ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 40 (2014), No. 2, pp. 481–504

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 2, pp. 481–504
Online ISSN: 1735-8515

APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS AND SOLVING SYSTEMS OF VARIATIONAL INEQUALITIES

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(Communicated by Antony To-Ming Lau)

Abstract. A new approximation method for the set of common fixed points of nonexpansive mappings and the set of solutions of systems of variational inequalities is introduced and studied. Moreover, we apply our main result to obtain strong convergence theorem to a common fixed point of a nonexpannsive mapping and solutions of a system of variational inequalities of an inverse strongly monotone mapping and strictly pseudo-contractive mapping of Browder-Petryshyn type.

Keywords: Fixed point, δ- strongly monotone, λ- strictly pseudo-contractive.
MSC(2010): Primary: 20D15; Secondary: 20D45, 11Y50.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $A\colon C \to H$ be a nonlinear mapping. The classical variational inequality problem is to fined $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solution of (1.1) is denoted by $VI(C, A)$, i.e.,

$$VI(C, A) = \{x \in C : \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C\}.$$

Article electronically published on April 30, 2014.
Received: 04 January 2013, Accepted: 27 March 2013.

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For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\| x - P_C x \| \leq \| x - y \|, \quad \forall y \in C.$$

P_C is called the projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\| P_C x - P_C y \|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H. \tag{1.2}$$

Moreover, P_C is characterized by the following properties:

$$\langle x - P_C x, y - P_C x \rangle \leq 0. \tag{1.3}$$

$$\| x - y \|^2 \geq \| x - P_C x \|^2 + \| y - P_C x \|^2, \tag{1.4}$$

for all $x \in H$, $y \in C$. It easy to see that the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \quad \lambda > 0. \tag{1.5}$$

Recall the following definitions,

(1) A is said to be strongly positive with a constant $\overline{\gamma}$ if there is a constant $\overline{\gamma} > 0$ such that
$$\langle Ax, x \rangle \geq \overline{\gamma} \| x \|^2, \qquad \forall x \in C.$$

(2) A is said to be monotone if
$$\langle Ax - Ay, x - y \rangle \geq 0, \qquad \forall x, y \in C.$$

(3) A is said to be $\eta-$strongly monotone if there exists a positive constant η such that
$$\langle Ax - Ay, x - y \rangle \geq \eta \| x - y \|^2, \qquad \forall x, y \in C. \tag{1.6}$$

(4) A is said to be $k-$Lipschitzian if there exists a positive constant k such that
$$\| Ax - Ay \| \leq k \| x - y \|, \qquad \forall x, y \in C.$$

(5) A is called $\alpha-$inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that
$$\langle Ax - Ay, x - y \rangle \geq \alpha \| Ax - Ay \|^2, \qquad \forall x, y \in C.$$
It is obvious that any $\alpha-$inverse strongly monotone mapping A is $\frac{1}{\alpha}-$Lipschitzian.

A set-valued mapping $U\colon H \to 2^H$ is called monotone if $x, y \in H, f \in Ux$ and $g \in Uy$ imply that $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $U\colon H \to 2^H$ is maximal if the graph of $G(U)$ of U is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping U is maximal if and only if for $(x, f) \in H\times H, \langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(U)$ implies that $f \in Ux$. Let A be a monotone mapping of C into H and let $N_C x$ be the normal cone to C at $x \in C$, that is, $N_C x = \{y \in H : \langle z-x, y\rangle \leq 0, \forall z \in C\}$ and define

$$Ux = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset & x \notin C. \end{cases} \tag{1.7}$$

Then U is maximal monotone and $0 \in Ux$ if and only if $x \in VI(C, A)$; see [11].

Let $T\colon C \to C$ be a mapping. In this paper, we use $Fix(T)$ to denote the set of fixed point of T. Recall the following definitions.

(1) T is said to be $\alpha-$contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\| Tx - Ty \| \leq \alpha \| x - y \|^2, \qquad \forall x, y \in C.$$

(2) T is said to be nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|^2, \qquad \forall x, y \in C.$$

(3) T is said to be λ-strictly pseudo-contractive of Browder and Petryshyn type [1] if there exists a constant $\alpha \in (0,1)$ such that

$$\| Tx - Ty \|^2 \leq \| x - y \|^2 + \lambda \| (I-T)x - (I-T)y \|^2, \qquad \forall x, y \in C. \tag{1.8}$$

It is well-known that the last inequality is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \| x - y \|^2 - \frac{1-\lambda}{2} \| (I-T)x - (I-T)y \|^2, \forall x, y \in C.$$

Marino and Xu [9] introduce the following general iterative methods:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \qquad n \geq 0, \tag{1.9}$$

where A is strongly positive with constant $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. They proved that if $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(C_1) $\alpha_n \to 0$,

(C_2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C_3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then, the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution of the variational inequality:

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \qquad \forall x \in Fix(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for all $x \in H$).

Further, Yao and Yao [14] introduced an iterative method for finding a common element of the set of fixed points of a single nonexpansive mapping and the set of solutions of a variational inequality for an $\alpha-$inverse strongly monotone mapping. To be more precise, they introduced the following iteration

$$(1.10) \qquad \begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T P_C(I - \lambda_n A)x_n, \ \forall n \geq 1. \end{cases}$$

where, P_C is a metric projection of H onto C, $A\colon H \to C$ an $\alpha-$inverse strongly monotone mapping, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. Under suitable conditions of these parameters they proved the strong convergence of the scheme (1.10) to $P_{\mathcal{F}}u$, where $\mathcal{F} = Fix(T) \cap VI(C, A)$.

On the other hand, Chen et al. [3] studied the following iterative process:

$$(1.11) \qquad x_{n+1} = \alpha_n f(x_n) + \beta_n T P_C(I - \lambda_n A)x_n, \ \forall n \geq 1,$$

and also obtained a strong convergence theorem by a viscosity approximation method.

In this paper, motivated and inspired by Marino and Xu [9], Katchang and Kumam [8], Jitpeera and Kumam [4, 5, 6], Chen et al. [3] and Yao and Yao [14], we introduce the iterative process below, with the initial guess $x_0 \in C$ chosen arbitrarily,

$$\begin{cases} t_n = P_C(I-\delta_{4,n}A_4)P_C(I-\delta_{3,n}A_3)P_C(I-\delta_{2,n}A_2)P_C(I-\delta_{1,n}A_1)x_n, \\ z_n = \delta_n P_C(I-\delta_{3,n}A_3)t_n + (1-\delta_n)P_C(I-\delta_{4,n}A_4)t_n, \\ y_n = \gamma_n P_C(I-\delta_{1,n}A_1)z_n + (1-\gamma_n)P_C(I-\delta_{2,n}A_2)z_n, \\ x_{n+1} = \alpha_n\gamma f(Ty_n) + \beta_n P_C(I-\delta_{1,n}A_1)x_n \\ \qquad +((1-\beta_n)I-\alpha_n F)TP_C(I-\delta_{2,n}A_2)y_n. \end{cases}$$

where, P_C is a metric projection of H onto C, for $i = 1,2,3,4.$, $A_i\colon H \to C$ a δ_i-inverse strongly monotone mapping, $F\colon C \to H$ is a mapping which is both δ- strongly monotone and λ- strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > \frac{1+\lambda}{2}$, f is a contraction on H with coefficient $0 < \alpha < 1$ and γ is a positive real number such that $\gamma < (1-\sqrt{\frac{2-2\delta}{1-\lambda}})/\alpha$. Our purpose in this paper is to introduce this general iterative algorithm for approximating a fixed point of a single nonexpansive mapping, which solves systems of variational inequalities. Our results improve and extend the results of Marino and Xu [9], Yao and Yao [14], Chen et al. [3] and many others.

2. **Preliminaries**

This section collects some lemmas which will be used in the proof of the main results in the next section.

Lemma 2.1. [13] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-b_n)a_n + b_n c_n, \qquad n \geq 0,$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.
Then, $\lim_{n\to\infty} a_n = 0$.

Let C be a nonempty subset of a Hilbert space H and $T\colon C \to H$ a mapping. Then T is said to be demiclosed at $v \in H$ if for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C \qquad \textit{and} \qquad Tx_n \to v \qquad \textit{imply} \qquad Tu = v,$$

where $\to$ (respectively, $\rightharpoonup$) denotes strong (respectively, weak) convergence.

Lemma 2.2. [7] *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T\colon C \to H$ is nonexpansive. Then, the mapping $I - T$ is demiclosed at zero.*

Lemma 2.3. [12] *Let H be a real Hilbert space. Then, for all $x, y \in H$*

(i) $\| x - y \|^2 \leq \| x \|^2 + 2\langle y, x + y\rangle$,

(ii) $\| x - y \|^2 \geq \| x \|^2 + 2\langle y, x\rangle$.

Lemma 2.4. [2, 10] *Let C be a nonempty closed convex subset of a real Hilbert space H.*

(i) *If $F\colon C \to C$ is a mapping which is both δ-strongly monotone and λ-strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > \frac{1+\lambda}{2}$, then, $I - F$ is contractive with constant $\sqrt{\frac{2-2\delta}{1-\lambda}}$.*

(ii) *If $F\colon C \to C$ is a mapping which is both δ-strongly monotone and λ-strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > \frac{1+\lambda}{2}$, then, for any fixed number $\tau \in (0, 1)$, $I - \tau F$ is contractive with constant $1 - \tau\left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}}\right)$.*

3. Strong convergence theorems

The following is our main result.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H, $F\colon C \to C$ a mapping which is both δ- strongly monotone and λ- strictly pseudo-contractive of Browder-Petryshyn type such that $\delta > (1 + \lambda)/2$, f a contraction on H with coefficient $0 < \alpha < 1$, and γ a positive real number such that $\gamma < (1 - \sqrt{\frac{2-2\delta}{1-\lambda}})/\alpha$. Let $T\colon C \to C$ be a nonexpansive mapping and for each $i = 1, 2, 3, 4$, $A_i\colon C \to H$ a δ_i-inverse strongly monotone mapping and $\mathcal{F} = \bigcap_{i=1}^{4} VI(C, A_i) \cap Fix(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\delta_{i,n}\}_{i=1,n=1}^{4,\ \infty}$ be sequences in $(0, 1)$, and $\{\beta_n\}_{n=1}^{\infty}$ a sequence in $[0, 1)$ satisfying the following conditions:*

(B_1) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,
$0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$.

(B_2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
$\lim_{n\to\infty} \beta_n = 0$ *and* $\lim_{n\to\infty} \delta_{i,n} = 0$, *for* $i = 1, 2, 3, 4$.

(B_3) $\sum_{n=1}^{\infty} \mid \alpha_{n+1} - \alpha_n \mid < \infty$, $\sum_{n=1}^{\infty} \mid \beta_{n+1} - \beta_n \mid < \infty$,
$\sum_{n=1}^{\infty} \mid \delta_{i,n+1} - \delta_{i,n} \mid < \infty$, *for* $i = 1, 2, 3, 4$.

If $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *are sequences generated by* $x_0 \in C$ *and*

$$\begin{cases} t_n = P_C(I-\delta_{4,n}A_4)P_C(I-\delta_{3,n}A_3)P_C(I-\delta_{2,n}A_2)P_C(I-\delta_{1,n}A_1)x_n, \\ z_n = \delta_n P_C(I-\delta_{3,n}A_3)t_n + (1-\delta_n)P_C(I-\delta_{4,n}A_4)t_n, \\ y_n = \gamma_n P_C(I-\delta_{1,n}A_1)z_n + (1-\gamma_n)P_C(I-\delta_{2,n}A_2)z_n, \\ x_{n+1} = \alpha_n\gamma f(Ty_n) + \beta_n P_C(I-\delta_{1,n}A_1)x_n \\ \qquad\quad +((1-\beta_n)I-\alpha_n F)TP_C(I-\delta_{2,n}A_2)y_n,\ \forall n \geq 1. \end{cases}$$

then $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *converge strongly to* $x^* \in \mathcal{F}$, *which is the unique solution of the system of the variational inequalities:*

$$\begin{cases} \langle (F-\gamma f)x^*, x-x^*\rangle \geq 0, \\ \langle A_i x^*, x-x^*\rangle \geq 0 \qquad , \end{cases} \qquad \forall x \in \mathcal{F}, i=1,2,3,4.$$

Proof. Since $\{\delta_{i,n}\}_{i=1,n=1}^{4,\ \infty}$ satisfies (B_2) and A_i is δ_i-inverse strongly monotone mapping, for any $x, y \in C$, we have

$$\begin{aligned} &\| (I-\delta_{i,n}A_i)x - (I-\delta_{i,n}A_i)y \|^2 \\ &\quad = \| (x-y) - \delta_{i,n}(A_i x - A_i y) \|^2 \\ &\quad = \| x-y \|^2 - 2\delta_{i,n}\langle x-y, A_i x - A_i y\rangle + \delta_{i,n}^2 \| A_i x - A_i y \|^2 \\ &\quad \leq \| x-y \|^2 - 2\delta_{i,n}\delta_i \| A_i x - A_i y \|^2 + \delta_{i,n}^2 \| A_i x - A_i y \|^2 \\ &\quad = \| x-y \|^2 + \delta_{i,n}(\delta_{i,n} - 2\delta_i) \| A_i x - A_i y \|^2 \\ &\quad \leq \| x-y \|^2 \end{aligned}$$

It follows that

$$\| (I-\delta_{i,n}A_i)x - (I-\delta_{i,n}A_i)y \| \leq \| x-y \|, \qquad i=1,2,3,4. \tag{3.1}$$

Let $p \in \mathcal{F}$. In the context of the variational inequality problem, the characterization of projection (1.5) implies that $p = P_C(I-\delta_{i,n}A_i)p$, $i = 1,2,3,4$. Using (1.5) and (3.1), we get

$$\begin{aligned} \| y_n - p \| &= \| \gamma_n P_C(I-\delta_{1,n}A_1)z_n + (1-\gamma_n)P_C(I-\delta_{2,n}A_2)z_n - p \| \\ &= \| \gamma_n[P_C(I-\delta_{1,n}A_1)z_n - P_C(I-\delta_{1,n}A_1)p] \\ &\quad + (1-\gamma_n)[P_C(I-\delta_{2,n}A_2)z_n - P_C(I-\delta_{2,n}A_2)p] \| \\ &\leq \gamma_n \| P_C(I-\delta_{1,n}A_1)z_n - P_C(I-\delta_{2,n}A_2)p \| \\ &\quad + (1-\gamma_n) \| P_C(I-\delta_{2,n}A_2)z_n - P_C(I-\delta_{2,n}A_2)p] \| \end{aligned}$$

$$\leq \gamma_n \parallel z_n - p \parallel +(1-\gamma_n) \parallel z_n - p \parallel = \parallel z_n - p \parallel .$$

Repeating the same argument as above, we can deduce that

$$\parallel z_n - p \parallel \leq \parallel t_n - p \parallel .$$

Since $\parallel y_n - p \parallel \leq \parallel z_n - p \parallel$, we have

$$\parallel y_n - p \parallel \leq \parallel z_n - p \parallel \leq \parallel t_n - p \parallel \leq \parallel x_n - p \parallel . \tag{3.2}$$

First we show that $\{x_n\}$ is bounded. Indeed, we take $p \in \mathcal{F}$. Then using (1.5) and Lemma 2.3, we have

$$\begin{aligned}
\parallel x_{n+1} - p \parallel &= \parallel \alpha_n \gamma f(Ty_n) + \beta_n P_C(I - \delta_{1,n}A_1)x_n \\
&\quad + ((1-\beta_n)I - \alpha_n F)TP_C(I - \delta_{2,n}A_2)y_n - p \parallel \\
&= \parallel [((1-\beta_n)I - \alpha_n F)TP_C(I - \delta_{2,n}A_2)y_n \\
&\quad - ((1-\beta_n)I - \alpha_n F)p] + \beta_n[P_C(I - \delta_{1,n}A_1)x_n - p] \\
&\quad + \alpha_n[\gamma f(Ty_n) - F(p)] \parallel \\
&\leq \left(1 - \beta_n - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}}\right)\right) \parallel y_n - p \parallel \\
&\quad + \beta_n \parallel x_n - p \parallel^2 + \alpha_n \parallel \gamma f(Ty_n) - \gamma f(p) \parallel \\
&\quad + \alpha_n \parallel \gamma f(p) - F(p) \parallel \\
&\leq \left(1 - \beta_n - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)\right) \parallel y_n - p \parallel \\
&\quad + \beta_n \parallel x_n - p \parallel^2 + \alpha_n \parallel \gamma f(p) - F(p) \parallel \\
&\leq \left(1 - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)\right) \parallel x_n - p \parallel \\
&\quad + \frac{\alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)}{\left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)} \parallel \gamma f(p) - F(p) \parallel \\
&\leq \max\left\{\left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)^{-1} \parallel \gamma f(p) - F(p) \parallel, \parallel x_n - p \parallel\right\}.
\end{aligned}$$

By induction,

$$\| x_n - p \| \leq \max\left\{ \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)^{-1} \| \gamma f(p) - F(p) \|, \| x_0 - p \| \right\}.$$

Therefore, $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{z_n\}$, $\{FT(y_n)\}$ and $\{f(Ty_n)\}$. Now we claim that

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{3.3}$$

Taking $v_{i,n} = P_C(I - \delta_{i,n}A_i)t_n$ for $i = 1, 2$ and $w_{i,n} = P_C(I - \delta_{i,n}A_i)t_n$ for $i = 3, 4$, and the definition of $\{y_n\}$, we have

$$\begin{aligned}
&\| z_{n+1} - z_n \| \\
&\quad = \| \delta_{n+1} w_{3,n+1} + (1-\delta_{n+1}) w_{4,n+1} - \delta_n w_{3,n} - (1-\delta_n) w_{4,n} \| \\
&\quad = \| \delta_{n+1}(w_{3,n+1} - w_{3,n}) + (\delta_{n+1} - \delta_n) w_{3,n} \\
&\qquad + (1-\delta_{n+1})(w_{4,n+1} - w_{4,n}) + (\delta_{n+1} - \delta_n) w_{4,n} \| \\
&\quad \leq \delta_{n+1} \| w_{3,n+1} - w_{3,n} \| + (1-\delta_{n+1}) \| w_{4,n+1} - w_{4,n} \| \\
&\qquad + | \delta_{n+1} - \delta_n | [\| w_{3,n} \| + \| w_{4,n} \|] \\
&\quad = \delta_{n+1} \| P_C(I - \delta_{3,n+1}A_3)t_{n+1} - P_C(I - \delta_{3,n+1}A_3)t_n \\
&\qquad + P_C(I - \delta_{3,n+1}A_3)t_n - P_C(I - \delta_{3,n}A_3)t_n \| \\
&\qquad + (1-\delta_{n+1}) \| P_C(I - \delta_{4,n+1}A_4)t_{n+1} - P_C(I - \delta_{4,n+1}A_4)t_n \\
&\qquad + P_C(I - \delta_{4,n+1}A_4)t_n - P_C(I - \delta_{4,n}A_4)t_n \| \\
&\qquad + | \delta_{n+1} - \delta_n | [\| w_{3,n} \| + \| v_{4,n} \|] \\
&\quad \leq \delta_{n+1} \| t_{n+1} - t_n \| + \delta_{n+1} | \delta_{3,n+1} - \delta_{3,n} | \| A_3 t_n \| \\
&\qquad + (1-\delta_{n+1}) \| t_{n+1} - t_n \| \\
&\qquad + (1-\delta_{n+1}) | \delta_{4,n+1} - \delta_{4,n} | \| A_4 t_n \| \\
&\qquad + | \delta_{n+1} - \delta_n | [\| w_{3,n} \| + \| w_{4,n} \|] \\
&\quad = \| t_{n+1} - t_n \| + \delta_{n+1} | \delta_{3,n+1} - \delta_{3,n} | \| A_3 t_n \| \\
&\qquad + (1-\delta_{n+1}) | \delta_{4,n+1} - \delta_{4,n} | \| A_4 t_n \| \\
&\qquad + | \delta_{n+1} - \delta_n | [\| w_{3,n} \| + \| w_{4,n} \|].
\end{aligned} \tag{3.4}$$

Repeating the same argument as above, we can deduce that

$$\begin{aligned}
&\| y_{n+1} - y_n \| \\
&\quad \leq \| z_{n+1} - z_n \| + \gamma_{n+1} | \delta_{1,n+1} - \delta_{1,n} | \| A_1 z_n \|
\end{aligned}$$

$$+ (1-\gamma_{n+1}) \mid \delta_{2,n+1} - \delta_{2,n} \mid\parallel A_2 z_n \parallel$$

$$+ \mid \gamma_{n+1} - \gamma_n \mid [\parallel v_{1,n} \parallel + \parallel v_{2,n} \parallel]. \tag{3.5}$$

Taking $P_n^k = P_C(I-\delta_{k,n}A_k)\cdots P_C(I-\delta_{1,n}A_1)$ for $k = 1, 2, \cdots, 4$ and $P_n^0 = I$, from the definition of $\{t_n\}_{n=1}^{\infty}$, we have

$$\begin{aligned}
&\parallel t_{n+1} - t_n \parallel \\
&=\parallel P_{n+1}^4 x_n - P_n^4 x_n \parallel \\
&=\parallel P_C(I-\delta_{4,n+1}A_4)P_{n+1}^3 x_{n+1} - P_C(I-\delta_{4,n}A_4)P_n^3 x_n \parallel \\
&\leq\parallel P_C(I-\delta_{4,n+1}A_4)P_{n+1}^3 x_{n+1} - P_C(I-\delta_{4,n+1}A_4)P_n^3 x_n \parallel \\
&\quad + \parallel P_C(I-\delta_{4,n+1}A_4)P_n^3 x_n - P_C(I-\delta_{4,n}A_4)P_n^3 x_n \parallel \\
&\leq\parallel P_{n+1}^3 x_{n+1} - P_n^3 x_n \parallel + \mid \delta_{4,n+1} - \delta_{4,n} \mid\parallel P_n^3 x_n \parallel \\
&=\parallel P_C(I-\delta_{3,n+1}A_3)P_{n+1}^2 x_{n+1} - P_C(I-\delta_{3,n}A_3)P_n^2 x_n \parallel \\
&\quad + \mid \delta_{4,n+1} - \delta_{4,n} \mid\parallel A_4 P_n^3 x_n \parallel \\
&\leq\parallel P_C(I-\delta_{3,n+1}A_3)P_{n+1}^2 x_{n+1} - P_C(I-\delta_{3,n+1}A_3)P_n^2 x_n \parallel \\
&\quad + \parallel P_C(I-\delta_{3,n+1}A_3)P_n^2 x_n - P_C(I-\delta_{3,n}A_3)P_n^2 x_n \parallel \\
&\quad + \mid \delta_{4,n+1} - \delta_{4,n} \mid\parallel A_4 P_n^3 x_n \parallel \\
&\leq\parallel P_{n+1}^2 x_{n+1} - P_n^2 x_n \parallel + \mid \delta_{3,n+1} - \delta_{3,n} \mid\parallel A_3 P_n^2 x_n \parallel \\
&\quad + \mid \delta_{4,n+1} - \delta_{4,n} \mid\parallel A_4 P_n^3 x_n \parallel \\
&=\parallel P_C(I-\delta_{2,n+1}A_2)P_{n+1}^1 x_{n+1} - P_C(I-\delta_{2,n}A_2)P_n^1 x_n \parallel \\
&\quad + \mid \delta_{3,n+1} - \delta_{3,n} \mid\parallel A_3 P_n^2 x_n \parallel + \mid \delta_{4,n+1} - \delta_{4,n} \mid\parallel A_4 P_n^3 x_n \parallel \\
&\leq\parallel P_C(I-\delta_{2,n+1}A_2)P_{n+1}^1 x_{n+1} - P_C(I-\delta_{2,n+1}A_2)P_n^1 x_n \parallel \\
&\quad + \parallel P_C(I-\delta_{2,n+1}A_2)P_n^1 x_n - P_C(I-\delta_{2,n}A_2)P_n^1 x_n \parallel \\
&\quad + \mid \delta_{3,n+1} - \delta_{3,n} \mid\parallel A_3 P_n^2 x_n \parallel + \mid \delta_{4,n+1} - \delta_{4,n} \mid\parallel A_4 P_n^3 x_n \parallel \\
&\leq\parallel P_{n+1}^1 x_{n+1} - P_n^1 x_n \parallel + \mid \delta_{2,n+1} - \delta_{2,n} \mid A_2 P_n^1 x_n \parallel \\
&\quad + \mid \delta_{3,n+1} - \delta_{3,n} \mid\parallel A_3 P_n^2 x_n \parallel + \mid \delta_{4,n+1} - \delta_{4,n} \mid\parallel A_4 P_n^3 x_n \parallel \\
&=\parallel P_C(I-\delta_{1,n+1}A_1)x_{n+1} - P_C(I-\delta_{1,n}A_1)x_n \parallel \\
&\quad + \mid \delta_{2,n+1} - \delta_{2,n} \mid A_2 P_n^1 x_n \parallel + \mid \delta_{3,n+1} - \delta_{3,n} \mid\parallel A_3 P_n^2 x_n \parallel \\
&\quad + \mid \delta_{4,n+1} - \delta_{4,n} \mid\parallel A_4 P_n^3 x_n \parallel \\
&\leq\parallel x_{n+1} - x_n \parallel + \mid \delta_{1,n+1} - \delta_{1,n} \mid\parallel A_1 x_n \parallel
\end{aligned}$$

$$+ \mid \delta_{2,n+1} - \delta_{2,n} \mid A_2 P_n^1 x_n \parallel + \mid \delta_{3,n+1} - \delta_{3,n} \mid\parallel A_3 P_n^2 x_n \parallel$$
$$+ \mid \delta_{4,n+1} - \delta_{4,n} \mid\parallel A_4 P_n^3 x_n \parallel$$
$$(3.6) \qquad = \parallel x_{n+1} - x_n \parallel + \sum_{i=1}^{4} \mid \delta_{i,n+1} - \delta_{i,n} \mid\parallel A_i P_n^{i-1} x_n \parallel .$$

Indeed, we have

$$\begin{aligned}
&\parallel x_{n+1} - x_n \parallel \\
&= \parallel \alpha_n \gamma f(Ty_n) + \beta_n P_C(I - \delta_{1,n} A_1) x_n + \mid \delta_{3,n+1} - \delta_{3,n} \mid\parallel A_3 P_n^2 x_n \parallel \\
&\quad + ((1-\beta_n) I - \alpha_n F) T P_C (I - \delta_{2,n} A_2) y_n \\
&\quad - \alpha_{n-1} \gamma f(Ty_{n-1}) - \beta_{n-1} P_C (I - \delta_{1,n-1} A_1) x_{n-1} \\
&\quad - ((1-\beta_{n-1}) I - \alpha_{n-1} F) T P_C (I - \delta_{2,n-1} A_2) y_{n-1} \parallel \\
&= \parallel [((1-\beta_n) I - \alpha_n F) T P_C (I - \delta_{2,n} A_2) y_n \\
&\quad - ((1-\beta_n) I - \alpha_n F) T P_C (I - \delta_{2,n-1} A_2) y_{n-1}] \\
&\quad + [((1-\beta_n) I - \alpha_n F) T P_C (I - \delta_{2,n-1} A_2) y_{n-1} \\
&\quad - ((1-\beta_{n-1}) I - \alpha_{n-1} F) T P_C (I - \delta_{2,n-1} A_2) y_{n-1}] \\
&\quad + \beta_n [P_C (I - \delta_{1,n} A_1) x_n - P_C (I - \delta_{1,n} A_1) x_{n-1}] \\
&\quad + \beta_n [P_C (I - \delta_{1,n} A_1) x_{n-1} - P_C (I - \delta_{1,n-1} A_1) x_{n-1}] \\
&\quad + (\beta_n - \beta_{n-1}) P_C (I - \delta_{1,n-1} A_1) x_{n-1} + \alpha_n \gamma [f(Ty_n) \\
&\quad - f(Ty_{n-1})] + (\alpha_n - \alpha_{n-1}) \gamma f(Ty_{n-1}) \parallel \\
&\leq \left(1 - \beta_n - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}}\right)\right) \parallel (I - \delta_{2,n} A_2) y_n \\
&\quad - (I - \delta_{2,n-1} A_2) y_{n-1} \parallel + \mid \beta_n - \beta_{n-1} \mid\parallel T P_C (I - \delta_{2,n-1} A_2) y_{n-1} \parallel \\
&\quad + \mid \alpha_n - \alpha_{n-1} \mid\parallel F T P_C (I - \beta_{2,n-1} A_2) y_{n-1} \parallel \\
&\quad + \beta_n \parallel x_n - x_{n-1} \parallel + \beta_n \mid \delta_{1,n} - \delta_{1,n-1} \mid\parallel A_1 x_{n-1} \parallel \\
&\quad + \mid \beta_n - \beta_{n-1} \mid\parallel P_C (I - \delta_{1,n-1} A_1) x_{n-1} \parallel + \alpha_n \gamma \alpha \parallel y_n - y_{n-1} \parallel \\
&\quad + \mid \alpha_n - \alpha_{n-1} \mid \gamma \parallel f(Ty_{n-1}) \parallel \\
&\leq \left(1 - \beta_n - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)\right) \parallel y_n - y_{n-1} \parallel \\
&\quad + \left(1 - \beta_n - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}}\right)\right) \mid \delta_{2,n} - \delta_{2,n-1} \mid\parallel A_2 y_{n-1} \parallel
\end{aligned}$$

$$+ \mid \beta_n - \beta_{n-1} \mid\parallel TP_C(I - \delta_{2,n-1}A_2)y_{n-1} \parallel$$
$$+ \mid \alpha_n - \alpha_{n-1} \mid\parallel FTP_C(I - \delta_{2,n-1}A_2)y_{n-1} \parallel$$
$$+ \beta_n \parallel x_n - x_{n-1} \parallel + \beta_n \mid \delta_{1,n} - \delta_{1,n-1} \mid\parallel Ax_{n-1} \parallel$$
$$+ \mid \beta_n - \beta_{n-1} \mid\parallel P_C(I - \delta_{1,n-1}A_1)x_{n-1} \parallel$$

(3.7)
$$+ \mid \alpha_n - \alpha_{n-1} \mid \gamma \parallel f(Ty_{n-1}) \parallel .$$

Substituting (3.4), (3.5) and (3.6) into (3.7), we obtain (for some constant $M > 0$)

$$\begin{aligned} &\parallel x_{n+1} - x_n \parallel \\ &\le \left(1 - \alpha_n \left(1 - \sqrt{\frac{2-2\delta}{1-\lambda}} - \gamma\alpha\right)\right) \parallel x_n - x_{n-1} \parallel \\ &\quad + [\mid \alpha_n - \alpha_{n-1} \mid + \mid \beta_n - \beta_{n-1} \mid + \mid \gamma_n - \gamma_{n-1} \mid \\ &\quad + \mid \delta_{1,n} - \delta_{1,n-1} \mid + \mid \delta_{2,n} - \delta_{2,n-1} \mid + \mid \delta_{3,n} - \delta_{3,n-1} \mid \\ &\quad + \mid \delta_{4,n} - \delta_{4,n-1} \mid]M. \end{aligned} \tag{3.8}$$

Thus, using condition (B_3) and Lemma 2.1 to (3.8), we conclude that $\parallel x_{n+1} - x_n \parallel \to 0$ as $n \to \infty$.
At this stage, we will show that

(3.9)
$$\lim_{n\to\infty} \parallel v_{i,n} - t_n \parallel 0, \quad i = 1,2 \ \ and \ \ \lim_{n\to\infty} \parallel w_{i,n} - t_n \parallel = 0, \quad i = 3,4.$$

Letting $p \in \mathcal{F}$, from the definition of $\{x_n\}$, we have

$$\begin{aligned} &\parallel x_{n+1} - p \parallel^2 \\ &= \parallel \alpha_n \gamma f(Ty_n) + \beta_n P_C(I - \delta_{1,n}A_1)x_n \\ &\quad + ((1-\beta_n)I - \alpha_n F)TP_C(I - \delta_{2,n}A_2)y_n - p \parallel^2 \\ &= \parallel [((1-\beta_n)I - \alpha_n F)TP_C(I - \delta_{2,n}A_2)y_n \\ &\quad - ((1-\beta_n)I - \alpha_n F)p] + \beta_n[P_C(I - \delta_{1,n}A_1)x_n - p] \\ &\quad + \alpha_n[\gamma f(Ty_n) - Fp] \parallel^2 \\ &\le \parallel ((1-\beta_n)I - \alpha_n F)TP_C(I - \delta_{2,n}A_2)y_n \\ &\quad - ((1-\beta_n)I - \alpha_n F)p \parallel^2 \\ &\quad + 2\beta_n \langle P_C(I - \delta_{1,n}A_1)x_n - p, x_{n+1} - p\rangle \\ &\quad + 2\alpha_n \langle \gamma f(Ty_n) - Fp, x_{n+1} - p\rangle \end{aligned}$$

$$\leq \left(1-\beta_n-\alpha_n\left(1-\sqrt{\frac{2-2\delta}{1-\lambda}}-\gamma\alpha\right)\right)\| y_n-p \|^2$$
$$+2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n-p, x_{n+1}-p\rangle$$
$$+2\alpha_n\langle \gamma f(Ty_n)-Fp, x_{n+1}-p\rangle$$
$$\leq\| y_n-p \|^2 +2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n-p, x_{n+1}-p\rangle$$
$$+2\alpha_n\langle \gamma f(Ty_n)-Fp, x_{n+1}-p\rangle. \tag{3.10}$$

From (1.5) and (3.10), we have

$$\begin{aligned}
&\| x_{n+1}-p \|^2\\
&\leq\| \gamma_n P_C(I-\delta_{1,n}A_1)z_n+(1-\gamma_n)P_C(I-\delta_{2,n}A_2)z_n-p \|^2\\
&\quad+2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n-p, x_{n+1}-p\rangle\\
&\quad+2\alpha_n\langle \gamma f(Ty_n)-Fp, x_{n+1}-p\rangle\\
&\leq \gamma_n \| P_C(I-\delta_{1,n}A_1)z_n-P_C(I-\delta_{1,n}A_1)p \|^2\\
&\quad+(1-\gamma_n)\| P_C(I-\delta_{2,n}A_2)z_n-P_C(I-\delta_{2,n}A_2)p \|^2\\
&\quad+2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n-p, x_{n+1}-p\rangle\\
&\quad+2\alpha_n\langle \gamma f(Ty_n)-Fp, x_{n+1}-p\rangle\\
&\leq \gamma_n \| (z_n-p)-\delta_{1,n}(A_1z_n-A_1p) \|^2\\
&\quad+(1-\gamma_n)\| z_n-p \|^2\\
&\quad+2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n-p, x_{n+1}-p\rangle\\
&\quad+2\alpha_n\langle \gamma f(Ty_n)-Fp, x_{n+1}-p\rangle\\
&= \gamma_n[\| z_n-p \|^2 +\delta_{1,n}^2 \| A_1z_n-A_1p \|^2\\
&\quad-2\delta_{1,n}\langle A_1z_n-A_1p, z_n-p\rangle]+(1-\gamma_n)\| z_n-p \|^2\\
&\quad+2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n-p, x_{n+1}-p\rangle\\
&\quad+2\alpha_n\langle \gamma f(Ty_n)-Fp, x_{n+1}-p\rangle\\
&\leq \gamma_n[\| z_n-p \|^2 +\delta_{1,n}^2 \| A_1z_n-A_1p \|^2\\
&\quad-2\delta_{1,n}\delta_1 \| A_1z_n-A_1p, \|^2]+(1-\gamma_n)\| z_n-p \|^2\\
&\quad+2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n-p, x_{n+1}-p\rangle\\
&\quad+2\alpha_n\langle \gamma f(Ty_n)-Fp, x_{n+1}-p\rangle\\
&\leq \gamma_n[\| z_n-p \|^2 +\delta_{1,n}(\delta_{1,n}-2\delta_1)\| A_1z_n-A_1p \|^2]\\
&\quad+(1-\gamma_n)\| z_n-p \|^2
\end{aligned}$$

$$+ 2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n - p, x_{n+1}-p\rangle$$
$$+ 2\alpha_n\langle \gamma f(Ty_n) - Fp, x_{n+1}-p\rangle.$$

From this and (3.2), we get

$$\begin{aligned}
&-\delta_{1,n}(\delta_{1,n}-2\delta_1)\parallel A_1z_n - A_1p\parallel^2\\
&\quad\leq\parallel z_n-p\parallel^2 - \parallel x_{n+1}-p\parallel^2\\
&\qquad+2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n-p, x_{n+1}-p\rangle\\
&\qquad+2\alpha_n\langle\gamma f(Ty_n)-Fp, x_{n+1}-p\rangle\\
&\quad\leq[\parallel x_n-p\parallel+\parallel x_{n+1}-p\parallel]\parallel x_{n+1}-x_n\parallel\\
&\qquad+2\beta_n\langle P_C(I-\delta_{1,n}A_1)x_n-p, x_{n+1}-p\rangle\\
&\qquad+2\alpha_n\langle\gamma f(Ty_n)-Fp, x_{n+1}-p\rangle. \qquad (3.11)
\end{aligned}$$

From (3.3), (3.11) and condition B_2, we get

$$\lim_{n\to\infty}\parallel A_1z_n - A_1p\parallel = 0. \qquad (3.12)$$

Repeating the same argument as above, we conclude that

(3.13)
$$\lim_{n\to\infty}\parallel A_2z_n - A_2p\parallel = \lim_{n\to\infty}\parallel A_3t_n - A_3p\parallel = \lim_{n\to\infty}\parallel A_4t_n - A_4p\parallel = 0.$$

From (1.2), we have

$$\begin{aligned}
&\parallel v_{i,n}-p\parallel^2\\
&\quad=\parallel P_C(I-\delta_{i,n}A_i)z_n - P_C(I-\delta_{i,n}A_i)p\parallel^2\\
&\quad\leq\langle(I-\delta_{i,n}A_i)z_n-(I-\delta_{i,n}A_i)p, v_{i,n}-p\rangle\\
&\quad=\frac{1}{2}[\parallel(I-\delta_{i,n}A_i)z_n-(I-\delta_{i,n}A_i)p\parallel^2+\parallel v_{i,n}-p\parallel^2\\
&\qquad-\parallel(I-\delta_{i,n}A_i)z_n-(I-\delta_{i,n}A_i)p-(v_{i,n}-p)\parallel^2]\\
&\quad\leq\frac{1}{2}[\parallel z_n-p\parallel^2+\parallel v_{i,n}-p\parallel^2\\
&\qquad-\parallel(I-\delta_{i,n}A_i)z_n-(I-\delta_{i,n}A_i)p-(v_{i,n}-p)\parallel^2]\\
&\quad=\frac{1}{2}[\parallel z_n-p\parallel^2+\parallel v_{i,n}-p\parallel^2-\parallel z_n-v_{i,n}\parallel^2\\
&\qquad+2\delta_{i,n}\langle z_n-v_{i,n}, A_iz_n-A_ip\rangle-\delta_{i,n}^2\parallel A_iz_n-A_ip\parallel^2].
\end{aligned}$$

So we obtain

$$\begin{aligned}\| v_{i,n} - p \|^2 \leq& \| z_n - p \|^2 - \| z_n - v_{i,n} \|^2 \\ &+ 2\delta_{i,n}\langle z_n - v_{i,n}, A_i z_n - A_i p\rangle \\ &- \delta_{i,n}^2 \| A_i z_n - A_i p \|^2, \qquad i = 1, 2. \end{aligned} \tag{3.14}$$

Using an argument similar to (3.14), we have

$$\begin{aligned}\| w_{i,n} - p \|^2 \leq& \| t_n - p \|^2 - \| t_n - w_{i,n} \|^2 \\ &+ 2\delta_{i,n}\langle t_n - w_{i,n}, A_i t_n - A_i p\rangle \\ &- \delta_{i,n}^2 \| A_i t_n - A_i p \|^2, \qquad i = 3, 4. \end{aligned} \tag{3.15}$$

From (3.2), (3.10) and (3.14), we have

$$\begin{aligned}&\| x_{n+1} - p \|^2 \\ &\leq \| y_n - p \|^2 + 2\beta_n \langle P_C(I - \delta_{1,n}A_1)x_n - p, x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle \gamma f(Ty_n) - Fp, x_{n+1} - p\rangle \\ &\leq \| \gamma_n(v_{1,n} - p) + (1-\gamma_n)(v_{2,n} - p) \|^2 \\ &\quad + 2\beta_n \langle P_C(I - \delta_{1,n}A_1)x_n - p, x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle \gamma f(Ty_n) - Fp, x_{n+1} - p\rangle \\ &\leq \gamma_n \| v_{1,n} - p \|^2 + (1-\gamma_n) \| v_{2,n} - p \|^2 \\ &\quad + 2\beta_n \langle P_C(I - \delta_{1,n}A_1)x_n - p, x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle \gamma f(Ty_n) - Fp, x_{n+1} - p\rangle \\ &\leq \gamma_n[\| z_n - p \|^2 - \| z_n - v_{1,n} \|^2 \\ &\quad + 2\delta_{1,n}\langle z_n - v_{1,n}, A_1 z_n - A_1 p\rangle - \delta_{1,n}^2 \| A_1 z_n - A_1 p \|^2] \\ &\quad + (1-\gamma_n)[\| z_n - p \|^2 - \| z_n - v_{2,n} \|^2 \\ &\quad + 2\delta_{2,n}\langle z_n - v_{2,n}, A_2 z_n - A_2 p\rangle - \delta_{2,n}^2 \| A_2 z_n - A_2 p \|^2] \\ &\quad + 2\beta_n \langle P_C(I - \delta_{1,n}A_1)x_n - p, x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle \gamma f(Ty_n) - Fp, x_{n+1} - p\rangle \\ &= \| z_n - p \|^2 + \gamma_n[- \| z_n - v_{1,n} \|^2 \\ &\quad + 2\delta_{1,n}\langle z_n - v_{1,n}, A_1 z_n - A_1 p\rangle - \delta_{1,n}^2 \| A_1 z_n - A_1 p \|^2] \end{aligned}$$

$$
\begin{aligned}
&+ (1-\gamma_n)[- \| z_n - v_{2,n} \|^2 + 2\delta_{2,n}\langle z_n - v_{2,n}, A_2 z_n - A_2 p\rangle \\
&- \delta_{2,n}^2 \| A_2 z_n - A_2 p \|^2] + 2\beta_n \langle P_C(I-\delta_{1,n}A_1)x_n - p, x_{n+1} - p\rangle \\
&+ 2\alpha_n \langle \gamma f(Ty_n) - Fp, x_{n+1} - p\rangle \\
\le& \| x_n - p \|^2 + \gamma_n[- \| z_n - v_{1,n} \|^2 + 2\delta_{1,n}\langle z_n - v_{1,n}, A_1 z_n - A_1 p\rangle \\
&- \delta_{1,n}^2 \| A_1 z_n - A_1 p \|^2] + (1-\gamma_n)[- \| z_n - v_{2,n} \|^2 \\
&+ 2\delta_{2,n}\langle z_n - v_{2,n}, A_2 z_n - A_2 p\rangle - \delta_{2,n}^2 \| A_2 z_n - A_2 p \|^2] \\
&+ 2\beta_n \langle P_C(I-\delta_{1,n}A_1)x_n - p, x_{n+1} - p\rangle \\
&+ 2\alpha_n \langle \gamma f(Ty_n) - Fp, x_{n+1} - p\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\gamma_n \| z_n - v_{1,n} \|^2 \\
&\le [\| x_n - p \| + \| x_{n+1} - p \|] \| x_{n+1} - x_n \| \\
&+ \gamma_n[2\delta_{1,n}\langle z_n - v_{1,n}, A_1 z_n - A_1 p\rangle - \delta_{1,n}^2 \| A_1 z_n - A_1 p \|^2] \\
&+ (1-\gamma_n)[- \| z_n - v_{2,n} \|^2 + 2\delta_{2,n}\langle z_n - v_{2,n}, A_2 z_n - A_2 p\rangle \\
&- \delta_{2,n}^2 \| A_2 z_n - A_2 p \|^2] + 2\beta_n \langle P_C(I-\delta_{1,n}A_1)x_n - p, x_{n+1} - p\rangle \\
&+ 2\alpha_n \langle \gamma f(Ty_n) - Fp, x_{n+1} - p\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&(1-\gamma_n) \| z_n - v_{2,n} \|^2 \\
&\le [\| x_n - p \| + \| x_{n+1} - p \|] \| x_{n+1} - x_n \| \\
&+ \gamma_n[- \| z_n - v_{1,n} \|^2 + 2\delta_{1,n}\langle z_n - v_{1,n}, A_1 z_n - A_1 p\rangle \\
&- \delta_{1,n}^2 \| A_1 z_n - A_1 p \|^2] + (1-\gamma_n)[2\delta_{2,n}\langle z_n - v_{2,n}, A_2 z_n - A_2 p\rangle \\
&- \delta_{2,n}^2 \| A_2 z_n - A_2 p \|^2] + 2\beta_n \langle P_C(I-\delta_{1,n}A_1)x_n - p, x_{n+1} - p\rangle \\
&+ 2\alpha_n \langle \gamma f(Ty_n) - Fp, x_{n+1} - p\rangle.
\end{aligned}
$$

Therefore using condition B_2, (3.3), and (3.12), we get

$$
\lim_{n\to\infty} \| z_n - v_{i,n} \| = 0 \qquad i = 1, 2. \tag{3.16}
$$

Using an argument similar to (3.14), we have

$$\lim_{n\to\infty} \parallel t_n - w_{i,n} \parallel = 0 \qquad i = 3, 4. \tag{3.17}$$

We now show that

$$\lim_{n\to\infty} \parallel x_n - Tx_n \parallel = 0. \tag{3.18}$$

Since T is nonexpansive, we get

$$\begin{aligned}
&\parallel x_n - Tx_n \parallel \\
&\leq \parallel x_n - x_{n+1} \parallel + \parallel x_{n+1} - TP_C(I - \delta_{2,n}A_2)y_n \parallel \\
&\quad + \parallel TP_C(I - \delta_{2,n}A_2)y_n - TP_C(I - \delta_{2,n}A_2)z_n \parallel \\
&\quad + \parallel TP_C(I - \delta_{2,n}A_2)z_n - Tz_n \parallel + \parallel Tz_n - Tx_n \parallel \\
&\leq \parallel x_n - x_{n+1} \parallel + \alpha_n \parallel \gamma f(Ty_n) - FTP_C(I - \delta_{2,n}A_2)y_n \parallel \\
&\quad + \beta_n \parallel P_C(I - \delta_{1,n}A_1)x_n - P_C(I - \delta_{2,n}A_2)y_n \parallel \\
&\quad + \parallel y_n - z_n \parallel + \parallel v_{2,n} - z_n \parallel + \parallel z_n - x_n \parallel \\
&\leq \parallel x_n - x_{n+1} \parallel + \alpha_n \parallel \gamma f(Ty_n) - FTP_C(I - \delta_{2,n}A_2)y_n \parallel \\
&\quad + \beta_n \parallel P_C(I - \delta_{1,n}A_1)x_n - P_C(I - \delta_{2,n}A_2)y_n \parallel \\
&\quad + \gamma_n \parallel v_{1,n} - z_n \parallel + (1 - \gamma_n) \parallel v_{2,n} - z_n \parallel \\
&\quad + \parallel v_{2,n} - z_n \parallel + \delta_n \parallel w_{3,n} - t_n \parallel \\
&\quad + (1 - \delta_n) \parallel w_{4,n} - t_n \parallel + \parallel t_n - x_n \parallel .
\end{aligned} \tag{3.19}$$

Since $x_n \in C$, $\parallel x_n - P_C x_n \parallel = 0$ and we have

$$\begin{aligned}
&\parallel t_n - x_n \parallel \\
&\quad \leq \parallel x_n - P_C x_n \parallel + \parallel P_C x_n - t_n \parallel \\
&\quad = \parallel P_C x_n - P_C(I - \delta_{4,n}A_4)P_n^3 x_n \parallel \\
&\quad \leq \parallel x_n - P_n^3 x_n \parallel + \delta_{4,n} \parallel A_4 P_n^3 x_n \parallel \\
&\quad \leq \parallel x_n - P_C x_n \parallel + \parallel P_C x_n - P_n^3 x_n \parallel + \delta_{4,n} \parallel A_4 P_n^3 x_n \parallel \\
&\quad = \parallel P_C x_n - P_C(I - \delta_{3,n}A_3)P_n^2 x_n \parallel + \delta_{4,n} \parallel A_4 P_n^3 x_n \parallel \\
&\quad \leq \parallel x_n - P_n^2 x_n \parallel + \delta_{3,n} \parallel A_3 P_n^2 x_n \parallel + \delta_{4,n} \parallel A_4 P_n^3 x_n \parallel \\
&\quad \leq \parallel x_n - P_C x_n \parallel + \parallel P_C x_n - P_n^2 x_n \parallel + \delta_{3,n} \parallel A_3 P_n^2 x_n \parallel \\
&\qquad + \delta_{4,n} \parallel A_4 P_n^3 x_n \parallel \\
&\quad = \parallel P_C x_n - P_C(I - \delta_{2,n}A_2)P_n^1 x_n \parallel + \delta_{3,n} \parallel A_3 P_n^2 x_n \parallel
\end{aligned}$$

$$
\begin{aligned}
&+\delta_{4,n}\parallel A_4P_n^3x_n\parallel\\
&\leq\parallel x_n-P_n^1x_n\parallel+\delta_{2,n}\parallel A_2P_n^1x_n\parallel+\delta_{3,n}\parallel A_3P_n^2x_n\parallel\\
&+\delta_{4,n}\parallel A_4P_n^3x_n\parallel\\
&\leq\parallel x_n-P_Cx_n\parallel+\parallel P_Cx_n-P_n^1x_n\parallel+\delta_{2,n}\parallel A_2P_n^1x_n\parallel\\
&+\delta_{3,n}\parallel A_3P_n^2x_n\parallel+\delta_{4,n}\parallel A_4P_n^3x_n\parallel\\
&=\parallel P_Cx_n-P_C(I-\delta_{1,n}A_1)x_n\parallel+\delta_{2,n}\parallel A_2P_n^1x_n\parallel\\
&+\delta_{3,n}\parallel A_3P_n^2x_n\parallel+\delta_{4,n}\parallel A_4P_n^3x_n\parallel\\
&\leq\delta_{1,n}\parallel A_1x_n\parallel+\delta_{2,n}\parallel A_2P_n^1x_n\parallel\\
&+\delta_{3,n}\parallel A_3P_n^2x_n\parallel+\delta_{4,n}\parallel A_4P_n^3x_n\parallel\\
&=\sum_{i=1}^{4}\delta_{i,n}\parallel A_iP_n^{i-1}x_n\parallel. \qquad (3.20)
\end{aligned}
$$

From (3.3), (3.16), (3.17), (3.20), and condition B_2, we get (3.18).
Next, let us show that, there exists a unique $x^*\in\mathcal{F}$ such that

$$\limsup_{n\to\infty}\langle(F-\gamma f)x^*,x^*-x_n\rangle\leq 0. \qquad (3.21)$$

Let $Q=P_{\mathcal{F}}$. Then $Q(I-F+\gamma f)$ is a contraction of H into itself. In fact, we see that

$$
\begin{aligned}
&\parallel Q(I-F+\gamma f)x-Q(I-F+\gamma f)y\parallel\\
&\quad\leq\parallel(I-F+\gamma f)x-(I-F+\gamma f)y\parallel\\
&\quad\leq\parallel(I-F)x-(I-F)y\parallel+\gamma\parallel f(x)-f(y)\parallel\\
&\quad=\lim_{n\to\infty}\parallel(I-(1-\frac{1}{n})F)x-(I-(1-\frac{1}{n})F)y\parallel+\gamma\parallel f(x)-f(y)\parallel\\
&\quad\leq\lim_{n\to\infty}(1-(1-\frac{1}{n})\tau)\parallel x-y\parallel+\gamma\alpha\parallel x-y\parallel\\
&\quad=(1-\tau)\parallel x-y\parallel+\gamma\alpha\parallel x-y\parallel,
\end{aligned}
$$

and hence $Q(I-F+\gamma f)$ is a contraction due to $(1-(\tau-\gamma\alpha))\in(0,1)$. Therefore, by Banach's contraction principle, $P_{\mathcal{F}}(I-F+\gamma f)$ has a unique fixed point x^*. Then using (1.5), x^* is the unique solution of the variational inequality:

$$\langle(\gamma f-F)x^*,x-x^*\rangle\leq 0,\qquad\forall x\in\mathcal{F}. \qquad (3.22)$$

We can choose a a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

(3.23)
$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Fx^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle \gamma f(x^*) - Fx^*, x_{n_j} - x^*\rangle.$$

As $\{x_{n_j}\}$ is bounded, $\{x_{n_j}\}$ has a subsequence $\{x_{n_{j_k}}\}$ such that $x_{n_{j_k}} \rightharpoonup z$. with no loss of generality, we may assume that $x_{n_j} \rightharpoonup z$. It follows from (3.18) and Lemma 2.2 that $z \in Fix(T)$.

Now, let us show that for $i = 1, 2$, $z \in VI(C, A_i)$. Let $U_i \colon H \to 2^H$ be a set-valued mapping defined by

$$U_i x = \begin{cases} A_i x + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

where $N_C x$ is the normal cone to C at $x \in C$. Since A_i is monotone, U_i is maximal monotone see [11]. Let $(x, y) \in G(U_i)$. Hence $y - A_i x \in N_C x$ and since $v_{i,n} = P_C(I - \delta_{i,n}A_i)z_n$, $\langle x - v_{i,n}, y - A_i x\rangle \geq 0$. On the other hand, from $v_{i,n} = P_C(z_n - \delta_{i,n}A_i z_n)$, we have

$$\langle x - v_{i,n}, v_{i,n} - (z_n - \delta_{i,n}A_i z_n)\rangle \geq 0,$$

that is

$$\langle x - v_{i,n}, \frac{v_{i,n} - z_n}{\delta_{i,n}} + A_i z_n\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
&\langle x - v_{i,n_j}, y\rangle \\
&\quad\geq \langle x - v_{i,n_j}, A_i x\rangle \\
&\quad\geq \langle x - v_{i,n_j}, A_i x\rangle - \langle x - v_{i,n_j}, \frac{v_{i,n_j} - z_{n_j}}{\delta_{i,n_j}} + A_i z_{n_j}\rangle \\
&\quad= \langle x - v_{i,n_j}, A_i x - \frac{v_{i,n_j} - z_{n_j}}{\delta_{i,n_j}} - A_i z_{n_j}\rangle \\
&\quad= \langle x - v_{i,n_j}, A_i x - A_i v_{i,n_j}\rangle + \langle x - v_{i,n_j}, A_i v_{i,n_j} - A_i z_{n_j}\rangle \\
&\qquad - \langle x - v_{i,n_j}, \frac{v_{i,n_j} - z_{n_j}}{\delta_{i,n_j}}\rangle \\
&\quad\geq \langle x - v_{i,n_j}, A_i v_{i,n_j} - A_i z_{n_j}\rangle - \langle x - v_{i,n_j}, \frac{v_{i,n_j} - z_{n_j}}{\delta_{i,n_j}}\rangle \\
&\quad\geq \langle x - v_{i,n_j}, A_i v_{i,n_j} - A_i z_{n_j}\rangle - \| x - v_{i,n_j} \| \| \frac{v_{i,n_j} - z_{n_j}}{\delta_{i,n_j}} \| .
\end{aligned}$$

Noting that $\lim_{i\to\infty} \| v_{i,n_j} - x_{n_j} \| = 0$, $\lim_{n\to\infty} \| z_n - x_n \| = 0$, $x_{n_j} \rightharpoonup z$ and A_i is $\frac{1}{\delta_i}-$ Lipschitzian, we obtain

$$\langle x - z, y \rangle \geq 0.$$

Since U_i is maximal monotone, we have $z \in U_i^{-1}0$, and hence

$$z \in VI(C, A_i), \quad i = 1, 2.$$

Repeating the same argument as above, we conclude that

$$z \in VI(C, A_i), \quad i = 3, 4.$$

Therefore $z \in \mathcal{F}$ and applying (3.22) and (3.23), we have

$$\limsup_{n\to\infty} \langle (\gamma f - F)x^*, x_n - x^* \rangle \leq 0.$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. Taking $\tau = 1 - \sqrt{\frac{2-2\delta}{1-\lambda}}$ and using (1.5), (3.2), and Lemma 2.4, we have

$$\begin{aligned}
&\| x_{n+1} - x^* \|^2 \\
&\quad = \| \alpha_n \gamma f(Ty_n) + \beta_n P_C(I - \delta_{1,n} A_1) x_n \\
&\qquad + \langle x - v_{i,n_j}, A_i v_{i,n_j} - A_i z_{n_j} \rangle \\
&\qquad + ((1 - \beta_n) I - \alpha_n F) T P_C (I - \delta_{2,n} A_2) y_n - x^* \|^2 \\
&\quad = \| [((1 - \beta_n) I - \alpha_n F) T P_C (I - \delta_{2,n} A_2) y_n \\
&\qquad - ((1 - \beta_n) I - \alpha_n F) T P_C (I - \delta_{2,n} A_2) x^*] \\
&\qquad + \beta_n [P_C (I - \delta_{1,n} A_1) x_n - P_C (I - \delta_{1,n} A_1) x^*] \\
&\qquad + \alpha_n [\gamma f(Ty_n) - F x^*] \|^2 \\
&\quad \leq \| [((1 - \beta_n) I - \alpha_n F) T P_C (I - \delta_{2,n} A_2) y_n \\
&\qquad - ((1 - \beta_n) I - \alpha_n F) T P_C (I - \delta_{2,n} A_2) x^*] \\
&\qquad + \beta_n [P_C (I - \delta_{1,n} A_1) x_n - P_C (I - \delta_{1,n} A_1) x^*] \|^2 \\
&\qquad + 2\alpha_n \langle \gamma f(Ty_n) - F x^*, x_{n+1} - x^* \rangle \\
&\quad \leq [(1 - \beta_n - \alpha_n \tau) \| y_n - x^* \| + \beta_n \| x_n - x^* \|]^2 \\
&\qquad + 2\alpha_n \langle \gamma f(Ty_n) - F x^*, x_{n+1} - x^* \rangle \\
&\quad \leq \| [(1 - \beta_n - \alpha_n \tau) \| x_n - x^* \| + \beta_n \| x_n - x^* \|]^2 \\
&\qquad + 2\alpha_n \gamma \langle f(Ty_n) - f(x^*), x_{n+1} - x^* \rangle
\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha_n\langle \gamma f(x^*) - Fx^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n\tau)^2 \parallel x_n - x^* \parallel^2 + \alpha_n\gamma\alpha[\parallel x_n - x^* \parallel^2 \\
&+ \parallel x_{n+1} - x^* \parallel^2] + 2\alpha_n\langle \gamma f(x^*) - Fx^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

So we reach the following

$$
\begin{aligned}
&\|x_{n+1} - x^* \|^2 \\
&\leq \frac{1+\alpha^2\tau^2 - 2\alpha_n\tau + \alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha} \parallel x_n - x^* \parallel^2 \\
&+ \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(x^*) - Fx^*, x_{n+1} - x^*\rangle \\
&\leq (1 - \alpha_n\frac{2(\tau-\gamma\alpha) - \alpha_n\tau^2}{1-\alpha_n\gamma\alpha}) \parallel x_n - x^* \parallel^2 \\
&+ \alpha_n\frac{2(\tau-\gamma\alpha) - \alpha_n\tau^2}{1-\alpha_n\gamma\alpha}\frac{2}{2(\tau-\gamma\alpha) - \alpha_n\tau^2}\langle \gamma f(x^*) - Fx^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

It follows that

$$
\parallel x_{n+1} - x^* \parallel^2 \leq (1-b_n) \parallel x_n - x^* \parallel^2 + b_n c_n, \tag{3.24}
$$

where

$$
b_n = \alpha_n\frac{2(\tau-\gamma\alpha) - \alpha_n\tau^2}{1-\alpha_n\gamma\alpha}
$$

and

$$
c_n = \frac{2}{2(\tau-\gamma\alpha) - \alpha_n\tau^2}\langle \gamma f(x^*) - Fx^*, x_{n+1} - x^*\rangle.
$$

Since $\{\alpha_n\}$ satisfies condition B_2, we have $\sum_{n=0}^{\infty} b_n = \infty$ and by condition B_2 and (3.21), we get $\limsup_{n\to\infty} c_n \leq 0$. Consequently, applying Lemma 2.1, to (3.24), we conclude that $x_n \to x^*$. From (3.2), we get $y_n \to x^*$, $z_n \to x^*$ and $t_n \to x^*$.

By a careful analysis of the proof of Theorem 3.1, we obtain the following theorem. Because its proof is much simpler than the proof of Theorem 3.1, we omit its proof

Theorem 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let $F\colon C \to C$ be a mapping which is both δ- strongly monotone* *and* λ- *strictly pseudo-contractive of Browder-Petryshyn type such that* $\delta > (1+\lambda)/2$, f *a contraction on* H *with coefficient* $0 < \alpha < 1$, *and* γ *a positive real number such that* $\gamma < (1 - \sqrt{\frac{2-2\delta}{1-\lambda}})/\alpha$. *Let* $T: C \to C$ *be a nonexpansive mapping and for each* $i = 1, 2, \ldots, m$, *let* $A_i: C \to C$ *be* δ_i*−inverse strongly monotone mapping and* $\mathcal{F} = \bigcap_{i=1}^{m} VI(C, A_i) \cap Fix(T) \neq \emptyset$. *Let* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, *and* $\{\delta_{i,n}\}_{i=1,n=1}^{m,\ \infty}$ *be sequences in* $(0,1)$, *and* $\{\beta_n\}_{n=1}^{\infty}$ *is a sequence in* $[0,1)$ *satisfying the following conditions:*

(B_1) $\lim_{n\to\infty} \delta_{i,n} = 0$ *for* $i = 1, 2, \ldots, m$.
(B_2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \beta_n = 0$.
(B_3) $\sum_{n=1}^{\infty} | \alpha_{n+1} - \alpha_n | < \infty$, $\sum_{n=1}^{\infty} | \beta_{n+1} - \beta_n | < \infty$, $\sum_{n=1}^{\infty} | \delta_{i,n+1} - \delta_{i,n} | < \infty$, *for* $i = 1, 2$.

If $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are sequences generated by* $x_0 \in C$ *and*

$$\begin{cases} y_n = P_C(I - \delta_{m,n}A_m)\cdots P_C(I - \delta_{4,n}A_4)P_C(I - \delta_{3,n}A_3)x_n, \\ x_{n+1} = \alpha_n \gamma f(Ty_n) + \beta_n P_C(I - \delta_{1,n}A_1)x_n \\ \qquad\quad +((1-\beta_n)I - \alpha_n F)TP_C(I - \delta_{2,n}A_2)y_n, \ \forall n \geq 1, \end{cases}$$

then $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *converge strongly to an* $x^* \in \mathcal{F}$, *which is the unique solution of the system of variational inequalities:*

$$\begin{cases} \langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \\ \langle A_i x^*, x - x^* \rangle \geq 0 \qquad , \qquad \forall x \in \mathcal{F}, i = 1, 2, \ldots, m. \end{cases}$$

Corollary 3.3. *(See Yao and Yao* [14]*) Let* C *be a closed convex subset of a real Hilbert space* H. *Let* A *be an* δ*-inverse strongly monotone mapping of* C *into* H *and let* T *be a nonexpansive mapping of* C *into itself such that* $Fix(T) \cap VI(C, A) \neq \emptyset$. *Suppose* $x_1 = u \in C$ *and* $\{x_n\}$ *and* $\{y_n\}$ *are given by*

$$\begin{cases} y_n = P_C(I - \lambda_n A)x_n \\ x_{n+1} = \alpha_n u + \beta_n x_n + \zeta_n TP_C(I - \lambda_n A)y_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\zeta_n\}$ *are three sequences in* $[0,1]$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 2\alpha]$. *If* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\zeta_n\}$, *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a,b]$ *for some* a, b *with* $0 < a < b < 2\delta$ *and*

(B_1) $\alpha_n + \beta_n + \zeta_n = 1$,
(B_2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(B_3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$,

(B_4) $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=0$,
then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $P_{Fix(S)\cap VI(C,A)}u$.

Proof. It suffices to take $F=I$, $f=\frac{1}{\gamma}u$, $A_2=A_3=A$, $\delta_{2,n}=\delta_{3,n}=\lambda_n$, $P_C(I-\delta_{i,n}A_i)=I$, for $i=1,4,\ldots,m$, and $\gamma_n=1$, for $n\in\mathbb{N}$, in Theorem 3.2.

Corollary 3.4. *(See Chen, Zhang and Fan* [3]*) Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $f\colon C\to C$ *be a contraction with coefficient* $\alpha\in(0,1)$, A *an* δ*-inverse strongly monotone mapping of* C *into* H, *and let* T *be a nonexpansive mapping of* C *into itself such that* $Fix(T)\cap VI(C,A)\neq\emptyset$. *Suppose* $\{x_n\}$ *be the sequence generated by* $x_0\in C$ *and*

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)TP_C(I-\lambda_nA)x_n,$$

where $\{\lambda_n\}\subset[a,b]$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$. *If* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n\in[a,b]$ *for some* a,b *with* $0<a<b<2\delta$ *and*

$$\lim_{n\to\infty}\alpha_n=0,\ \sum_{n=1}^{\infty}\alpha_n=\infty,\ \sum_{n=1}^{\infty}\mid\alpha_{n+1}-\alpha_n\mid<\infty,\ \sum_{n=1}^{\infty}\mid\lambda_{n+1}-\lambda_n\mid<\infty.$$

then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to an* $x^*\in Fix(T)\cap VI(C,A)$, *which is the unique solution of the variational inequality;*

$$\langle(I-f)x^*,x-x^*\rangle\geq 0,\quad\forall x\in Fix(T)\cap VI(C,A),$$

Proof. It suffices to take $F=I$, $\gamma=1$, $A_2=A$, $\delta_{2,n}=\lambda_n$, $P_C(I-\delta_{i,n}A_i)=I$, for $i=1,3,\ldots,m$ and $\beta_n=0$, for $n\in\mathbb{N}$, in Theorem 3.2.

Acknowledgments

The author would like to thank the editor. Also, special thanks are due to the anonymous referee.

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