ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 40 (2014), No. 2, pp. 459–472

Title:

On I-statistical and I-lacunary statistical convergence of order α

Author(s):

P. Das and E. Savas

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 2, pp. 459–472
Online ISSN: 1735-8515

ON I-STATISTICAL AND I-LACUNARY STATISTICAL CONVERGENCE OF ORDER α

P. DAS AND E. SAVAS*

(Communicated by Gholam Hossein Esslamzadeh)

ABSTRACT. In this paper, following a very recent and new approach, we further generalize recently introduced summability methods, namely, I-statistical convergence and I-lacunary statistical convergence (which extend the important summability methods, statistical convergence and lacunary statistical convergence using ideals of $\mathbb{N}$) and introduce the notions of I-statistical convergence of order α and I-lacunary statistical convergence of order α, where $0 < \alpha < 1$. We mainly investigate their relationship and also make some observations about these classes and in the way try to give an answer to an open problem posed By Das, Savas and Ghosal in 2011. The study leaves a lot of interesting open problems.
Keywords: Ideal, filter, I-statistical convergence of order α, I-lacunary statistical convergence of order α, closed subspace.
MSC(2010): Primary: 40A05; Secondary: 40D25.

1. Introduction

The idea of convergence of a real sequence was extended to statistical convergence by Fast [6] (see also Schoenberg [28]) as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$, then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the set K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \text{ and } \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

Article electronically published on April 30, 2014.
Received: 3 August 2012, Accepted: 17 March 2013.
*Corresponding author.

If $\overline{d}(K) = \underline{d}(K)$, then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \dfrac{K(1,n)}{n}$ (one can see [7, 8] for more details of density function)

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [9] and Šalát [21]. For some very interesting investigations concerning statistical convergence one may consult the papers of Moricz [18, 19], Miller and Orhan [20], Savas [23] where more references on this important summability method can be found.

The idea of statistical convergence was further extended to I-convergence in [11] using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [4, 5, 13, 14, 26, 27] where many important references can be found.

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [10] as follows. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L) if for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [10] the relation between lacunary statistical convergence and statistical convergence was established among other things. More results on this convergence can be seen in [15, 22, 24, 25].

Recently in [5, 26] we used ideals to introduce the concepts of I-statistical convergence and I-lacunary statistical convergence which naturally extend the notions of the above mentioned convergences.

On the other hand, in [1, 2] a different direction was given to the study of statistical convergence, where the notion of statistical convergence of order α $(0 < \alpha < 1)$ was introduced by using the notion of natural density of order α (where n is replaced by n^α in the denominator in the definition of natural density). It was observed in [1] that the behaviour of this new convergence was not exactly parallel to that of statistical convergence and some basic properties were obtained. One can also see [3] for related works.

In a natural way, in this paper we combine the approaches of [5] and [1] and introduce new and more general summability methods, namely, I-statistical convergence of order α and I-lacunary statistical convergence of order α. In this context it should be mentioned that the concept of lacunary statistical convergence of order α (which happens to be a special case of I-lacunary statistical convergence of order α) has also not been studied till now. We mainly investigate their relationship and also make some observations about these classes and in the way try to give an answer to an open problem posed in [5] (Problem 1, [5]). Due to the presence of both ideals and the number α the proofs do not seem to be exactly analogous to those of [5] or [1, 2] and most importantly the study leaves a lot of interesting open problems.

Throughout by a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ we shall mean a sequences of real numbers.

2. **Main results**

The following definitions and notions will be needed in the sequel.

Definition 2.1. *A non-empty family $I \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:*
(a) $A, B \in I$ imply $A \cup B \in I$,
(b) $A \in I,\ B \subset A$ imply $B \in I$.

Definition 2.2. *A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:*
(a) $\phi \notin F$,
(b) $A, B \in F$ imply $A \cap B \in F$,
(c) $A \in F,\ A \subset B$ imply $B \in F$.

If I is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin I$), then the family of sets $F(I) = \{M \subset \mathbb{N} :\ \exists\ A \in I : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal I.

Definition 2.3. *A proper ideal I is said to be admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.*

Throughout I will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 2.4. *([11], See also [13]) Let $I \subset 2^{\mathbb{N}}$ be a proper admissible ideal of $\mathbb{N}$.*

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be I-convergent to $L \in \mathbb{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\} \in I$.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be I^*-convergent to $L \in \mathbb{R}$ if there exists $M \in F(I)$ such that $\{x_n\}_{n\in M}$ converges to L.

We now introduce our main definitions.

Definition 2.5. *A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be I-statistically convergent of order α to L or $S(I)^\alpha$-convergent to L, where $0 < \alpha \leq 1$, if for each $\epsilon > 0$ and $\delta > 0$*

$$\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta\} \in I.$$

In this case we write $x_k \to L(S(I)^\alpha)$. The class of all sequences that are I-statistically convergent of order α will be denoted simply by $S(I)^\alpha$.

Remark 2.6. *For $I = I_{fin}, S(I)^\alpha$-convergence coincides with statistical convergence of order α [1, 2]. For an arbitrary ideal I and for $\alpha = 1$ it coincides with I-statistical convergence [5]. When $I = I_{fin}$ and $\alpha = 1$ it reduces to statistical convergence.*

Example 2.7. *Let us take the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ where $\lambda_n = 1$ for $n = 1$ to 10 and $\lambda_n = n-10$ for all $n \geq 10$, and take $I = I_d$ (the ideal of density zero sets of N) and let $A = \{1^2, 2^2, 3^2, 4^2, 5^2, ...\}$.*

Define $x = \{x_k\}_{k\in\mathbb{N}}$ by

$$x_k = \begin{cases} k & \text{for } n - [\sqrt{\lambda_n^\alpha}] + 1 \leq k \leq n, n \notin A, \\ k & \text{for } n - \lambda_n + 1 \leq k \leq n, n \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then for every $\epsilon > 0$ $(0 < \epsilon < 1)$ since

$$\frac{1}{\lambda_n^\alpha}|\{k \in I_n : |x_k| \geq \epsilon\}| = \frac{[\sqrt{\lambda_n^\alpha}]}{\lambda_n^\alpha} \to 0$$

as $n \to \infty$ and $n \notin A$, where $I_n = [n - \lambda_n + 1, n]$, so for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha}|\{k \in I_n : |x_k| \geq \epsilon\}| \geq \delta\right\} \subset A \cup \{1, 2, \ldots, m_1\} \; (1)$$

for some $m_1 \in N$. Now let $\delta > 0$ be given. Observe that $\lim_n \frac{n - \lambda_n}{n^\alpha} = 0$, and so we can choose $m_2 \in N$ such that $\frac{n-\lambda_n}{n^\alpha} < \frac{\delta}{2}$ for all $n \geq m_2$. Now *note that for the above $\varepsilon > 0$,*

$$\begin{aligned}\frac{1}{n^{\alpha}}\left|\{k \le n : |x_k| \ge \epsilon\}\right| &= \frac{1}{n^{\alpha}}\left|\{k \le n-\lambda_n : |x_k| \ge \epsilon\}\right| \\ &\quad + \frac{1}{n^{\alpha}}\left|\{k \in I_n : |x_k| \ge \epsilon\}\right| \\ &\le \frac{n-\lambda_n}{n^{\alpha}} + \frac{1}{n^{\alpha}}\left|\{k \in I_n : |x_k| \ge \epsilon\}\right| \\ &\le \frac{\delta}{2} + \frac{1}{{\lambda_n}^{\alpha}}\left|\{k \in I_n : |x_k| \ge \epsilon\}\right|\end{aligned}$$

for all $n \ge m_2$. Hence

$$\begin{aligned}&\left\{n \in N : \frac{1}{n^{\alpha}}\left|\,\{k \le n : |x_k| \ge \epsilon\}\,\right| \ge \delta\right\} \\ &\subset \left\{n \in N : \frac{1}{{\lambda_n}^{\alpha}}\left|\{k \in I_n : |x_k| \ge \epsilon\}\right| \ge \frac{\delta}{2}\right\} \cup \{1,2,3,\ldots,m_2\} \\ &\subset A \cup \{1,2,\ldots,m\}\end{aligned}$$

from (1) where $m = max\{m_1, m_2\}$. Clearly the set on the right hand side belongs to I and so the set on the left hand side also belongs to I. This shows that $x = \{x_k\}_{k\in\mathbb{N}}$ is I- statistically convergent of order α to 0. Note that x is not statistically convergent of order α to 0.

Definition 2.8. *Let θ be a lacunary sequence. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be I-lacunary statistically convergent of order α to L or $S_\theta(I)^\alpha$-convergent to L if for any $\epsilon > 0$ and $\delta > 0$*

$$\{r \in \mathbb{N} : \frac{1}{h_r^{\alpha}}|\{k \in I_r : |x_k - L| \ge \epsilon\}| \ge \delta\} \in I.$$

In this case we write $x_k \to L(S_\theta(I)^\alpha)$. The class of all I-lacunary statistically convergent sequences of order α will be denoted by $S_\theta(I)^\alpha$.

Remark 2.9. *For $\alpha = 1$ the definition coincides with I-lacunary statistical convergence* [5]*. Further it must be noted in this context that lacunary statistical convergence of order α has not been studied till now. Clearly lacunary statistical convergence of order α is a special case of I-lacunary statistical convergence of order α when we take $I = I_{fin}$. So properties of lacunary statistical convergence of order α can be easily obtained from our results with obvious modifications.*

Theorem 2.10. *Let $0 < \alpha \le \beta \le 1$. Then $S(I)^\alpha \subset S(I)^\beta$ and the inclusion is strict for at least those α, β for which there is a $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$ and when $I = I_{fin}$.*

Proof. Let $0 < \alpha \leq \beta \leq 1$. Then

$$\frac{|\{k\leq n:|x_k-L|\geq\epsilon\}|}{n^{\beta}} \leq \frac{|\{k\leq n:|x_k-L|\geq\epsilon\}|}{n^{\alpha}}$$

and so for any $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{|\{k \leq n : |x_k - L| \geq \epsilon\}|}{n^{\beta}} \geq \delta\} \subset \{n \in \mathbb{N} : \frac{|\{k \leq n : |x_k - L| \geq \epsilon\}|}{n^{\alpha}} \geq \delta\}.$$

Hence if the set on the right hand side belongs to the ideal I, then obviously the set on the left hand side also belongs to I. This shows that $S(I)^{\alpha} \subset S(I)^{\beta}$. To prove that the inclusion is strict for the above mentioned α, β consider the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ defined by

$$x_n = 1 \text{if } n = J^k$$

$$x_n = 0 \text{if } n \neq j^k, j \in \mathbb{N}.$$

Then $S(I)^{\beta} - \lim x_n = 0$, i.e., $x \in S(I)^{\beta}$ but $x \notin S(I)^{\alpha}$ where $I = I_{fin}$. □

Corollary 2.11. *If a sequence is I-statistically convergent of order α to L for some $0 < \alpha \leq 1$, then it is I-statistically convergent to L, i.e., $S(I)^{\alpha} \subset S(I)$.*

Similarly we can prove the following result.

Theorem 2.12. *Let $0 < \alpha \leq \beta \leq 1$. Then*
(i) $S_{\theta}(I)^{\alpha} \subset S_{\theta}(I)^{\beta}$.
(ii) In particular $S_{\theta}(I)^{\alpha} \subset S_{\theta}(I)$.
and the inclusion is strict for at least those α, β for which there is a $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$ and when $I = I_{fin}$.

It can be checked that $S(I)^{\alpha}$ and $S_{\theta}(I)^{\alpha}$ are both linear subspaces the space of all real sequences. We now prove the following result which gives a topological characterization of these spaces. As the line of proofs for both the results are similar so we give the detailed proof for the class $S_{\theta}(I)^{\alpha}$ only.

Theorem 2.13. *$S_{\theta}(I)^{\alpha} \cap l_{\infty}$ is a closed subset of l_{∞} where as usual l_{∞} is the Banach space of all bounded real sequences endowed with the sup norm.*

Proof. Suppose that $\{x^n\}_{n\in\mathbb{N}} \subseteq S_{\theta}(I)^{\alpha} \cap l_{\infty}$, $0 < \alpha \leq 1$, is a convergent sequence and it converges to $x \in l_{\infty}$. We need to prove that $x \in S_{\theta}(I)^{\alpha} \cap l_{\infty}$. Since $S_{\theta}(I)^{\alpha} \subset S_{\theta}(I)$ (by Theorem 2 (iii)) and $S_{\theta}(I) \cap l_{\infty}$ is closed in l_{∞} (see Theorem 1 [5]) so $x \in S_{\theta}(I) \cap l_{\infty}$. Again since $x^n \in S_{\theta}(I)^{\alpha} \subset$

$S_\theta(I)$ for all $n = 1, 2, 3, \ldots,$ x^n is I-statistically convergent to some number L_n for $n = 1, 2, 3, \ldots.$ We shall first show that the sequence $\{L_n\}_{n\in\mathbb{N}}$ is convergent to some number L and the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is I-statistically convergent of order α to L. Take a strictly decreasing sequence of positive numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$ converging to 0. Choose a positive integer n such that $||x - x^n||_\infty < \frac{\epsilon_n}{4}$. Let $0 < \delta < 1$. Then

$$A = \{r \in \mathbb{N} : \ \frac{1}{h_r} \ |\{k \in I_r : \ |x_k^n - L_n| \geq \frac{\epsilon_n}{4}\}| < \frac{\delta}{3}\} \in F(I)$$

$$\text{and } B = \{r \in \mathbb{N} : \ \frac{1}{h_r} \ |\{k \in I_r : \ |x_k^{n+1} - L_{n+1}| \geq \frac{\epsilon_{n+1}}{4}\}| < \frac{\delta}{3}\} \in F(I).$$

Since $A \cap B \in F(I)$ and $\phi \notin F(I)$, so we can choose $r \in A \cap B$. Then

$$\frac{1}{h_r} \ |\{k \in I_r : \ |x_k^n - L_n| \geq \frac{\epsilon_n}{4}\}| < \frac{\delta}{3}$$

$$\text{and } \frac{1}{h_r} \ |\{k \in I_r : \ |x_k^{n+1} - L_{n+1}| \geq \frac{\epsilon_{n+1}}{4}\}| < \frac{\delta}{3}$$

and so $\frac{1}{h_r} \ |\{k \in I_r : \ |x_k^n - L_n| \geq \frac{\epsilon_n}{4} \ \vee \ |x_k^{n+1} - L_{n+1}| \geq \frac{\epsilon_{n+1}}{4}\}| < \delta < 1.$

Hence there exists a $k \in I_r$ for which $|x_k^n - L_n| < \frac{\epsilon_n}{4}$ and $|x_k^{n+1} - L_{n+1}| < \frac{\epsilon_{n+1}}{4}$. Then we can write

$$\begin{aligned}
|L_n - L_{n+1}| &\leq |L_n - x_k^n| + |x_k^n - x_k^{n+1}| + |x_k^{n+1} - L_{n+1}| \\
&\leq |x_k^n - L_n| + |x_k^{n+1} - L_{n+1}| + ||x - x^n||_\infty + ||x - x^{n+1}||_\infty \\
&\leq \frac{\epsilon_n}{4} + \frac{\epsilon_{n+1}}{4} + \frac{\epsilon_n}{4} + \frac{\epsilon_{n+1}}{4} \leq \epsilon_n.
\end{aligned}$$

This implies that $\{L_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and so there is a real number L such that $L_n \to L$ as $n \to \infty$. We need to prove that $x \to L(S_\theta(I)^\alpha)$. For any $\epsilon > 0$, choose $n \in \mathbb{N}$ such that $\epsilon_n < \frac{\epsilon}{4}$, $||x - x^n||_\infty < \frac{\epsilon}{4}$, $|L_n - L| < \frac{\epsilon}{4}$. Then

$$\begin{aligned}
&\frac{1}{h_r^\alpha}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \\
&\quad\leq \frac{1}{h_r^\alpha}|\{k \in I_r : \ |x_k^n - L_n| + ||x_k - x_k^n||_\infty + |L_n - L| \geq \epsilon\}| \\
&\quad\leq \frac{1}{h_r^\alpha}|\{k \in I_r : \ |x_k^n - L_n| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \geq \epsilon\}| \\
&\quad\leq \frac{1}{h_r^\alpha}|\{k \in I_r : \ |x_k^n - L_n| \geq \frac{\epsilon}{2}\}|.
\end{aligned}$$

This implies $\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \epsilon\}| < \delta\} \supseteq \{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k^n - L_n| \geq \frac{\epsilon}{2}\}| < \delta\} \in F(I)$. So $\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \epsilon\}| < \delta\} \in F(I)$ and so $\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\} \in I$. This establishes the fact that $x \to L(S_\theta(I)^\alpha)$ which completes the proof of the theorem.

□

Theorem 2.14. *$S(I)^\alpha \cap l_\infty$ is a closed subset of l_∞.*

Remark 2.15. *Theorem 3* [1] *is a special case of Theorem 4 when we take $I = I_{fin}$. Also Theorem 1* [5] *is a special case of Theorem 3 when we take $\alpha = 1$.*

In the following we prove a result in line of Theorem 2 [5] regarding I-lacunary statistical convergence of order α.

Definition 2.16. *(cf.* [16, 17]*) Let θ be a lacunary sequence. Then $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be $N_\theta(I)^\alpha$-convergent to L if for any $\epsilon > 0$*

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k\in I_r} |x_k - L| \geq \epsilon\} \in I.$$

It is denoted by $x_k \to L(N_\theta(I)^\alpha)$ and the class of such sequences will be denoted simply by $N_\theta(I)^\alpha$.

Theorem 2.17. *Let $\theta = \{k_r\}_{r\in\mathbb{N}}$ be a lacunary sequence. Then*
(a) $x_k \to L(N_\theta(I)^\alpha) \Rightarrow x_k \to L(S_\theta(I)^\alpha)$, and
(b) $N_\theta(I)^\alpha$ is a proper subset of $S_\theta(I)^\alpha$.

Proof. (a) If $\epsilon > 0$ and $x_k \to L(N_\theta(I)^\alpha)$, we can write

$$\sum_{k\in I_r} |x_k - L| \geq \sum_{k\in I_r, |x_k - L|\geq\epsilon} |x_k - L| \geq \epsilon |\{k \in I_r : |x_k - L| \geq \epsilon\}|$$

and so $\frac{1}{\epsilon . h_r^\alpha} \sum_{k\in I_r} |x_k - L| \geq \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \epsilon\}|$.

Then for any $\delta > 0$

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\} \subseteq \{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k\in I_r} |x_k - L| \geq \epsilon . \delta\} \in I.$$

This proves the result.

(b) In order to establish that the inclusion $N_\theta(I)^\alpha \subseteq S_\theta(I)^\alpha$ is proper, let θ be given and define x_k to be $1, 2, \ldots, [\sqrt{h_r^\alpha}]$ at the first $[\sqrt{h_r^\alpha}]$ integers in I_r and $x_k = 0$ otherwise for all $r = 1, 2, 3, \ldots$. Then for any $\epsilon > 0$,

$$\frac{1}{h_r^\alpha}|\{k \in I_r : |x_k - 0| \geq \epsilon\}| \leq \frac{[\sqrt{h_r^\alpha}]}{h_r^\alpha}$$

and for any $\delta > 0$ we get

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}|\{k \in I_r : |x_k - 0| \geq \epsilon\}| \geq \delta\} \subseteq \{r \in \mathbb{N} : \frac{[\sqrt{h_r^\alpha}]}{h_r^\alpha} \geq \delta\}.$$

Since the set on the right hand side is a finite set and so belongs to I, it follows that $x_k \to 0(S_\theta(I)^\alpha)$. On the other hand

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} |x_k - 0| = \frac{1}{h_r^\alpha}. \frac{[\sqrt{h_r^\alpha}]([\sqrt{h_r^\alpha}] + 1)}{2} .$$

Then

$$\begin{aligned} \{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |x_k - 0| \geq \frac{1}{4}\} &= \{r \in \mathbb{N} : \frac{[\sqrt{h_r^\alpha}]([\sqrt{h_r^\alpha}] + 1)}{h_r} \geq \frac{1}{2}\} \\ &= \{m, m+1, m+2, \ldots\} \end{aligned}$$

for some $m \in \mathbb{N}$ which belongs to $F(I)$ since I is admissible. So $x_k \nrightarrow 0(N_\theta(I)^\alpha)$.

□

Remark 2.18. *In Theorem 2* [5] *it was further proved that*

(i) $x \in l_\infty$ *and* $x_k \to L(S_\theta(I)) \Rightarrow x_k \to L(N_\theta(I)$,

(ii) $S_\theta(I) \cap l_\infty = N_\theta(I) \cap l_\infty$.

However whether these results remain true for $0 < \alpha < 1$ *is not clear and we leave them as open problems.*

We will now investigate the relationship between I-statistical and I-lacunary statistical convergence of order α.

Theorem 2.19. *For any lacunary sequence* θ, *I-statistical convergence of order α implies I-lacunary statistical convergence of order α if* $\liminf_r q_r^\alpha > 1$.

Proof. Suppose first that $\liminf_r q_r^\alpha > 1$. Then there exists $\sigma > 0$ such that $q_r^\alpha \geq 1+\sigma$ for sufficiently large r which implies that

$$\frac{h_r^\alpha}{k_r^\alpha} \geq \frac{\sigma}{1+\sigma}.$$

Since $x_k \to L(S(I)^\alpha)$, then for every $\epsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r^\alpha}|\{k \leq k_r : |x_k - L| \geq \epsilon\}| &\geq \frac{1}{k_r^\alpha}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \\ &\geq \frac{\sigma}{1+\sigma}.\frac{1}{h_r^\alpha}|\{k \in I_r : |x_k - L| \geq \epsilon\}|.\end{aligned}$$

Then for any $\delta > 0$, we get

$$\begin{aligned}&\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\} \\ &\subseteq \{r \in \mathbb{N} : \frac{1}{k_r^\alpha}|\{k \leq k_r : |x_k - L| \geq \epsilon\}| \geq \frac{\delta\sigma}{(1+\sigma)}\} \in I.\end{aligned}$$

This proves the result. □

Remark 2.20. *The converse of this result is true for $\alpha = 1$ (see Theorem 3* [5]*). However for $\alpha < 1$ it is not clear and we leave it as an open problem.*

We now present two theorems which specify sufficient conditions for the converse relation of Theorem 4 to be true. In this context it should be mentioned that it was left as an open problem for the case $\alpha = 1$ (Problem 1 [5]). For the next two results we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F(I)$, $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \in F(I)$.

Theorem 2.21. *For a lacunary sequence θ satisfying the above condition, I-lacunary statistical convergence implies I-statistical convergence if $\limsup_r q_r < \infty$.*

Proof. If $\limsup_r q_r < \infty$, then without any loss of generality we can assume that there exists a $0 < B < \infty$ such that $q_r < B$ for all $r \geq 1$. Suppose that $x_k \to L(S_\theta(I))$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| < \delta\}$$

and

$$T = \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| < \delta_1\}.$$

It is obvious from our assumption that $C \in F(I)$, the filter associated with the ideal I. Further observe that

$$A_j = \frac{1}{h_j}|\{k \in I_j : |x_k - L| \geq \epsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
&\frac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| \\
&\leq \frac{1}{k_{r-1}}|\{k \leq k_r : |x_k - L| \geq \epsilon\}| \\
&= \frac{1}{k_{r-1}}|\{k \in I_1 : |x_k - L| \geq \epsilon\}| + \cdots + \frac{1}{k_{r-1}}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \\
&= \frac{k_1}{k_{r-1}}\frac{1}{h_1}|\{k \in I_1 : |x_k - L| \geq \epsilon\}| + \frac{k_2 - k_1}{k_{r-1}}\frac{1}{h_2}|\{k \in I_2 : |x_k - L| \geq \epsilon\}| + \ldots \\
&\quad + \frac{k_r - k_{r-1}}{k_{r-1}}\frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \\
&= \frac{k_1}{k_{r-1}}A_1 + \frac{k_2 - k_1}{k_{r-1}}A_2 + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}A_r \\
&\leq \sup_{j \in C} A_j . \frac{k_r}{k_{r-1}} \\
&< B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(I)$, it follows from our assumption on θ that the set T also belongs to $F(I)$, and this completes the proof of the theorem. □

Theorem 2.22. *For a lacunary sequence θ satisfying the above condition, I-lacunary statistical convergence of order α implies I-statistical convergence of order α, $0 < \alpha < 1$, if* $\sup_r \sum_{i=0}^{r-1} \frac{h_{i+1}^{\alpha}}{(k_{r-1})^{\alpha}} = B < \infty$.

Proof. Suppose that $x_k \rightarrow L(S_\theta(I)^\alpha)$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \{r \in \mathbb{N} : \frac{1}{h_r{}^{\alpha}}|\{k \in I_r : |x_k - L| \geq \epsilon\}| < \delta\}$$

and

$$T = \{n \in \mathbb{N} : \frac{1}{n^{\alpha}}|\{k \leq n : |x_k - L| \geq \epsilon\}| < \delta_1\}.$$

It is obvious from our assumption that $C \in F(I)$, the filter associated with the ideal I. Further observe that

$$A_j = \frac{1}{{h_j}^{\alpha}}|\{k \in I_j : |x_k - L| \geq \epsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
&\frac{1}{n^{\alpha}}|\{k \leq n : |x_k - L| \geq \epsilon\}| \\
\leq\ & \frac{1}{k_{r-1}^{\alpha}}|\{k \leq k_r : |x_k - L| \geq \epsilon\}| \\
=\ & \frac{1}{k_{r-1}^{\alpha}}|\{k \in I_1 : |x_k - L| \geq \epsilon\}| + \cdots + \frac{1}{k_{r-1}^{\alpha}}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \\
=\ & \frac{k_1^{\alpha}}{k_{r-1}^{\alpha}}\frac{1}{h_1^{\alpha}}|\{k \in I_1 : |x_k - L| \geq \epsilon\}| + \frac{(k_2 - k_1)^{\alpha}}{k_{r-1}^{\alpha}}\frac{1}{h_2^{\alpha}}|\{k \in I_2 : |x_k - L| \geq \epsilon\}| + \ldots \\
& + \frac{(k_r - k_{r-1})^{\alpha}}{k_{r-1}^{\alpha}}\frac{1}{h_r^{\alpha}}|\{k \in I_r : |x_k - L| \geq \epsilon\}| \\
=\ & \frac{k_1^{\alpha}}{k_{r-1}^{\alpha}}A_1 + \frac{(k_2 - k_1)^{\alpha}}{k_{r-1}^{\alpha}}A_2 + \cdots + \frac{(k_r - k_{r-1})^{\alpha}}{k_{r-1}^{\alpha}}A_r \\
\leq\ & \sup_{j \in C} A_j . \sup_r \sum_{i=0}^{r-1} \frac{(k_{i+1} - k_i)^{\alpha}}{k_{r-1}^{\alpha}} \\
<\ & B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(I)$, it follows from our assumption on θ that the set T also belongs to $F(I)$ and this completes the proof of the theorem. □

Acknowledgments

The first author is thankful to the Council of Scientific and Industrial Research, HRDG, India for the research project No. 25(0186)/10/EMR-II during the tenure of which this work was done. This work was done when the first author visited Istanbul Commerce University in 2012.
We are thankful to The Referees and The Editor for valuable suggestions which improved the presentation of the paper.

References

[1] S. Bhunia, P. Das, S. Pal, Restricting statistical convergence, *Acta Math. Hungar.* **134** (2012), no. 1-2, 153–161.

[2] R. Colak, Statistical convergence of order α, Modern Methods in Analysis and its Applications, Anamaya Pub., New Delhi, 2010.
[3] R. Colak and C. A. Bektas, λ-statistical convergence of order α, *Acta Math. Sci. Ser. B Engl. Ed.* **31** (2011), no. 3, 953–959.
[4] P. Das and S. Ghosal, Some further results on I-Cauchy sequences and condition (AP), *Comput. Math. Appl.* **59** (2010), no. 8, 2597–2600.
[5] Pratulananda Das, E. Savas, S. K. Ghosal, On generalizations of certain summability methods using ideals, *Appl. Math. Lett.* **24** (2011) 1509 - 1514.
[6] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951) 241–244.
[7] A. R. Freedman, J. J. Sember and M. Rapnael, Some Cesaro type summability spaces, *Proc. London. Math. Soc. (3)* **37** (1978), no. 3, 508–520.
[8] A. R. Freedman and J. J. Sember, Densities and summability, *Pacific. J. Math.* **95** (1981), no. 2, 293–305.
[9] J. A. Fridy, On statistical convergence, *Analysis* **5** (1985), no. 4, 301–313.
[10] J. A. Fridy and C. Orhan, Lacunary statistical convergence, *Pacific J. Math.* **160** (1993), no. 1, 43–51.
[11] P. Kostyrko, T. Šalát and W. Wilczynki, I-convergence, *Real Anal. Exchange* **26** (2000/2001), no. 2, 669–685.
[12] K. Kuratowski, Topology I, PWN, Warszawa, 1961.
[13] B. K. Lahiri and P. Das, I and I^*-convergence in topological spaces, *Math. Bohem.* **130** (2005), no. 2, 153–160.
[14] B. K. Lahiri and P. Das, I and I^*-convergence of nets, *Real Anal. Exchange* **33** (2008), no. 2, 431–442.
[15] J. Li, Lacunary statistical convergence and inclusion properties between lacunary methods, *Int. J. Math. Math. Sc.* **23** (2000), no. 3, 175–180.
[16] I. J. Maddox, A new type of convergence, *Math. Proc. Cambridge Phil. Soc.* **83** (1978), no. 1, 61–64.
[17] I. J. Maddox, Space of strongly summable sequence, *Quart. J. Math. Oxford Ser. (2)* **18** (1967) 345–355.
[18] F. Moricz and C. Orhan, Tauberian conditions under which statistical convergence follows from statistical summability by weighted means, *Studia Sci. Math. Hungar.* **41** (2004), no. 4, 391–403.
[19] F. Moricz, Tauberian conditions, under which statistical convergence follows from statistical summability $(C, 1)$, *J. Math. Anal. Appl.* **275** (2002), no. 1, 277–287.
[20] H. I. Miller and C. Orhan, On almost convergent and statistically convergent subsequences, *Acta Math. Hungar.* **93** (2001), no. 1-2, 135–151.
[21] T. Šalát, On Statistically convergent sequences of real numbers, *Math. Slovaca* **30** (1980), no. 2, 139–150.
[22] E. Savas, On lacunary strong σ-convergence, *Indian J. Pure Appl. Math.* **21** (1990), no. 4, 359–365.
[23] E. Savas, Some sequence spaces and statistical convergence, *Int. J. Math. Math. Sci.* **29** (2002), no. 5, 303–306.
[24] E. Savas and V. Karakaya, Some new sequence spaces defined by lacunary sequences, *Math. Slovaca* **57** (2007), no. 4, 393–399.
[25] E. Savas and R. F. Patterson, Double σ-convergence lacunary statistical sequences, *J. Comput. Anal. Appl.* **11** (2009), no. 4, 610–615.

[26] E. Savas and P. Das, A generalized statistical convergence via ideals, *Appl. Math. Lett.* **24** (2011), no. 6, 826–830.
[27] E. Savas, P. Das and S. Dutta, A note on strong matrix summability via ideals, *Appl. Math. Lett.* **25** (2012), no. 4, 733–738.
[28] I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* **66** (1959) 361–375.

(Pratulananda Das) Department of Mathematics, Jadavpur University, Kolkata-700032, Kolkata, India
E-mail address: pratulananda@yahoo.co.in

(Ekrem Savas) Department of Mathematics, Istanbul Ticaret University, Üsküdar, Istanbul, Turkey
E-mail address: ekremsavas@yahoo.com