semigroups by the Cayley graphs which studied by Kelarev in 2006 based on a fixed subset T of the semigroup.
Let S be a finite semigroup. The left Green graph $\Gamma_{\mathcal{L}}(S)$ for semigroup S, is an undirected graph with t vertices $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_t$ where $\mathcal{L}_i$'s are the left Green classes of semigroup S and two vertices $\mathcal{L}_i$ and $\mathcal{L}_j$ are adjacent in $\Gamma_{\mathcal{L}}(S)$ if and only if $g.c.d......(|\mathcal{L}_i|, |\mathcal{L}_j|) > 1$, ("g.c.d......" is used for the greatest common divisor). Following the article [5], the Green graphs for all of the Green relations in a finite semigroup have been studied.

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Indeed, the Green graphs are the generalization of conjugacy graphs (in finite groups) for the finite semigroups and these graphs may be use as a tool for classification of finite semigroups. In this paper we intend to study properties of the Green graphs for the finite regular and finite inverse semigroups. Our notations are fairly standard. In the group and semigroup presentations one may consult [4, 7, 12] and for the theoretical semigroup ideas one may see [6]. Also our computational examples will be in continuations of the articles [1, 2, 3, 5]. For the investigation of graphs related to algebraic structures and to find many more references on graphs associated to semigroups and applications of such graphs, we refer the readers to the articles [8, 9, 10]. The article [11] of Kelarev succeeded to identify the inverse semigroups by using the Cayley graphs $Cay(S,T)$, where S is a semigroup and T is any subset of S. Our identification and comparison of regular and inverse semigroups in terms of Green graphs dose not use any subset of the semigroup and exactly based on the intrinsic property of the Green relations on the semigroup. The notation K_n is used for the complete graph with n vertices.

For a finite semigroup S, an equivalence relation $\mathcal{L}$ on S is defined by the rule that $a\mathcal{L}b$, if and only if, $S^1a = S^1b$, where $S^1 = S$ if S possess an identity element, otherwise $S^1 = S \cup \{1\}$, such that, $1s = s1 = s$, for every $s \in S$. Similarly we define the equivalence relation $\mathcal{R}$ by the rule that $a\mathcal{R}b$, if and only if, $aS^1 = bS^1$. It is well known that $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence. The intersection of $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{H}$ and the join of $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{D}$. Also the two sided Green relation $\mathcal{J}$ on S is defined by the rule that $a\mathcal{J}b$, if and only if, $S^1aS^1 = S^1bS^1$. The $\mathcal{L}$-class (resp. $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{J}$-class and $\mathcal{D}$-class) containing the element a will be denoted by $\mathcal{L}_a = [a]_{\mathcal{L}}$ (resp. $\mathcal{R}_a = [a]_{\mathcal{R}}$, $\mathcal{H}_a = [a]_{\mathcal{H}}$, $\mathcal{J}_a = [a]_{\mathcal{J}}$ and $\mathcal{D}_a = [a]_{\mathcal{D}}$).

We collect the main results as follows:

Proposition A. A $\mathcal{D}$-class of a finite semigroup S, is a regular class if and only if it contains an idempotent element.

Proposition B. Let S be a finite semigroup. The following statements are equivalent:
1) S is an inverse semigroup.
2) S is a regular semigroup and its idempotents commute.

3) Each $\mathcal{L}$-class and each $\mathcal{R}$-class contains an idempotent element.

Proposition C. Let S be a finite regular semigroup. Then, $\Gamma_{\mathcal{D}}(S) \simeq \Gamma_{\mathcal{J}}(S) \simeq \Gamma_{\mathcal{R}}(S)$. Moreover, $\Gamma_{\mathcal{L}}(S) \simeq \Gamma_{\mathcal{H}}(S)$ which is a complete graph.

Proposition D. For every finite inverse semigroup S, all of the five Green graphs are of degree $|E(S)|$ which are complete isomorphic graphs, where $E(S)$ is the set of idempotents of S.

2. **Proofs of propositions A and B**

Let S be a finite semigroup. An element a of S is said to be regular if there exists $x \in S$ such that $axa = a$. A semigroup S is called regular if every element of S is regular. An element x of S is said to be idempotent, if and only if, $x^2 = x$. The set of all idempotent elements of S, denoted by $E(S)$. If $e \in E(S)$ is an idempotent element, then $e = eee$. So all idempotent elements of S are regular elements. Two elements a and b of a semigroup S are said to be inverse of each other, if and only if, $aba = a$ and $bab = b$. By an inverse semigroup we mean a semigroup in which every element has a unique inverse.

By these definitions, first we give some elementary results many of which of them will be used in the next section as well.

Lemma 2.1. *Let $e \in E(S)$ be an idempotent element. If $x\mathcal{L}e$, then $xe = x$, and if $x\mathcal{R}e$, then $ex = x$.*

Proof. If $x\mathcal{L}e$, then $x \in \mathcal{L}_e$ and there exists $s \in S^1$ such that $x = se$. So by $ee = e$, we get $x = se = see = se.e = xe$. The proof of the second part is similar. □

Lemma 2.2. *Let S be a finite semigroup and $x \in S$ be a regular element. Then xy and yx are idempotent elements.*

Proof. By the definition of regular element is easy. □

Lemma 2.3. *Let $x \in S$. Then the following statements are equivalent:*
(1) x is a regular element.
(2) For some idempotent element e, $x\mathcal{R}e$.
(3) For some idempotent element f, $x\mathcal{L}f$.

Proof. Let $x \in S$ be a regular element and for some $y \in S$, $x = xyx$. Then by Lemma 2.2 the elements $e = xy$ and $f = yx$ are idempotent elements and satisfy, $x\mathcal{R}e$ and $x\mathcal{L}f$. Suppose that $x\mathcal{R}e$. Now, by Lemma 2.1, $x = ex$ and there exists a $y \in S^1$ such that $e = xy$. Therefore, $x = ex = xyx$ and x is a regular element. Similarly if $x\mathcal{L}f$, then by Lemma 2.1, $x = xf$ and there exists a $z \in S^1$ such that $f = zx$. So $x = xf = xzx$ and x is a regular element. □

Lemma 2.4. *Let $x \in S$ be a regular element. Then the $\mathcal{D} -$ class, $\mathcal{D}_x$ is a regular class.*

Proof. We say the $\mathcal{D}$-class, $\mathcal{D}_x$ is a regular class, if all elements of this class are regular. Let x be a regular element, then by the Lemma 2.3 the class $\mathcal{D}_x$ contains an idempotent e, that is, $\mathcal{D}_x = \mathcal{D}_e$. Now, let $z \in \mathcal{D}_x$, $z\mathcal{D}e$, then there exists $u \in \mathcal{D}_x$ such that $e\mathcal{R}u$ and $u\mathcal{L}z$. Therefore, there exist $r, s, s^{'} \in S$ such that $e = ur$, $u = eu$, $u = sz$, $z = s^{'}u$. Consequently, $z = s^{'}u = s^{'}eu = s^{'}esz = s^{'}ursz = z.rs.z$, and z is a regular element. □

In particular, $\mathcal{D}_e$ is a regular class for each idempotent $e \in E(S)$.

Lemma 2.5. *Let D be a regular $\mathcal{D}$-class, then each $\mathcal{L}$-class $\mathcal{L}_x \subseteq D$ and each $\mathcal{R}$-class $\mathcal{R}_x \subseteq D$ contains an idempotent.*

Proof. Let $x \in D$. Then for some $y \in S$, $x = xyx$. Moreover xy and yx are idempotent elements. So, $x\mathcal{L}yx$ and $x\mathcal{R}xy$. This yields $yx \in \mathcal{L}_x$ and $xy \in \mathcal{R}_x$. □

These lemmas are useful to give detailed proofs of Propositions A and B. However one may also consult Propositions 2.3.1, 2.3.2 and Theorem 5.1.1 of Howie [6].

Proof of Proposition A. Using Lemmas 2.2, 2.4 and 2.5 or consult the Propositions 2.3.1 and 2.3.2 of [6]. □

Lemma 2.6. *Each regular element $x \in S$ has an inverse element.*

Proof. Let $x \in S$ be a regular element. Then for some $y \in S$, $x = xyx$ and $yxy = y(xyx)y = yxy.xy = yxy.xyxy = yxy.x.yxy$. So yxy is a regular element. Also $x = xyx = xyx.yx = x.yxy.x$. Therefore, yxy is an inverse element of x. □

Let S be a finite inverse semigroup and $e \in E(S)$. Then $eee = e$. So, for all idempotents e of an inverse semigroup S, $e^{-1} = e$ and $E(S)$ is a semilattice.

Lemma 2.7. *Let S be an inverse semigroup. Then the set of idempotents $E(S)$ forms a subsemigroup of S. Moreover, $E(S)$ is a semilattice, that is, the idempotents of S commute.*

Proof. Let $e, f \in E(S)$, and x be (unique) inverse of ef $(x = (ef)^{-1})$. Then $ef = ef.x.ef = ef.xe.ef$ or $ef = ef.x.ef = ef.fx.ef$. And $xe.ef.xe = xefx.e = xe$, $fx.ef.fx = f.xefx = fx$. So $x = (ef)^{-1} = xe = fx$. Also $x \in E(S)$ since, $x^2 = xe.fx = x.ef.x = x.x^{-1}.x = x$. Thus, for all $e, f \in E(S)$, $ef \in E(S)$, that is, $E(S)$ is a subsemigroup of S. Furthermore for all $e, f \in E(S)$, $ef, fe \in E(S)$ and $ef.fe.ef = efef = (ef)^2 = ef$, $fe.ef.fe = fefe = (fe)^2 = fe$. Therefore $fe = (ef)^{-1} = ef$ and $E(S)$ is commutative. □

Proof of Proposition B. Use Lemmas 2.5, 2.6, and 2.7 or consult Theorem 5.1.1 of [6]. □

Note that for an inverse semigroup S, $x^{-1}E(S)x \subseteq E(S)$ holds for every $x \in S$. Moreover, the Green relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{D}$ may be redefined as follows:
(1) $x\mathcal{L}y \Longleftrightarrow x^{-1}x = y^{-1}y$.
(2) $x\mathcal{R}y \Longleftrightarrow xx^{-1} = yy^{-1}$.
(3) $e\mathcal{D}f \Longleftrightarrow \exists z \in S$ such that $e = zz^{-1}$ and $f = z^{-1}z$, for the idempotents e and f.

3. Proofs of Propositions C and D

Let S be a finite semigroup then, $\Gamma_{\mathcal{D}}(S) \simeq \Gamma_{\mathcal{J}}(S)$ is a quick consequence of the identification of the Green relations $\mathcal{D}$ and $\mathcal{J}$ (see [6]). Now for a finite regular semigroup S, we can prove Proposition C.

Proof of Proposition C. By the above comment, $\Gamma_{\mathcal{J}}(S) \simeq \Gamma_{\mathcal{D}}(S)$. So it is sufficient to show that $\Gamma_{\mathcal{R}}(S) \simeq \Gamma_{\mathcal{J}}(S)$. For every $\mathcal{J}$-class $[x]_{\mathcal{J}}$, if $y \in [x]_{\mathcal{J}}$, there exist $u_1, v_1, u_2, v_2 \in S$ such that $y = u_1xv_1$ and $x = u_2yv_2$. So, $y = u_1xv_1 = (u_1x)v_1 = x'v_1$ and then $y \in [x']_{\mathcal{R}}$. Also if $y \in [z]_{\mathcal{R}}$, then there exists $k \in S$ such that $y = zk$, and by the regularity of S, there exists $r \in S$ such that $z = zrz$. So $y = zrzk = z(rz)k = zlk \in [l]_{\mathcal{J}}$. Thus, every $\mathcal{J}$-class is equal to an $\mathcal{R}$-class. Now, if $[x_1]_{\mathcal{J}} = [x'_1]_{\mathcal{R}}$ and $[x_2]_{\mathcal{J}} = [x'_2]_{\mathcal{R}}$ such that $g.c.d.(|[x_1]_{\mathcal{J}}|, |[x_2]_{\mathcal{J}}|) > 1$, then, $g.c.d.|[x'_1]_{\mathcal{R}}|, |[x'_2]_{\mathcal{R}}|) > 1$. Thus there is a bijection between the vertex sets of $\Gamma_{\mathcal{R}}(S)$ and $\Gamma_{\mathcal{J}}(S)$ such that two adjacent vertices of $\Gamma_{\mathcal{R}}(S)$ map two the adjacent vertices of $\Gamma_{\mathcal{J}}(S)$. This shows that $\Gamma_{\mathcal{R}}(S) \simeq \Gamma_{\mathcal{J}}(S)$. For the other part of the proposition it is obvious that each $\mathcal{H}$-class of the semigroup S is a subset of a $\mathcal{L}$-class. Let $x \in [a]_{\mathcal{L}}$. Then, there exist $u, v \in S$ such that $x = ua$ and $a = vx$. By regularity of S, there exists $r \in S$ such that $a = ara$. So $x = ua = uara = ua(ra) = uak \in [a]_{\mathcal{J}} = [a]_{\mathcal{R}}$, and $x \in [a]_{\mathcal{L}} \cap [a]_{\mathcal{R}} = [a]_{\mathcal{H}}$. This proves that $\Gamma_{\mathcal{H}}(S) \simeq \Gamma_{\mathcal{L}}(S)$ which is a complete graph by considering the regularity condition and definition of $\mathcal{H}$ relation. $\square$

We note that if all of the five Green graphs of a semigroup S are isomorphic, then the semigroup S is regular. For, let S be a finite semigroup that all of its five Green graphs are isomorphic. Then for a fixed element $x \in S$, there exists a $\mathcal{J}$-class $[a]_{\mathcal{J}}$ such that $x \in [a]_{\mathcal{J}}$. So, there are $u, v, u', v' \in S$ such that $x = uav$ and $a = u'xv'$. The relations $x = uav$ and $a = u'xv'$ imply $u \in [x]_{\mathcal{R}}$ and $v \in [x]_{\mathcal{L}}$. Consequently, there exist $r, s \in S$ such that $u = xr$ and $v = sx$. So $x = uav = (xr)a(sx) = x(ras)x = xkx$. This implies that x is a regular element and consequently S is a regular semigroup.

Note that the reverse of the above comment, does not hold in general. But for the inverse semigroups this is true, as in Proposition D.

Proof of Proposition D. Let S be a finite inverse semigroup. By Proposition B, the semigroup S is regular and each $\mathcal{L}$-class and each $\mathcal{R}$-class has exactly one idempotent. Furthermore, for each $a \in S$, we have $a\mathcal{R}aa^{-1}\mathcal{L}a^{-1}\mathcal{R}a^{-1}a\mathcal{L}a$, and in the inverse semigroup, $a\mathcal{R}b$, if and only if, $aa^{-1} = bb^{-1}$ ($a\mathcal{L}b$, if and only if $a^{-1}a = b^{-1}b$). The map $a \longrightarrow a^{-1}$ induces a bijection between the elements of $\mathcal{R}$-class $\mathcal{R}_{aa^{-1}}$ and the elements of $\mathcal{L}$-class $\mathcal{L}_{aa^{-1}}$. So we may conclude that in an inverse semigroup S, $\Gamma_{\mathcal{L}}(S) \simeq \Gamma_{\mathcal{R}}(S)$. Now, by the Proposition C we see that all of the five Green graphs are isomorphic, and these graphs of S are complete graphs of degree $|E(S)|$ (by the Lemma 2.2 and that $|E(S)| \geq 2$). $\square$

4. **Conclusion**

Certain finitely presented finite semigroups will be studied here as examples of the results of the last section. For a detailed investigation on the structure and order of the studied semigroups one may see [1, 2, 4, 5].

(1). Consider the non-commutative finitely presented semigroup

$$S_1 =< a, b | ab = ba^{1+p^{\alpha-\gamma}}, a = a^{1+p^{\alpha}}, b = b^{1+p^{\beta}} > .$$

Where, $p \geq 2$ is a prime, α, β and γ are integers such that $\alpha \geq 2\gamma$, $\beta \geq \gamma \geq 1$, and $\alpha + \beta \geq 3$.

The semigroup S_1 is an example of a regular semigroup such that $E(S_1)$ is a semilattice. For verifying the regularity of S_1, we note that the elements of S_1 have one of the three forms, a^i, b^j and $b^j.a^i$ where $1 \leq i \leq p^\alpha$, $1 \leq j \leq p^\beta$. By the definition of regularity and using the relations of the semigroup S_1 we may see that:

$a^i.a^{p^\alpha - i}.a^i = a^i, \quad (1 \leq i \leq p^\alpha - 1),$

$b^j.b^{p^\beta - j}.b^j = b^j, \quad (1 \leq j \leq p^\beta - 1),$

$(b^j.a^i).(b^{j(p^\beta - 1)}.a^{i(j.p^{\alpha-\gamma} - 1)}).(b^j.a^i) = b^j.a^i, \quad (1 \leq i \leq p^\alpha,\ 1 \leq j \leq p^\beta).$

Also three elements a^{p^α}, b^{p^β} and $b^{p^\beta}.a^{p^\alpha}$ are idempotent elements and so are regular. Thus by Proposition B, this semigroup is an inverse semigroup and the cardinality of $E(S_1)$ is 3. On the other hand, in an inverse semigroup every $\mathcal{L}$-class and every $\mathcal{R}$-class contains exactly one idempotent. So, the semigroup S_1 has three $\mathcal{L}$-classes and three $\mathcal{R}$-classes. Indeed, we can easily show that $\Gamma_{\mathcal{L}}(S_1) \simeq \Gamma_{\mathcal{R}}(S_1) \simeq \Gamma_{\mathcal{J}}(S_1) \simeq \Gamma_{\mathcal{H}}(S_1) \simeq \Gamma_{\mathcal{D}}(S_1) \simeq K_3$.

(2). For every integer $n \geq 2$, consider two classes of non-commutative finitely presented semigroups as follows:

$$S_2 = < a, b | a^3 = a, b^n a = a, abab^2 = b >,$$
$$S_3 = < a, b | a^3 = a, b^{n+1} = b, abab^2 = b > .$$

S_2 and S_3 are examples of finite regular semigroups. Proving the regularity is easy by using the relations of the semigroups, and using GAP [13], we see that

$\Gamma_{\mathcal{R}}(S_2) = \Gamma_{\mathcal{J}}(S_2) = \Gamma_{\mathcal{D}}(S_2) = K_1,$

$\Gamma_{\mathcal{L}}(S_2) = \Gamma_{\mathcal{H}}(S_2) = K_2.$

$\Gamma_{\mathcal{R}}(S_3) = \Gamma_{\mathcal{J}}(S_3) = \Gamma_{\mathcal{D}}(S_3) = K_2,$

$\Gamma_{\mathcal{L}}(S_3) = \Gamma_{\mathcal{H}}(S_3) = K_4.$

(3). For every integer $n \geq 2$, consider the following two classes of non-commutative finitely presented semigroups:

$$S_4 = < a, b | a^3 = a, b^{2n+1} = b, ab^2ab^{n-1} = ba >,$$
$$S_5 = < a, b | a^3 = a, b^{2n+1} = b, ab^{n-1}ab^2 = ba > .$$

The semigroups S_4 and S_5 are not regular, for, we may use the general forms of their elements to show that the regularity condition does not hold for at least one element in each of them. The Green graphs of these semigroups have been computed in [5] and shows taht Propositions C and D for these semigroups does not hold.

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