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Author(s):

R. Farnoosh and M. Aalaei

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A STOCHASTIC ALGORITHM TO SOLVE MULTIPLE DIMENSIONAL FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

R. FARNOOSH* AND M. AALAEI

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Abstract. In the present work, a new stochastic algorithm is proposed to solve multiple dimensional Fredholm integral equations of the second kind. The solution of the integral equation is described by the Neumann series expansion. Each term of this expansion can be considered as an expectation which is approximated by a continuous Markov chain Monte Carlo method. An algorithm is proposed to simulate a continuous Markov chain with probability density function arisen from an importance sampling technique. Theoretical results are established in a normed space to justify the convergence of the proposed method. The method has a simple structure and it is a good candidate for parallelization because of the fact that many independent sample paths are used to estimate the solution. Numerical results are performed in order to confirm the efficiency and accuracy of the present work.
Keywords: Fredholm integral equations, Monte Carlo method, continuous Markov chain, Neumann series expansion, importance sampling.
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1. Introduction

Multiple dimensional Fredholm integral equations of the second kind are arisen directly from mathematical modeling of real world problems in applied sciences such as computer graphics, mathematical physics

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*Corresponding author.

and engineering [5, 6]. For example, continuous particle transport problems can be formulated as Fredholm integral equations of the second kind with dimensions up to 7, [7]. Therefore, the choice of appropriate approach to solve multiple dimensional Fredholm integral equations is very important. But it is difficult to find the exact solution of these equations. So far, many numerical algorithms and various stochastic algorithms are introduced, discussed and developed to solve

$$\psi(\boldsymbol{x}) = \int_{\Gamma} k(\boldsymbol{x}, \boldsymbol{y})\psi(\boldsymbol{y})d\boldsymbol{y} + S(\boldsymbol{x}), \tag{1.1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_s), \boldsymbol{y} = (y_1, y_2, \ldots, y_s) \in \Gamma = \prod_{i=1}^{s}[a_i, b_i]$, $S(\boldsymbol{x})$ and $k(\boldsymbol{x}, \boldsymbol{y})$ are given functions, respectively on $D = L^2(\prod_{i=1}^{s}[a_i, b_i])$, $E = D \times D$ and $\psi(\boldsymbol{x})$ is unknown on D.

Numerical algorithms obtain deterministic approximated solutions. In [10], a fast wavelet collocation method is developed to solve integral equations on polygons. Two dimensional triangular orthogonal functions have been used to approximate the solution in [8]. A numerical algorithm based on discrete Galerkin method and Richardson- type extrapolation schemes is considered to solve these equations in [3]. Some iterative corrections for finite element solutions of two dimensional Fredholm integral equations of the second kind have been done in [11].

The approximated solutions based on stochastic algorithms are not deterministic. Monte Carlo methods are stochastic tools for solving multiple dimensional integral equations. In the work [7], adaptive Monte Carlo methods are applied to solve matrix equations arisen from multiple integral equations. Also, it is well known that Monte Carlo methods are preferable for solving multiple integrals, such as those arising from approximations of integral equations [2, 4, 9]. The error of Monte Carlo approximations for solving integrals is proportional to $N^{-1/2}$ for N sample points and the error of standard composite numerical integration procedures is proportional to $N^{-\frac{\alpha+1}{p}}$ where p is the dimension of integral and α is the precision of the numerical integration method, for example $\alpha = 1$ for the trapezoidal rule and $\alpha = 3$ for Simpson's rule. It is clear that Monte Carlo methods can have significant accuracy advantages over other methods especially for large dimensions p [2]. In this article, instead of approximating the multiple integrals with classical MC algorithms, we approximate them with a simple structure Markov Chain Monte Carlo method.

The goal of this paper is to introduce a new stochastic method to solve

multiple dimensional integral equations of the second kind. Our main idea is to use continuous Markov chain for solving these equations. The proposed method has an acceptable accuracy, low cost, desirable speed and simple structure simultaneously. Theoretical results are established in normed spaces to justify the convergence of the method. According to our best knowledge of research works, the proposed method has not been studied before and it is applied to multiple integral equations for the first time.

The paper is organized as follows: Introducing an integral operator, the Neumann series expansion of the solution of multiple dimensional Fredholm integral equations of the second kind is described in Section 2. We also discuss the theoretical results on the convergence condition and the error estimation in normed space. Continuous Markov chain Monte Carlo method via importance sampling technique for solving these equations is described in Section 3 and the convergence of the method is discussed theoretically. Furthermore, an algorithm is considered to simulate the random pathes. Numerical experiments are given in Section 4 and our conclusions and future directions are given in Section 5.

2. Neumann Series Expansion to Solve Fredholm integral equations

Let us define the integral operator K such that

$$(K\psi)(\boldsymbol{x}) = \int_{\Gamma} k(\boldsymbol{x}, \boldsymbol{y})\psi(\boldsymbol{y})d\boldsymbol{y},$$

and

$$(K^n\psi)(\boldsymbol{x}) = \int_{\Gamma} k(\boldsymbol{x}, \boldsymbol{y}_{n-1})(K^{n-1}\psi)(\boldsymbol{y}_{n-1})d\boldsymbol{y}_{n-1}. \tag{2.1}$$

where K^0 is supposed to be the identity function. Equation (1.1) may be written in the operational form as

$$\psi = S + K\psi. \tag{2.2}$$

To solve (1.1), we apply the following recursive equation:

$$\psi^{(n+1)} = K\psi^{(n)} + S, \quad n = 0, 1, 2, \ldots. \tag{2.3}$$

Assuming $\psi^0 = 0$ and $K^0 \equiv I$, we have

$$\psi^{(n+1)} = S + KS + \ldots + K^nS = \sum_{m=0}^{n} K^mS. \tag{2.4}$$

Taking the limit, we obtain the Neumann series

$$\lim_{n\to\infty} \psi^{(n)} = \sum_{m=0}^{\infty} K^m S. \tag{2.5}$$

Before proceeding further, we discuss the error estimation and the convergence condition of Equation (2.5).

2.1. **Convergence condition and error estimation.** To discuss the convergence and the error of the estimated solution, we define the following norm for the linear operator K.

Definition 2.1. *Let $K : U \to W$ be a linear operator between two normed spaces. The norm of K is defined by*

$$\|K\| = \sup\{\|K\psi\| : \|\psi\| = 1\}.$$

Lemma 2.2. *For all $\psi \in U$, the following inequality holds:*

$$\|K\psi\| \leq \|K\|.\|\psi\|. \tag{2.6}$$

The proof of lemma (2.2) is clear. The condition for the convergence of recursive equation (2.3) is given by the following theorem. Note that the infinity norm is used.

Theorem 2.3. *If the inequality*

$$\sup\{|k(\boldsymbol{x},\boldsymbol{y})| : (\boldsymbol{x},\boldsymbol{y}) \in (\prod_{i=1}^{s}[a_i,b_i])^2\} < \frac{1}{\prod_{i=1}^{s}(b_i-a_i)} \tag{2.7}$$

holds, the recursive equation (2.3) *is convergent.*

Proof. If $M = \sup\{|k(\boldsymbol{x},\boldsymbol{y})| : (\boldsymbol{x},\boldsymbol{y}) \in (\prod_{i=1}^{s}[a_i,b_i])^2\}$, then

$$|(K\psi)(\boldsymbol{x},\boldsymbol{y})| \leq M\prod_{i=1}^{s}(b_i-a_i)\|\psi\|_\infty.$$

Therefore we have $\|K\psi\|_\infty \leq M\prod_{i=1}^{s}(b_i-a_i)\|\psi\|_\infty$. Considering the definition (2.1) and the inequality (2.7), we can write

$$\|K\| \leq M\prod_{i=1}^{s}(b_i-a_i) < 1. \tag{2.8}$$

Now subtracting the formulae (2.3) and (2.2), we have

$$\psi^{(n+1)} - \psi = K(\psi^{(n)} - \psi),$$

and applying Lemma (2.2) recursively, we get

(2.9)
$$\begin{aligned}\|\psi^{(n+1)}-\psi\|_\infty &= \|K(\psi^{(n)}-\psi)\|_\infty \le \|K\|.\|\psi^{(n)}-\psi\|_\infty \\ &\le \|K\|^2.\|\psi^{(n-1)}-\psi\|_\infty \le \cdots \le \|K\|^{n+1}.\|\psi\|_\infty.\end{aligned}$$

Finally using Equation (2.8), we obtain

$$\lim_{n\to\infty}\|\psi^{(n+1)}-\psi\|_\infty = 0.$$

□

Theorem 2.4. *Assume that $\|K\| < 1$. Then for $n \ge 1$ the absolute and relative errors of the estimated solution $\psi^{(n)}$ satisfy the following inequalities*

$$\|\psi^{(n)}-\psi\|_\infty \le \frac{\|K\|^n}{1-\|K\|}\|\psi^{(1)}\|_\infty, \tag{2.10}$$

$$\frac{\|\psi^{(n)}-\psi\|_\infty}{\|\psi\|_\infty} \le \|K\|^n. \tag{2.11}$$

Proof. The following inequalities hold clearly,

$$\begin{aligned}\|\psi^{(n+1)}-\psi^{(n)}\|_\infty &= \|\psi^{(n+1)}-\psi-(\psi^{(n)}-\psi)\|_\infty \\ &\ge \|\psi^{(n)}-\psi\|_\infty - \|\psi^{(n+1)}-\psi\|_\infty \\ &\ge (1-\|K\|)\|\psi^{(n)}-\psi\|_\infty\end{aligned} \tag{2.12}$$

On the other hand, by using Lemma (2.2) and Equation (2.3) recursively, we can write

(2.13)
$$\|\psi^{(n+1)}-\psi^{(n)}\|_\infty \le \|K\|.\|\psi^{(n)}-\psi^{(n-1)}\|_\infty \le \cdots \le \|K\|^n.\|\psi^{(1)}\|_\infty$$

Considering equations (2.13) and (2.12) together, one can get

$$(1-\|K\|)\|\psi^{(n)}-\psi\|_\infty \le \|K\|^n.\|\psi^{(1)}\|_\infty$$

and therefore we obtain Equation (2.10). To prove Equation (2.11), note that it is another form of Equation (2.9). □

In the next section, the solution of multiple dimensional integral equation is estimated using Monte Carlo method. It means that a random variable is defined as the unbiased estimator of $\psi^{(n)}$. Note that the simulation strategy, described in the next section, estimate the sum of the first $n+1$ terms of the Neumann series.

3. Continuous Markov chain Monte Carlo method

The Monte Carlo methods share the characteristic that they rely upon random paths generated by a proposal density. They differ only by how they use these random paths. Our proposed Monte Carlo method is estimating the function $\psi(\boldsymbol{x})$ by an importance sampling technique using a continuous probability density function to simulate these random paths.

3.1. Importance sampling technique.

Suppose we are estimating the multiple integral

$$\xi = \int_{\Gamma} h(\boldsymbol{x}) d\boldsymbol{x} \tag{3.1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_s)$ and $h(\boldsymbol{x})$ is any function such that the integral exists. Let $g(\boldsymbol{x})$ be any probability density function on Γ. Then Equation (3.1) can be considered as

$$\xi = \int_{\Gamma} \frac{h(\boldsymbol{x})}{g(\boldsymbol{x})} g(\boldsymbol{x}) d\boldsymbol{x} = E[\frac{h(\boldsymbol{x})}{g(\boldsymbol{x})}]. \tag{3.2}$$

An unbiased estimator of ξ is

$$\hat{\xi} = \frac{1}{m} \sum_{i=1}^{m} \frac{h(\boldsymbol{X}_i)}{g(\boldsymbol{X}_i)}$$

where $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_m$ are identically and independently distributed random vectors with density g. Then the variance of ξ can be obtained by:

$$var\hat{\xi} = \frac{1}{m} var(\frac{h(\boldsymbol{X})}{g(\boldsymbol{X})}) = \frac{1}{m}(\int \frac{h^2(\boldsymbol{x})}{g(\boldsymbol{x})} d\boldsymbol{x} - \xi^2)$$

Theorem 3.1. *The minimum of $var\hat{\xi}$ is equal to*

$$var\hat{\xi}_0 = \frac{1}{m}[(\int |h(\boldsymbol{x})| d\boldsymbol{x})^2 - \xi^2]$$

where the random variable $\boldsymbol{X}$ is distributed with probability density

$$g(\boldsymbol{x}) = \frac{|h(\boldsymbol{x})|}{\int |h(\boldsymbol{x})| d\boldsymbol{x}} \tag{3.3}$$

To prove Theorem (3.1), see [1]. It is clear that if $h(\boldsymbol{x}) > 0$, then $var\hat{\xi}_0 = 0$. However, Under this assumption, the denominator in Equation (3.3) is ξ which is required to be found, so this ideal can not be achieved in practice. Nevertheless, it does indicate that a good choice of the importance sampling distribution, $g(x)$, is one that is similar in shape to $h(x)$.
Furthermore, we combine the Neumann series expansion, Monte Carlo method and importance sampling technique to solve multiple dimensional integral equations.

3.2. **Monte Carlo method via importance sampling technique.**

Consider a truncated first finite sum of terms in the Neumann series expansion of the solution (2.4) as an approximate solution of (1.1). Using iterative equation (2.1) for each term $(K^m S)(\boldsymbol{x})$, we have

$$(K^m S)(\boldsymbol{x}) = \int_\Gamma \int_\Gamma \cdots \int_\Gamma k(\boldsymbol{x}, \boldsymbol{y}_1) k(\boldsymbol{y}_1, \boldsymbol{y}_2) \cdots k(\boldsymbol{y}_{m-1}, \boldsymbol{y}_m) S(\boldsymbol{y}_m) d\boldsymbol{y}_1 d\boldsymbol{y}_2 \cdots d\boldsymbol{y}_m.$$

Let us define probability transition P, the initial probability density p and

$$P(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{s} P(x_i, y_i), \quad p(\boldsymbol{x}) = \prod_{i=1}^{s} p(x_i).$$

Under some assumption, we can consider the combined probability density function as follows

$$P(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m) = p(\boldsymbol{x}) P(\boldsymbol{x}, \boldsymbol{y}_1) \prod_{i=2}^{m} P(\boldsymbol{y}_{i-1}, \boldsymbol{y}_i).$$

Using the importance sampling technique, we have

$$(K^m S)(\boldsymbol{x}) = \frac{1}{p(\boldsymbol{x})} \int_\Gamma \cdots \int_\Gamma \frac{k(\boldsymbol{x}, \boldsymbol{y}_1)}{P(\boldsymbol{x}, \boldsymbol{y}_1)} \prod_{i=2}^{m} \frac{k(\boldsymbol{y}_{i-1}, \boldsymbol{y}_i)}{P(\boldsymbol{y}_{i-1}, \boldsymbol{y}_i)} P(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m) S(\boldsymbol{y}_m) d\boldsymbol{y}_1 d\boldsymbol{y}_2 \cdots \boldsymbol{y}_m.$$

Therefore

$$(K^m S)(\boldsymbol{x}) = E[\frac{1}{p(\boldsymbol{x})} \frac{k(\boldsymbol{x}, \boldsymbol{y}_1)}{P(\boldsymbol{x}, \boldsymbol{y}_1)} \prod_{i=2}^{m} \frac{k(\boldsymbol{y}_{i-1}, \boldsymbol{y}_i)}{P(\boldsymbol{y}_{i-1}, \boldsymbol{y}_i)} S(\boldsymbol{y}_m)]$$

Now using Monte Carlo method to estimate this expectation, we have an approximate solution based on the combination of Monte Carlo method and the importance sampling technique. We then discuss this procedure in details.
Consider independent Markov chains $t_i^0 \to t_i^1 \to \ldots$ with state space

$[a_i, b_i]$, for $i = 1, \cdots, s$. Let us define $\boldsymbol{t}^m = (t_1^m, t_2^m, \cdots, t_s^m)$ and the weight function w_m with the following recursive formula

$$w_m = w_{m-1} \frac{k(\boldsymbol{t}^{m-1}, \boldsymbol{t}^m)}{P(\boldsymbol{t}^{m-1}, \boldsymbol{t}^m)}, \quad w_0 = 1.$$

The following theorem shows that the random variable

$$\eta_n(r) = \frac{r(\boldsymbol{t}^0)}{p(\boldsymbol{t}^0)} \sum_{m=0}^{n} w_m S(\boldsymbol{t}^m),$$

is an unbiased Monte Carlo approximation using the importance sampling technique to solve Fredholm integral equation.

Definition 3.2. *The inner product of two continuous function f and g with domain Γ is defined by*

$$\langle f, g \rangle = \int_\Gamma f(x) g(x) dx.$$

Theorem 3.3. *The mathematical expectation value of the random variable $\eta_n(r)$ is equal to the inner product $\langle r, \psi^{(n+1)} \rangle$ i.e.,*

$$E[\eta_n(r)] = \langle r, \psi^{(n+1)} \rangle.$$

Proof. Since the random paths $t_i^0 \to t_i^1 \to \ldots \to t_i^n$, for $i = 1, \cdots, s$ are considered to be independent, the combined probability density can be obtained by

$$P(\boldsymbol{t}^0, \boldsymbol{t}^1, \cdots, \boldsymbol{t}^n) = \prod_{i=1}^{s} p(t_i^0) P(t_i^0, t_i^1) \cdots P(t_i^{n-1}, t_i^n),$$

The desired expectation is calculated as follows:

$$\begin{aligned} E[\eta_n(r)] &= E[\tfrac{r(\boldsymbol{t}^0)}{p(\boldsymbol{t}^0)} \textstyle\sum_{m=0}^{n} w_m S(\boldsymbol{t}^m)] \\ &= \int \int \cdots \int \tfrac{r(\boldsymbol{t}^0)}{p(\boldsymbol{t}^0)} \textstyle\sum_{m=0}^{n} w_m S(\boldsymbol{t}^m) P(\boldsymbol{t}^0, \boldsymbol{t}^1, \cdots, \boldsymbol{t}^n) d\boldsymbol{t}^0 d\boldsymbol{t}^1 \ldots d\boldsymbol{t}^n, \end{aligned}$$

where $d\boldsymbol{t}^0 d\boldsymbol{t}^1 \ldots d\boldsymbol{t}^n = \prod_{i=1}^{s} dt_i^0 dt_i^1 \ldots dt_i^n$. Because $\int_0^1 P(x, y) dy = 1$, we have

$$E[\eta_n(r)] = \sum_{m=0}^{n} \int \ldots \int r(\boldsymbol{t}^0) k(\boldsymbol{t}^0, \boldsymbol{t}^1) \ldots k(\boldsymbol{t}^{m-1}, \boldsymbol{t}^m) S(\boldsymbol{t}^m) d\boldsymbol{t}^0 d\boldsymbol{t}^1 \ldots d\boldsymbol{t}^n.$$

Thus

$$E[\eta_n(r)] = \langle r, \sum_{m=0}^{n} K^m S \rangle = \langle r, \psi^{(n+1)} \rangle.$$

□

To estimate, $\langle r, u^{(n+1)}\rangle$, we consider sN random paths of length n and calculate the sample mean

$$\gamma_n(r) = \frac{1}{N}\sum_{s=1}^{N}\eta_n^{(s)}(r). \tag{3.4}$$

Assuming $r(\boldsymbol{x}) = \prod_{i=1}^{s}\delta(x_i - z_i)$, we will obtain the approximated solution in point $\boldsymbol{z} = (z_1, z_2, \ldots, z_s)$:

$$\begin{aligned}\langle r(\boldsymbol{x}), \psi^{(n+1)}(\boldsymbol{x})\rangle &= \langle r(\boldsymbol{x}), (\textstyle\sum_{m=0}^{n} K^m S)(\boldsymbol{x})\rangle \\ &= \int\int\cdots\int \textstyle\prod_{i=1}^{s}\delta(x_i - z_i)(\sum_{m=0}^{n} K^m S)(x_1, \ldots, x_s)\prod_{i=1}^{s} dx_i.\end{aligned}$$

It is clear that $\int \delta(x_i - z_i)g(x_i)dx_i = g(z_i)$ for an arbitrary function g and therefore

$$\langle r(\boldsymbol{x}), \psi^{(n+1)}(\boldsymbol{x})\rangle = (\sum_{m=0}^{n} K^m S)(\boldsymbol{z}) = \psi^{(n+1)}(\boldsymbol{z}),$$

Under this assumption, equation (3.4) approximates the solution of multiple dimensional Fredholm integral equation of the second kind.

3.3. **Simulation of random paths.** To apply the proposed method, N continuous random paths $t_i^0 \to t_i^1 \to \cdots \to t_i^n$ are simulated, for $i = 1, 2, \ldots, s$, with the transition kernel

$$P(t_i^{m-1}, t_i^m) = \rho(t_i^{m-1})\delta(t_i^m - t_i^{m-1}) + (1 - \rho(t_i^{m-1}))g(t_i^m)$$

where $\delta(t_i^m - t_i^{m-1})$ is the Dirac delta function at t_i^{m-1}, $g(x)$ is a probability density function on $[a_i, b_i]$ and $\rho(x) \in (0, 1)$. To simulate the random paths with initial density function p and transition kernel P, the following algorithm is implemented for $j = 1, 2, \ldots, N$:

1. Generate $t_i^{0,j}$ from density p.
2. for $m = 1$ to $n - 1$
   - 2.1. Generate a uniformly distributed τ random number.
     - 2.1.1. If $\rho(t_i^{m,j}) > \tau$ then $t_i^{m+1,j} = t_i^{m,j}$, else
     - 2.1.2. Generate $t_i^{m+1,j}$ from probability density g.

4. **Numerical experiments**

In this section, we demonstrate some numerical results of approximating the solution of two dimensional integral equations to verify the proposed method. As it was mentioned a good choice of importance sampling function is one that is similar in shape to the integrand. For the following examples, let us consider $p(x) = \delta(x - z_i)$ and $g(x)$ the

Beta distribution with parameters α and β.

Example 1. Consider the two dimensional Fredholm integral equation

$$\psi(x_1, x_2) = \int_0^1 \int_0^1 \frac{x_1}{(8 + x_2)(1 + y_1 + y_2)} \psi(y_1, y_2) dy_1 dy_2 + S(x_1, x_2),$$

where

$$S(x_1, x_2) = \frac{1}{(1 + x_1 + x_2)^2} - \frac{x_1}{6(8 + x_2)}.$$

Its exact solution is $\psi(x_1, x_2) = \frac{1}{(1+x_1+x_2)^2}$, [8]. Table 1 shows the performance of the Monte Carlo estimator when used for estimating ψ point-wise. The absolute difference between Mont carlo and the exact solution is derived in table 1.

TABLE 1. The computed errors with $N = 1000$, $\alpha = 1.8$ and $\beta = 1$.

(x, y)	|Monte Carlo - Exact|	Variance
$(0.5, 0.5)$	2.5203×10^{-4}	2.5961×10^{-6}
$(0.4, 0.9)$	7.0302×10^{-4}	7.4279×10^{-7}
$(0.3, 0.7)$	7.5532×10^{-4}	6.2406×10^{-6}
$(0.6, 0.4)$	5.3977×10^{-4}	1.7975×10^{-6}
$(0.7, 0.8)$	7.2548×10^{-4}	3.6040×10^{-6}
$(0.5, 0.4)$	9.4982×10^{-4}	3.8144×10^{-6}

Example 2. Consider the two dimensional Fredholm integral equation

$$\psi(x_1, x_2) = \int_0^1 \int_0^1 \frac{1}{50} e^{x_1^2 + x_2} y_2 y_1^2 \cos(y_1^3) \psi(y_1, y_2) dy_1 dy_2 + S(x_1, x_2),$$

where $S(x_1, x_2)$ is chosen such that its exact solution is $\psi(x_1, x_2) = e^{x_2 \sin(x_1^3)}$. Table 2 shows the performance of the Monte Carlo estimator when used for estimating ψ point-wise.

TABLE 2. The computed errors with $N = 1000$, $\alpha = 1.8$ and $\beta = 1$.

(x, y)	|Monte Carlo - Exact|	Variance
$(0.5, 0.5)$	6.4682×10^{-4}	1.9086×10^{-7}
$(0.6, 0.4)$	5.4024×10^{-5}	1.9770×10^{-7}
$(0.3, 0.7)$	6.0471×10^{-5}	2.4937×10^{-7}
$(0.4, 0.9)$	2.3383×10^{-4}	2.1737×10^{-7}
$(0.7, 0.8)$	4.1495×10^{-4}	1.9247×10^{-7}
$(0.8, 0.3)$	6.5777×10^{-4}	2.6885×10^{-7}

5. **Conclusion and future directions**

Multiple dimensional Fredholm integral equations of the second kind are usually difficult to solve analytically. In this paper, we explained a stochastic algorithm to solve these equations. The idea was to use the described continuous Monte Carlo method via importance sampling to estimate each term of the truncated first finite sum of terms in the Neumann series expansion of the solution. Required theories were proved and some numerical examples were solved to show the efficiency and accuracy of the method.

The proposed method has two significant advantages: it has a simple structure and it is a good candidate for parallelization because of the fact that many independent sample paths are used to estimate the solution. In the proposed algorithm, it is enough to choose a proper probability density function to have an efficient approximation of the solution pointwise. The choice of optimal probability density function for importance sampling technique is one of the main problems which is left as one of our future research topics.

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(Rahman Farnoosh) School of Mathematics, Iran University of Science & Technology, Tehran, Iran
E-mail address: rfarnoosh@iust.ac.ir

(Mahboubeh Aalaei) School of Mathematics, Iran University of Science & Technology, Tehran, Iran
E-mail address: aalaei@iust.ac.ir