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A CHARACTERIZATION OF THE SYMMETRIC GROUP OF PRIME DEGREE

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Abstract. Let G be a finite group and $\Gamma(G)$ the prime graph of G. Recently people have been using prime graphs to study simple groups. Naturally we pose a question: can we use prime graphs to study almost simple groups or non-simple groups? In this paper some results in this respect are obtained and as follows: $G \cong S_p$ if and only if $|G| = |S_p|$ and $\Gamma(G) = \Gamma(S_p)$, where p is a prime.
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MSC(2010): Primary: 20B30; Secondary: 20F28, 20D60.

1. Introduction

Let G be a finite group, $\pi(G)$ the set of all prime divisors of the order of G and $\omega(G)$ the spectrum of G, that is the set of element orders of G. The prime graph of G which is denoted by $\Gamma(G)$ is defined as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if $pq \in \omega(G)$.

Denote by $t(G)$ the maximal number of prime primes in $\pi(G)$ that are pairwise non-adjacent in $\Gamma(G)$. In other words, $t(G)$ is the size of some independent set with the maximal number of vertices in $\Gamma(G)$. Recall that a vertex set is said to be independent if its elements are pairwise non-adjacent. In graph theory, this number is usually called the independence number of the graph. By analogy, we denote by $t(r, G)$ the size of some independent set of $\Gamma(G)$ containing r, with the maximal number of

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elements. This number is called the r-independence number. We denote by $\rho(G)(\rho(r,G))$ some independent set in $\Gamma(G)$ (containing r) with the maximal number of vertices. Thus $|\rho(G)| = t(G)$ and $|\rho(r,G)| = t(r,G)$. And we write $[x]$ for the integer part of a rational number x.

Gruenberg and Kegel introduced prime graphs (it is also called the Gruenberg-Kegel graphs) in the middle of 1970's and gave a characterization of finite groups with a disconnected prime graph. We denote the number of connected components of $\Gamma(G)$ by $s(G)$. This deep result and a classification of finite simple groups with $s(G) > 1$, obtained by Williams and Kondrat'ev (see [16] and [9]), imply a series of important corollaries. Vasil'ev and his colleagues proved a series of important results on prime graphs (see [12-15]) from 2005.

In recent years the known results on prime graphs have been used to study finite simple groups. There are a series of papers especially on the recognition of finite groups by spectrum and the recognition or quasirecognition of finite simple groups by prime graphs, see for example [1, 6, 7, 8, 10]. Naturally we pose a question: Can we use prime graphs to study almost simple groups or non-simple groups? Later we found a paper related to this question (see [2]). In this paper some results in this respect are obtained as follows: Let G be a finite group. Then $G \cong S_p$ if and only if $|G| = |S_p|$ and $\Gamma(G) = \Gamma(S_p)$, where p is a prime.

In this paper, all groups are finite. And further unexplained notations are standard for which we refer the reader to [5], for example.

2. **Preliminaries**

Lemma 2.1. ([12]). *Let G be a finite group with $t(G) \geq 3$ and $t(2,G) \geq 2$. Then then there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for a maximal normalsoluoble subgroup K of G and $t(S) \geq t(G) - 1$. Moreover, for every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ devides the product $|K|.|\overline{G}/S|$. And one of thhe following statements hold:*

(1) $S \cong A_7$ or $L_2(q)$ for some q, and $t(S) = t(2,S) = 3$.

(2) For every prime $p]in\pi(G)$ nonadjoint to 2 in $\Gamma(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2,S) \geq t(2,G)$.

(1) *$S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2,S) = 3$.*

(2) *For every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2,S) \geq t(2,G)$.*

Lemma 2.2. ([3]). *Let H be a finite group, $N \trianglelefteq H$, $P \in Syl_p(N)$ and $|P| = p^k$. If H/N is a simple group, $t \in \omega(H/N)$, $(p,t) = 1$ and $pt \notin \omega(H)$. Then $|H/N| \mid \prod_{i=0}^{k-1}(p^k - p^i)$.*

Lemma 2.3. ([4]). $\omega(L_2(q)) = \{p, r \mid \frac{q+1}{2}, s \mid \frac{q-1}{2}\}$, *where $q = p^n$ for some odd prime p.*

3. Main Results

Theorem 3.1. *Let G be a finite group. Then $G \cong S_p$ if and only if $|G| = |S_p|$ and $\Gamma(G) = \Gamma(S_p)$, where p is a prime.*

Proof. We first claim that the theorem holds for $2 \leq p \leq 107$. If $G \cong S_p$, then the conclusion is obvious. Now we assume that $|G| = |S_p|$ and $\Gamma(G) = \Gamma(S_p)$. And we discuss the following cases.

Case 1. If p is equal to 2, 3, or 5, then it is easy to see the truth of the theorem.

Case 2. Let p be equal to 7. Then $\rho(G) = \{3, 5, 7\}$ and $\rho(2, G) = \{2, 7\}$. Thus the conditions of Lemma 2.1 are satisfied and so there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for a maximal normal soluble subgroup K of G. Note that $\pi(S) \subseteq \{2, 3, 5, 7\}$. Then $S \cong A_5$, A_6, $U_4(2)$, $L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_2(49)$, $U_3(5)$, $L_3(4)$, A_8, A_9, J_2, A_{10}, $U_4(3)$, $S_4(7)$, $S_6(2)$ or $O_8^+(2)$ according to Table 1 in [17]. If $S \cong A_5$, then $7 \mid |K|$ and it follows that $5 \sim 7$ in $\Gamma(G)$ by Lemma 2.2, which is a contradiction. Similarly we can show that $S \ncong A_6$, $L_2(7)$ and $L_2(8)$. Note that $|S| \mid |G|$. Then we have $S \cong A_7$. Hence $A_7 \leq G/K \leq S_7$. If $K \cong Z_2$, then every Sylow 7-subgroup of G acts fixed-point-freely on K and so $7 \mid 1$, which is a contradiction. Therefore $K = 1$ and $G \cong S_7$.

Case 3. Let p be equal to 11. Then $\rho(G) = \{5, 7, 11\}$ and $\rho(2, G) = \{2, 11\}$. Thus the conditions of Lemma 2.1 are satisfied and it follows that there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for a maximal normal soluble subgroup K of G and one of the following statements holds:

(1) $S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, S) = 3$.

(2) For every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.

Suppose that (1) holds. If $S \cong A_7$, then $A_7 \leq G/K \leq S_7$. So $11 \mid |K|$. By Lemma 2.2, it follows that $|A_7| \mid 10$, which is a contradiction. If $S \cong L_2(q)$, then $q = 7$ or 11 according to Table 1 in [17]. We know that at least $|\rho(G)| - 1$ primes in $\rho(G)$ divide $|S|$ by Lemma 2.1 and so $q = 11$. Hence $L_2(11) \leq G/K \leq Aut(L_2(11))$. Therefore $7 \mid |K|$ and we have $|L_2(11)| \mid (7 - 1)$ by Lemma 2.2. Thus we get a contradiction. Consequently (1) does not hold and (2) holds.

Let $S = H/K$. We claim that $7 \nmid |K|$ and $7 \nmid |G/H|$. If $7 \mid |K|$, then $7 \sim 11$ in $\Gamma(G)$ by Lemma 2.2 and this is impossible. If $7 \mid |G/H|$, then $|M/H| = 7$, where $M \leq G$. Hence $7 \mid (11 - 1)$ by Lemma 2.2, which is a contradiction. Thus $7 \nmid |K|$ and $7 \nmid |G/H|$, which implies that $7 \mid |S|$. Note that 11 is the maximal prime divisor of $|S|$ and by Table 1 in [17], it follows that S is isomorphic to one of the following simple groups: $L_2(11)$, M_{11}, M_{12}, $U_5(2)$, M_{22}, A_{11}, McL, HS, A_{12}, $U_6(2)$. Therefore $S \cong M_{22}$ or A_{11} since $7 \mid |S|$ and $|S| \mid |G|$. If $S \cong M_{22}$, then $M_{22} \leq G/K \leq Aut(M_{22})$ and so $5 \mid |K|$. Thus $5 \sim 11$ in $\Gamma(G)$ by Lemma 2.2, which is a contradiction. Therefore $A_{11} \leq G/K \leq S_{11}$. If $K \cong Z_2$, then every Sylow 11-subgroup of G acts fixed-point-freely on K and so $11 \mid 1$, which is a contradiction. And it follows that $K = 1$ and $G \cong S_{11}$.

Case 4. Let p be equal to 19. Then $\rho(G) = \{11, 13, 17, 19\}$ and $\rho(2, G) = \{2, 19\}$. Thus the conditions of Lemma 2.1 are satisfied. So there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for a maximal normal soluble subgroup K of G and one of the following statements holds:

(1) $S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, S) = 3$.

(2) For every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.

Suppose that (1) holds. If $S \cong A_7$, then $A_7 \leq G/K \leq S_7$. By Lemma 2.1, we know that at least $|\rho(G)| - 1$ primes in $\rho(G)$ divide $|S|$, which is a contradiction. If $S \cong L_2(q)$, where $q = r^\alpha$ and r is an odd prime. Since $\pi(L_2(q)) \subseteq \pi(G)$, we get that $r \in \{3, 5, 7, 11, 13, 17, 19\}$. And from Lemma 2.1, we know that at least $|\rho(G)| - 1$ primes in $\rho(G)$ divide $|S|$. In the following we will discuss the possibilities of r.

If $r = 3$, 5, or 7, then at least three primes in $\rho(G)$ divide $\frac{q^2-1}{2}$ and it follows that at least two primes in $\rho(G)$ divide $\frac{q+1}{2}$ or $\frac{q-1}{2}$. Hence $l \sim s$ in $\Gamma(G)$ for some $l, s \in \{11, 13, 17, 19\}$ by using Lemma 2.3, which is a contradiction. If $r = 11$, 13, 17, or 19, then $\alpha \leq 1$ since $|S| \mid |G|$. It is evident to see that this is impossible by Lemma 2.1 (b).

Consequently (1) does not hold and (2) holds. Since $\Gamma(G) = \Gamma(S_{19})$, we get that 19 is the maximal prime divisor of $|G|$ and by Table 1 in [17], it follows that S is isomorphic to one of the following simple groups: $L_2(19)$, $U_3(19)$, $U_3(8)$, $L_3(7)$, $L_4(7)$, J_1, J_3, $L_3(11)$, HN, $U_4(8)$, A_{19}, A_{20}, A_{21}, A_{22} and ${}^2E_6(2)$. On the other hand, at least three primes in $\rho(G)$ divide $|S|$ by Lemma 2.1 and so $S \ncong L_2(19)$, $L_3(7)$, $U_3(19)$, $U_3(8)$, $L_4(7)$, J_1, J_3, $L_3(11)$, HN and $U_4(8)$. If $S \cong {}^2E_6(2)$, then $2^{36} \mid |G|$, which is a contradiction. By the same reason, $S \ncong A_{20}$, A_{21} and A_{22}. Therefore $S \cong A_{19}$ and $A_{19} \leq G/K \leq S_{19}$. If $K \cong Z_2$, then every Sylow 19-subgroup of G acts fixed-point-freely on K and so $19 \mid 1$, which is a contradiction. And it follows that $K = 1$ and $G \cong S_{19}$.

Similar to Case 3 and Case 4 we can prove the cases for $p = 13$, 17.

Case 5. Assume that $23 \leq p \leq 107$. It is not difficult to see that $\rho(G) \geq 5$ and $\rho(2, G) = \{2, p\}$. Thus the conditions of Lemma 2.1 are satisfied. So there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for a maximal normal soluble subgroup K of G. Also $p \mid |S|$. We claim that $\prod r \mid |S|$, where r is a prime and $\frac{p}{2} < r \leq p$. If not, then there exists a prime r such that $\frac{p}{2} < r < p$ and $r \nmid |S|$. Let $S = H/K$. Then $r \mid |G/H|$ or $r \mid |K|$. If $r \mid |K|$, then by Lemma 2.2 we get that $|S| \mid (r-1)$ since G does not have any element of order rp. Therefore $p \mid (r-1)$, which is a contradiction. If $r \mid |G/H|$, then G has a subgroup L such that $H < L$ and L/H is a simple group of order r. Consequently we have $r \mid (p-1)$ by Lemma 2.2 and so $p - 1 = r$ for $\frac{p}{2} < r < p$, which is impossible since $23 \leq p \leq 107$. Therefore $\prod r \mid |S|$, where r is a prime and $\frac{p}{2} < r \leq p$. So S is not a sporadic simple group by comparing the order of all the sporadic simple groups. By the same reason and $|S| \mid |G|$ we can get that S is not a simple group of Lie type. And consequently $S \cong A_m$. Since p is the maximal prime divisor of $|S|$ and $|S| \mid |G|$, we have $m = p$. Then $A_p \leq G/K \leq Aut(A_p) = S_p$. If $K \cong Z_2$, then every Sylow p-subgroup of G acts fixed-point-freely on K and so $p \mid 1$, which is impossible. Thus $K = 1$ and $G \cong S_p$ for $23 \leq p \leq 107$.

Now we claim that the theorem holds for $p > 107$. If $G \cong S_p$, then the conclusion is obvious. In the following we assume that $|G| = |S_p|$ and $\Gamma(G) = \Gamma(S_p)$. And we have the following case.

Case 6. Let $p > 107$ be a prime. By Corollary 3 of Theorem 2 in [11], it follows that $k(p) - k(p/2) \geq \frac{3p}{10\log(p/2)}$ and so $k(p) - k(p/2) \geq 14$ for $p \geq 211$, where $k(p)$ denotes the number of prime numbers not exceeding p. And it follows that $t(G) \geq 14$ for $p \geq 211$. In fact, we can get that $t(G) \geq 14$ for $107 < p < 211$ by easy calculations. Thus $t(G) \geq 14$ for all $p > 107$. Note that $\rho(2, G) = \{2, p\}$. Then the conditions of Lemma 2.1 are satisfied. So there exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for a maximal normal soluble subgroup K of G. Also $p \mid |S|$.

First, S is not a sporadic simple group. Otherwise, according to Table 1 in [15], we know that $t(S) \leq 11$. On the other hand, by Lemma 2.1, it follows that $t(S) \geq t(G) - 1 = |\rho(G)| - 1 \geq 14 - 1 = 13$, which is a contradiction. Second, S is not a simple group of Lie type. If not, according to Tables 2-4 in [15], we obtain that S is isomorphic to one of the following simple groups: ${}^2A_{n-1}(q)(n \geq 7)$, $A_{n-1}(q)(n \geq 7)$, $B_n(q)$, $C_n(q)$, $D_n(q)$ and ${}^2D_n(q)$, where n and q should satisfy the corresponding conditions in [15, Tables 2-4]. If $S \cong {}^2A_{n-1}(q)$, then $t(S) = [\frac{n+1}{2}] \geq [\frac{3p}{10\log(p/2)}] - 1$. Hence $n > \frac{3p}{10\log(p/2)}$. By calculations and similar to the above, it follows that $q^l || |S| (l > p)$ for all $p > 107$. And we have $q^l || |G|$ for some $l > p$ since $|S| \mid |G|$. On the other hand, we know that the prime divisor with the maximal exponent of $|S_p|$ is 2 and it is easy to see that $2^p \nmid |S_p|$. Thus we get a contradiction. Similarly we can prove that $S \ncong A_{n-1}(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$ and ${}^2D_n(q)$. Consequently S is an alternating group. Furthermore, $S \cong A_p$ and $A_p \leq G/K \leq S_p$. If $K \cong Z_2$, then every Sylow p-subgroup of G acts fixed-point-freely on

K and so $p \mid 1$, which is impossible. Thus $K = 1$ and it follows that $G \cong S_p$ for $p > 107$.

Now the proof of the theorem is complete. □

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