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ARENS REGULARITY OF BILINEAR FORMS AND UNITAL BANACH MODULE SPACES

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ABSTRACT. Assume that A, B are Banach algebras and that $m : A \times B \rightarrow B, m' : A \times A \rightarrow B$ are bounded bilinear mappings. We study the relationships between Arens regularity of m, m' and the Banach algebras A, B . For a Banach A -bimodule B , we show that B factors with respect to A if and only if B^{**} is unital as an A^{**} -module. Let $Z_{e''}(B^{**}) = B^{**}$ where e'' is a mixed unit of A^{**} . Then B^* factors on both sides with respect to A if and only if B^{**} has a unit as A^{**} -module.

Keywords: Arens regularity, bilinear mappings, topological center, unital A -module, module action.

MSC(2010): Primary: 46L06, 46L07, 46L10; Secondary: 47L25, 47L50.

1. Introduction

Throughout this paper, A is a Banach algebra and A^*, A^{**} , respectively, are the first and second dual of A . Recall that a left approximate identity, abbreviated as LAI , (respectively, right approximate identity, abbreviated as RAI) in a Banach algebra A is a net $(e_\alpha)_{\alpha \in I}$ in A such that $e_\alpha a \rightarrow a$ (respectively, $ae_\alpha \rightarrow a$) for each $a \in A$. We say that a net $(e_\alpha)_{\alpha \in I} \subseteq A$ is a approximate identity, abbreviated as AI , for A if it is both an LAI and a RAI for A . If $(e_\alpha)_{\alpha \in I}$ in A is bounded and is an AI for A , then we say that $(e_\alpha)_{\alpha \in I}$ is a bounded approximate identity, abbreviated as BAI , for A . For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by

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$\langle a'a, b \rangle = \langle a', ab \rangle = a'(ab)$ and $\langle aa', b \rangle = \langle a', ba \rangle = a'(ba)$ for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a, a' \rangle = \langle a', a \rangle$ for every $a \in A$ and $a' \in A^*$. We denote the set $\{a'a : a \in A \text{ and } a' \in A^*\}$ and $\{aa' : a \in A \text{ and } a' \in A^*\}$ by A^*A and AA^* , respectively. It is clear that these two sets are subsets of A^* . Assume that A has a *BAI*. If the equality $A^*A = A^*$ ($AA^* = A^*$) holds, then we say that A^* factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that A^* factors on both sides.

It is well-known that (see for instance [1]), the second dual A^{**} of A endowed with the either Arens multiplications is a Banach algebra. The constructions of the two Arens multiplications in A^{**} lead us to the definition of topological centers for A^{**} with respect to Arens multiplications. The topological centers of Banach algebras, module actions and applications of them have been introduced and discussed in many papers such as [7, 8, 10, 11, 12]. The extension of bilinear maps on normed spaces and the concept of regularity of bilinear maps have taken a great deal of attention by many researchers (see for example [1, 2, 5, 7, 12]). We commence by recalling some definitions as follows.

Let X, Y and Z be normed spaces and $m : X \times Y \rightarrow Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***} of m from $X^{**} \times Y^{**}$ into Z^{**} as follows

1. $m^* : Z^* \times X \rightarrow Y^*$, given by $\langle m^*(z', x), y \rangle = \langle z', m(x, y) \rangle$ where $x \in X, y \in Y, z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \rightarrow X^*$, given by $\langle m^{**}(y'', z'), x \rangle = \langle y'', m^*(z', x) \rangle$ where $x \in X, y'' \in Y^{**}, z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \rightarrow Z^{**}$, given by $\langle m^{***}(x'', y''), z' \rangle = \langle x'', m^{**}(y'', z') \rangle$ where $x'' \in X^{**}, y'' \in Y^{**}, z' \in Z^*$.

The mapping m^{***} is the unique extension of m such that $x'' \rightarrow m^{***}(x'', y'')$ from X^{**} into Z^{**} is *weak*-weak** continuous for every $y'' \in Y^{**}$, but the mapping $y'' \rightarrow m^{***}(x'', y'')$ is not in general *weak*-weak** continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as follows

$$Z_1(m) = \{x'' \in X^{**} : y'' \rightarrow m^{***}(x'', y'') \text{ is } \textit{weak}^*\text{-to-}\textit{weak}^* \text{ continuous}\}.$$

Let now $m^t : Y \times X \rightarrow Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z , and so it may be extended as above to

$m^{t***} : Y^{**} \times X^{**} \rightarrow Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \rightarrow Z^{**}$ in general is not equal to m^{***} , see [1]. If $m^{***} = m^{t***t}$, then m is called Arens regular. The bounded bilinear mapping m is said to be left (respectively, right) strongly Arens irregular whenever m^{***} and m^{t***t} are equal only on $X \times Y^{**}$ (respectively, $X^{**} \times Y$). The mapping $y'' \rightarrow m^{t***t}(x'', y'')$ is *weak*-weak** continuous for every $y'' \in Y^{**}$, but the mapping $x'' \rightarrow m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general *weak* - weak** continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : x'' \rightarrow m^{t***t}(x'', y'') \text{ is } \textit{weak*}-\textit{weak*} \text{ continuous}\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j) \rangle = \lim_j \lim_i \langle z', m(x_i, y_j) \rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [13].

The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.

Let now B be a Banach A -bimodule, and let

$$\pi_\ell : A \times B \rightarrow B \text{ and } \pi_r : B \times A \rightarrow B.$$

be the left and right module actions of A on B , respectively. Then B^{**} is a Banach A^{**} -bimodule with module actions

$$\pi_\ell^{***} : A^{**} \times B^{**} \rightarrow B^{**} \text{ and } \pi_r^{***} : B^{**} \times A^{**} \rightarrow B^{**}.$$

Similarly, B^{**} is a Banach A^{**} -bimodule with module actions

$$\pi_\ell^{t***t} : A^{**} \times B^{**} \rightarrow B^{**} \text{ and } \pi_r^{t***t} : B^{**} \times A^{**} \rightarrow B^{**}.$$

We may therefore define the topological centers of the left and right module actions of A on B as follows:

$$Z_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{a'' \in A^{**} : \text{the map } b'' \rightarrow \pi_\ell^{***}(a'', b'') : B^{**} \rightarrow B^{**} \\ \text{is } \textit{weak*}-\textit{weak*} \text{ continuous}\}$$

$$Z_{B^{**}}^t(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} : \text{the map } b'' \rightarrow \pi_r^{t***}(a'', b'') : B^{**} \rightarrow B^{**} \\ \text{is } \textit{weak*}-\textit{weak*} \text{ continuous}\}$$

$$Z_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} : \text{the map } a'' \rightarrow \pi_r^{***}(b'', a'') : A^{**} \rightarrow B^{**} \\ \text{is } \textit{weak*}-\textit{weak*} \text{ continuous}\}$$

$$Z_{A^{**}}^t(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} : \text{the map } a'' \rightarrow \pi_\ell^{t***}(b'', a'') : A^{**} \rightarrow B^{**} \\ \text{is weak}^*\text{-weak}^* \text{ continuous}\}$$

One can also see ready that if B is a left (respectively, right) Banach A -module and $\pi_\ell : A \times B \rightarrow B$ (respectively, $\pi_r : B \times A \rightarrow B$) is left (respectively, right) module action of A on B , then B^* is a right (respectively, left) Banach A -module.

We write

$$\begin{aligned} ab &= \pi_\ell(a, b), ba = \pi_r(b, a), \pi_\ell(a_1 a_2, b) = \pi_\ell(a_1, a_2 b), \\ \pi_r(b, a_1 a_2) &= \pi_r(b a_1, a_2), \pi_\ell^*(a_1 b', a_2) = \pi_\ell^*(b', a_2 a_1), \\ \pi_r^*(b' a, b) &= \pi_r^*(b', ab), \end{aligned}$$

for all $a_1, a_2, a \in A, b \in B$ and $b' \in B^*$ when there is no confusion. Regarding A as a Banach A -bimodule, the operation $\pi : A \times A \rightarrow A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (respectively, left) and the second (respectively, right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. In this situation, we shall also simplify our notations. So the first (respectively, left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$\begin{aligned} \langle a'a, b \rangle &= \langle a', ab \rangle, \\ \langle a''a', a \rangle &= \langle a'', a'a \rangle, \\ \langle a''b'', a' \rangle &= \langle a'', b''a' \rangle. \end{aligned}$$

for every $a, b \in A$ and $a' \in A^*$. Similarly, the second (respectively, right) Arens product of $a'', b'' \in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$\begin{aligned} \langle aoa', b \rangle &= \langle a', ba \rangle, \\ \langle a'oa'', a \rangle &= \langle a'', aoa' \rangle, \\ \langle a''ob'', a' \rangle &= \langle b'', a'oa'' \rangle. \end{aligned}$$

for all $a, b \in A$ and $a' \in A^*$.

The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**} , the second dual of A . By Goldstine's Theorem ([6], page 424), there are bounded nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = \text{weak}^*\text{-}\lim_\alpha a_\alpha$ and $b'' = \text{weak}^*\text{-}\lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_\alpha \lim_\beta \langle a', \pi(a_\alpha, b_\beta) \rangle = \langle a''b'', a' \rangle$$

and

$$\lim_{\beta} \lim_{\alpha} \langle a', \pi(a_{\alpha}, b_{\beta}) \rangle = \langle a''ob'', a' \rangle,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**} , respectively, see [5, 11, 12].

We find the usual first and second topological center of A^{**} , which are

$$Z_{A^{**}}(A^{**}) = Z(\pi) = \{a'' \in A^{**} : b'' \rightarrow a''b'' \text{ is } weak^*-weak^* \\ \text{continuous}\},$$

$$Z_{A^{**}}^t(A^{**}) = Z(\pi^t) = \{a'' \in A^{**} : a'' \rightarrow a''ob'' \text{ is } weak^*-weak^* \\ \text{continuous}\}.$$

An element e'' of A^{**} is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if, for each $a'' \in A^{**}$, $a''e'' = e''oa'' = a''$. By ([3], page 146), an element e'' of A^{**} is a mixed unit if and only if it is a $weak^*$ cluster point of some BAI $(e_{\alpha})_{\alpha \in I}$ in A . A functional a' in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a \rightarrow a'a$ from A into A^* is weakly compact. In [13], Pym showed that this definition is equivalent to the following condition.

For every two nets $(a_i)_i$ and $(b_j)_j$ in $A_1 = \{a \in A : \|a\| \leq 1\}$, we have

$$\lim_i \lim_j \langle a', a_i b_j \rangle = \lim_j \lim_i \langle a', a_i b_j \rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on A is denoted by $wap(A)$. Also we have $a' \in wap(A)$ if and only if $\langle a''b'', a' \rangle = \langle a''ob'', a' \rangle$ for every $a'', b'' \in A^{**}$.

Throughout the paper, for two normed spaces A and B , $\mathbf{B}(A, B)$ is the set of all bounded linear operators from A into B .

In the next section, we investigate the relationships between the Arens regularity of some bilinear mappings and Banach algebras, particularly, under some conditions, for bounded bilinear mappings $m : A \times B \rightarrow B$, $m' : A \times A \rightarrow B$, we show that m or m' are Arens regular (respectively, irregular) if and only if A or B are Arens regular (respectively, irregular). We also give some results in special Banach algebras such as $L^1(G)$, $M(G)$, $L^\infty(X)$ and $C(X)$ whenever G is a locally compact group and X is a semigroup. In Section 3, we extend some results from [11] into module actions along with some new results.

2. Arens regularity of some bilinear forms

In Theorem 2.1 and Theorem 2.6 of this section, we introduce some special bilinear mappings from $A \times B$ or $A \times A$ into B . We establish some relationships between the Arens regularity of these bilinear mappings and Arens regularity of A or B with some applications in group algebras.

Theorem 2.1. *Let A be a normed space and B be a Banach algebra. Let $T \in \mathbf{B}(A, B)$ and m be the bilinear mapping from $A \times B$ into B such that for every $a \in A$ and $b \in B$ we have $m(a, b) = T(a)b$. Then the following statements are true.*

- a) *If B is Arens regular, then m is Arens regular.*
- b) *If T is surjective, then we have*
 - i) *B is Arens regular if and only if m is Arens regular.*
 - ii) *If m is left strongly Arens irregular, then B is left strongly Arens irregular.*
 - iii) *If T is injective, then B is left strongly Arens irregular if and only if m is left strongly Arens irregular.*

Proof. a) By the definition of m^{***} , we have $m^{***}(a'', b'') = T^{**}(a'')b''$ and $m^{t****}(a'', b'') = T^{**}(a'')ob''$ where $a'' \in A^{**}$ and $b'' \in B^{**}$. Since $Z_1(B^{**}) = B^{**}$, the mapping $b'' \rightarrow T^{**}(a'')b'' = m^{***}(a'', b'')$ is $weak^*$ - $weak^*$ continuous for all $a'' \in A^{**}$. Also since $Z_2(B^{**}) = B^{**}$, the mapping $a'' \rightarrow T^{**}(a'')ob'' = m^{t****}(a'', b'')$ is $weak^*$ - $weak^*$ continuous for all $b'' \in B^{**}$. Hence m is Arens regular.

b) i) Let m be Arens regular. Then $Z_1(m) = A^{**}$. Let $b''_1, b''_2 \in B^{**}$ and $(b''_\alpha)_\alpha \in B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b''_2$. Since T is surjective, there is an $a'' \in A^{**}$ such that $T^{**}(a'') = b''_1$. Then we have

$$\begin{aligned} b''_1 b''_2 &= T^{**}(a'')b''_2 = m^{***}(a'', b''_2) = weak^* - \lim_{\alpha} m^{***}(a'', b''_\alpha) \\ &= weak^* - \lim_{\alpha} T^{**}(a'')b''_\alpha = weak^* - \lim_{\alpha} b''_1 b''_\alpha. \end{aligned}$$

Hence $Z_1(B^{**}) = B^{**}$ consequently B is Arens regular.

b) ii) Let m be left strongly Arens irregular. Then $Z_1(m) = A$. For $b''_1 \in Z_1(B^{**})$ the mapping $b''_2 \rightarrow b''_1 b''_2$ is $weak^*$ - $weak^*$ continuous. Also since T is surjective, there exists $a'' \in A^{**}$ such that $T^{**}(a'') = b''_1$ and the mapping $b''_2 \rightarrow T^{**}(a'')b''_2 = m^{***}(a'', b''_2)$ is $weak^*$ - $weak^*$ continuous. Hence $a'' \in Z_1(m) = A$. Consequently we have $b''_1 = T^{**}(a'') \in B$. It follows that $Z_1(B^{**}) = B$.

b) iii) Let B be left strongly Arens irregular. So $Z_1(B^{**}) = B$. For $a'' \in Z_1(m)$ the mapping $b'' \rightarrow m^{***}(a'', b'')$ is $weak^*$ - $weak^*$ continuous

and consequently $T^{**}(a'') \in Z_1(B^{**}) = B$. Since T is bijective, $a'' \in A$. We conclude $Z_1(m) = A$. \square

In Theorem 2.1, if we replace the left strongly Arens irregularity of A , B and m with the right strongly Arens irregularity of them, then the results hold.

The following definition, introduced by Ülger [16], has an important role in showing some sufficient conditions for the Arens regularity of tensor product $A \hat{\otimes} B$ where A and B are Banach algebra.

Suppose that A and B are Banach algebras. We recall that a bilinear form $m : A \times B \rightarrow \mathbb{C}$ is biregular if for any two pairs of sequences $(a_i)_i$, $(\tilde{a}_j)_j$ in A_1 and $(b_i)_i$, $(\tilde{b}_j)_j$ in B_1 , we have

$$\lim_i \lim_j m(a_i \tilde{a}_j, b_i \tilde{b}_j) = \lim_j \lim_i m(a_i \tilde{a}_j, b_i \tilde{b}_j)$$

provided that these limits exist.

Corollary 2.2. *Let B be a unital Banach algebra and suppose that A is subalgebra of B . If A is not Arens regular, then $A \hat{\otimes} B$ is not Arens regular.*

Proof. Let $m : A \times B \rightarrow \mathbb{C}$ be the bilinear form that was introduced in Theorem 2.1 where $T : A \rightarrow B$ is natural inclusion. Since A is not Arens regular, m is not biregular. Consequently by ([16], Theorem 3.4), $A \hat{\otimes} B$ is not Arens regular. \square

Example 2.3. *Let $C(X)$ be the Banach algebra of all continuous bounded functions on $X = [0, 1]$ with the supremum norm and the convolution as multiplication defined by*

$$f * g(x) = \int_0^x f(x-t)g(t)dt \text{ where } 0 \leq x \leq 1.$$

*Let $T : C(X) \rightarrow L^\infty(X)$ be the natural inclusion and $m : C(X) \times L^\infty(X) \rightarrow L^\infty(X)$ be defined by $m(f, g) = f * g$ where $f \in C(X)$ and $g \in L^\infty(X)$. By [2], $L^\infty(X)$ is Arens regular and by Theorem 2.1, we conclude that m is Arens regular.*

*Similarly since c_0 is Arens regular, see [1, 5], by using Theorem 2.1, it is clear that the bounded bilinear mapping $(f, g) \rightarrow f * g$ from $\ell^1 \times c_0$ into c_0 is Arens regular.*

For a Banach algebra A , we recall that a bounded linear operator $T : A \rightarrow A$ is said to be a left (respectively, right) multiplier if for all $a, b \in A$, $T(ab) = T(a)b$ (respectively, $T(ab) = aT(b)$). We denote

by $LM(A)$ (respectively, $RM(A)$) the set of all left (respectively, right) multipliers of A . The set $LM(A)$ (respectively, $RM(A)$) is a normed subalgebra of the algebra $L(A)$ of bounded linear operator on A .

Now, we define a new concept which is an extension of the left (respectively, right) multiplier from a Banach algebra to module actions. We show that some relationships hold between this concept and Arens regularity of some bilinear mappings.

Definition 2.4. Let B be a left Banach (respectively, right) A -module and $T \in \mathbf{B}(A, B)$. Then T is called extended left (respectively, right) multiplier if

$$T(a_1 a_2) = \pi_r(T(a_1), a_2) \text{ (resp. } T(a_1 a_2) = \pi_\ell(a_1, T(a_2))) \text{ for all } a_1, a_2 \in A$$

We denote by $LM(A, B)$ [respectively, $RM(A, B)$] the set of all left (respectively, right) multiplier extension from A into B .

Example 2.5. Let $a' \in A^*$. Then the mapping $T_{a'} : a \rightarrow a'a$ (respectively $R_{a'} a \rightarrow aa'$) from A into A^* is a left (respectively, right) multiplier, that is, $T_{a'} \in LM(A, A^*)$ ($R_{a'} \in RM(A, A^*)$). It is also clear that $T_{a'}$ is weakly compact if and only if $a' \in \text{wap}(A)$.

Theorem 2.6. Let B be a left (respectively, right) Banach A -module and $T \in \mathbf{B}(A, B)$ be a continuous map. Assume that $m : A \times A \rightarrow B$ is the bilinear mapping such that $m(a_1, a_2) = T(a_1 a_2)$. Then we have the following assertions

- i) If A is Arens regular, then m is Arens regular.
- ii) If m is left (right) strongly Arens irregular, then A is left (respectively, right) strongly Arens irregular.
- iii) If $T \in LM(A, B)$, then $T^{**}(Z_1(m)) \subseteq Z_{A^{**}}(B^{**})$.
- iv) If $T \in LM(A, B)$, then $T^{**} \in LM(A^{**}, B^{**})$.
- v) Suppose that B is Banach algebra and T is an embedding. Then, B is Arens regular if and only if m is Arens regular.

Proof. i) An easy calculation shows that

$$m^{***}(a_1'', a_2'') = T^{**}(a_1'' a_2'') , \quad m^{t***t}(a_1'', a_2'') = T^{**}(a_1'' o a_2'').$$

Since A is Arens regular, the mapping $a_2'' \rightarrow a_1'' a_2''$ is *weak*-weak** continuous for all $a_1'' \in A^{**}$. Also the mapping $a_1'' \rightarrow a_1'' o a_2''$ is *weak*-weak** continuous for all $a_2'' \in A^{**}$. Hence both mappings $a_2'' \rightarrow T^{**}(a_1'' a_2'') = m^{***}(a_1'', a_2'')$ and $a_1'' \rightarrow T^{**}(a_1'' o a_2'') = m^{t***t}(a_1'', a_2'')$ are *weak*-weak** continuous for all $a_1'' \in A^{**}$ and $a_2'' \in A^{**}$, respectively. We conclude that $Z_1(m) = Z_2(m) = A^{**}$.

- ii) Let $a_1'' \in Z_1(A^{**})$. Then the mapping $a_2'' \rightarrow a_1''a_2''$ is *weak*-weak** continuous. Consequently, the mapping $a_2'' \rightarrow T^{**}(a_1''a_2'') = m^{***}(a_1'', a_2'')$ is *weak*-weak** continuous. Hence $a_1'' \in Z_1(m) = A$.
- iii) Let $a_1'' \in Z_1(m)$. Then the mapping

$$a_2'' \rightarrow m^{***}(a_1'', a_2'') = T^{**}(a_1'')a_2''$$

is *weak*-weak** continuous from A^{**} into B^{**} . It follows that $T^{**}(a_1'') \in Z_{A^{**}}(B^{**})$.

- iv) If we set $m(a_1, a_2) = T(a_1a_2)$ (*respectively*, $= T(a_1)a_2$) for all $a_1, a_2 \in A$, then $m^{***}(a_1'', a_2'') = T^{**}(a_1''a_2'')$ (*respectively*, $= T^{**}(a_1'')a_2''$) for all $a_1'', a_2'' \in A^{**}$. Thus, we conclude that $T^{**}(a_1''a_2'') = T^{**}(a_1'')a_2''$ for all $a_1'', a_2'' \in A^{**}$.

- v) Let m be Arens regular and $b_1'', b_2'' \in B^{**}$ and let $(b''_\alpha)_\alpha \in B^{**}$ such that $b''_\alpha \xrightarrow{w^*} b_2''$. We set $a_1'', a_2'' \in A^{**}$ and $(a''_\alpha)_\alpha \in A^{**}$ such that $T^{**}(a_1'') = b_1''$, $T^{**}(a_2'') = b_2''$ and $T^{**}(a''_\alpha) = b''_\alpha$. Then

$$\begin{aligned} b_1''b_2'' &= T^{**}(a_1'')T^{**}(a_2'') = T^{**}(a_1''a_2'') = m^{***}(a_1'', a_2'') \\ &= \text{weak}^* - \lim_{\alpha} m^{***}(a_1'', a''_\alpha) = \text{weak}^* - \lim_{\alpha} T^{**}(a_1''a''_\alpha) \\ &= \text{weak}^* - \lim_{\alpha} T^{**}(a_1'')T^{**}(a''_\alpha) = \text{weak}^* - \lim_{\alpha} b_1''b''_\alpha, \end{aligned}$$

where by the open mapping theorem, we have $a''_\alpha \xrightarrow{w^*} a_2''$. Consequently $Z_1(B^{**}) = B^{**}$.

Conversely, let B be Arens regular and $a_1'', a_2'' \in A^{**}$ and $(a_\alpha)_\alpha \in A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a_2''$. Then

$$\begin{aligned} m^{***}(a_1'', a_2'') &= T^{**}(a_1''a_2'') = \text{weak}^* - \lim_{\alpha} T^{**}(a_1''a''_\alpha) \\ &= \text{weak}^* - \lim_{\alpha} m^{***}(a_1'', a''_\alpha). \end{aligned}$$

It follow that $Z_1(m) = A^{**}$. Thus m is Arens regular. \square

Example 2.7. Assume that $T : c_0 \rightarrow \ell^\infty$ is the natural inclusion map and $m : c_0 \times c_0 \rightarrow \ell^\infty$ be the bilinear mapping such that $m(f, g) = f * g$, for all $f, g \in c_0$. Since c_0 is Arens regular, m is Arens regular. Similarly the bilinear mapping $m : C(G) \times C(G) \rightarrow L^\infty(G)$ defined by formula $(f, g) \rightarrow f * g$, for all $f, g \in C(G)$ is Arens regular whenever G is compact.

For normed spaces X, Y, Z, W let $m_1 : X \times Y \rightarrow Z$ and $m_2 : X \times W \rightarrow Z$ be bounded bilinear mappings. If $h : Y \rightarrow W$ is a continuous linear mapping such that $m_1(x, y) = m_2(x, h(y))$ for all $x \in X$ and $y \in Y$, then

we say that m_1 factors through m_2 , see [2]. We say that the continuous bilinear mapping $m : X \times Y \rightarrow Z$ factors if m is onto Z , see [7].

Theorem 2.8. *Let A and B be Banach algebras and B be a Banach A -bimodule. Let $T \in \mathbf{B}(A, B)$ be a continuous homomorphism. If T is weakly compact, then the bilinear mapping $m(a_1, a_2) = T(a_1 a_2)$ from $A \times A$ into B is Arens regular.*

Proof. Let m' be the bilinear mapping that we introduced in Theorem 2.1. Then $m(a_1, a_2) = m'(a_1, T(a_2))$ for all $a_1, a_2 \in A$. Consequently m factors through m' . So by ([2], Theorem 2), we conclude that m is Arens regular. \square

Example 2.9. *Suppose that $T : L^1(G) \rightarrow M(G)$ is the natural inclusion. Then the bilinear mapping $m : L^1(G) \times L^1(G) \rightarrow M(G)$ defined by $m(f, g) = f * g$ for all $f, g \in L^1(G)$ is Arens regular whenever G is finite, see [18]. Also the left strongly Arens irregularity of m implies that $L^1(G)$ is also left strongly Arens irregular, see [10, 11].*

3. Unital A -modules and module actions

In [11], Lau and Ülger showed that for a Banach algebra A , A^* factors on the left if and only if A^{**} is unital with respect to the first Arens product. In this section we extend this result to module actions.

We say that A^{**} has a *weak* bounded left approximate identity*, abbreviated as W^*BLAI with respect to the first Arens product, if there is a bounded net $(e_\alpha)_\alpha \subseteq A$ such that for all $a'' \in A^{**}$ and $a' \in A^*$, we have $\langle e_\alpha a'', a' \rangle \rightarrow \langle a'', a' \rangle$. The definition of W^*RBAI is similar and if A^{**} has both W^*LBAI and W^*RBAI , then we say that A^{**} has W^*BAI .

Assume that B is a Banach A -bimodule. Then we define the set AB as the linear span of the set $\{ab : a \in A, b \in B\}$. We say that B factors on the left (right) if $B = BA$ ($B = AB$) and B factors on both sides, if $B = BA = AB$.

Definition 3.1. *Let B be a left Banach A -module and e be a left unit element of A . We say that e is a left unit (respectively, weak left unit) for A -module B if $\pi_\ell(e, b) = b$ (respectively, $\langle b', \pi_\ell(e, b) \rangle = \langle b', b \rangle$ for all $b' \in B^*$) where $b \in B$. The definition of right unit (respectively, weak right unit) is similar.*

We say that a Banach A -bimodule B is a unital as an A -module if B has the same left and right unit. Thus in this case, we say that B is

unital as A -module.

Let B be a left Banach A -module and $(e_\alpha)_\alpha \subseteq A$ be a LAI (respectively, weakly left approximate identity, abbreviated as (WLAI)) for A . Then $(e_\alpha)_\alpha$ is said to be a left approximate identity, abbreviated as (LAI) (respectively, weakly left approximate identity, abbreviated as (WLAI)) for B , if for each $b \in B$, we have $\pi_\ell(e_\alpha, b) \rightarrow b$ (respectively, $\pi_\ell(e_\alpha, b) \xrightarrow{w} b$). The definition of the right approximate identity (= RAI) (respectively, weakly right approximate identity, abbreviated as (WRAI)) is similar. We say that $(e_\alpha)_\alpha$ is an approximate identity, abbreviated as (AI) (respectively, weakly approximate identity, abbreviated as (WAI)) for B , if B has the same left and right approximate identity (respectively, weakly left and right approximate identity).

Let $(e_\alpha)_\alpha \subseteq A$ be weak* left approximate identity for A^{**} . Then $(e_\alpha)_\alpha$ is weak* left approximate identity (W^* LAI) as A^{**} -module for B^{**} , if for all $b'' \in B^{**}$, we have $\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w^*} b''$. The definition of the weak* right approximate identity, abbreviated as (W^* RAI) is similar. We say that $(e_\alpha)_\alpha$ is a weak* approximate identity, abbreviated as (W^* AI) for B^{**} , if B^{**} has the same weak* left and right approximate identity.

Example 3.2. Let G be a locally compact group. We know that $L^p(G)$, for $1 \leq p \leq \infty$, is a Banach $M(G)$ – bimodule under the convolution as multiplication. Thus $L^p(G)$ is a unital $M(G)$ – bimodule.

Theorem 3.3. Assume that A is a Banach algebra and A has a BAI $(e_\alpha)_\alpha$. Then we have the following assertions.

- i) Let B be a right Banach A -module. Then B factors on the left if and only if B has a WRAI.
- ii) Let B be a left Banach A -module. Then B factors on the right if and only if B has a WLAI.
- iii) B factors on both side if and only if B has a WAI.

Proof. i) Suppose that $B = BA$. Let $b \in B$ and $b' \in B^*$. Then there are $x \in B$ and $a \in A$ such that $b = xa$. Therefore

$$\begin{aligned} \langle b', \pi_r(b, e_\alpha) \rangle &= \langle b', \pi_r(xa, e_\alpha) \rangle = \langle \pi_r^*(b', x), ae_\alpha \rangle \rightarrow \langle \pi_r^*(b', x), a \rangle \\ &= \langle b', \pi_r(x, a) \rangle = \langle b', b \rangle. \end{aligned}$$

It follows that $\pi_r(b, e_\alpha) \xrightarrow{w} b$, and so B has a WRAI.

For the converse, since BA is a weakly closed subspace of B , so by Cohen factorization theorem, see [5], the result is immediate.

ii) The proof is similar to that of (i).

iii) This is clear. □

In Theorem 3.3, if we set $B = A$, then we obtain Lemma 2.1 from [11].

Theorem 3.4. *Assume that B is a left Banach A -module and A^{**} has a right unit e'' . Then, B factors on the left if and only if e'' is a right unit for A^{**} -module B^{**} .*

Proof. Since A^{**} has a right unit e'' , there is a BRAI $(e_\alpha)_\alpha$ for A such that $e_\alpha \xrightarrow{w^*} e''$, see [3]. Let $BA = B$ and $b \in B$. Thus, there is $x \in B$ and $a \in A$ such that $b = xa$. Then for all $b' \in B^*$, we have

$$\begin{aligned} \langle \pi_r^{**}(e'', b'), b \rangle &= \langle e'', \pi_r^*(b', b) \rangle = \lim_\alpha \langle e_\alpha, \pi_r^*(b', b) \rangle \\ &= \lim_\alpha \langle \pi_r^*(b', b), e_\alpha \rangle = \lim_\alpha \langle b', \pi_r(b, e_\alpha) \rangle \\ &= \lim_\alpha \langle b', \pi_r(xa, e_\alpha) \rangle = \lim_\alpha \langle \pi_r^*(b', x), ae_\alpha \rangle \\ &= \langle \pi_r^*(b', x), a \rangle = \langle b', \pi_r(x, a) \rangle = \langle b', b \rangle. \end{aligned}$$

Thus $\pi_r^{**}(e'', b') = b'$. Now let $b'' \in B^{**}$. Then we have

$$\langle \pi_r^{***}(b'', e''), b' \rangle = \langle b'', \pi_r^{**}(e'', b') \rangle = \langle b'', b' \rangle.$$

We conclude that $\pi_r^{***}(b'', e'') = b''$. Hence it follows that B^{**} has a right unit.

Conversely, assume that e'' is a right unit for B^{**} . Let $b \in B$ and $b' \in B^*$. Then we have

$$\begin{aligned} \langle b', \pi_r(b, e_\alpha) \rangle &= \langle \pi_r^*(b', b), e_\alpha \rangle \rightarrow \langle \pi_r^*(b', b), e'' \rangle = \langle e'', \pi_r^*(b', b) \rangle \\ &= \langle \pi_r^{**}(e'', b'), b \rangle = \langle b, \pi_r^{**}(e'', b') \rangle = \langle \pi_r^{***}(b, e''), b' \rangle \\ &= \langle b', b \rangle. \end{aligned}$$

Consequently $\pi_r(b, e_\alpha) \xrightarrow{w} \pi_r(b, e'') = b$, and so $b \in \overline{BA}^w$. Since BA is a weakly closed subspace of B , so by the Cohen factorization theorem, $b \in BA$. \square

Definition 3.5. *Let B be a Banach A -bimodule and $a'' \in A^{**}$. We define the sets $Z_{a''}^t(B^{**})$ (or $Z_{a''}^t(\pi_\ell^t)$) and $Z_{a''}(B^{**})$ (or $Z_{a''}(\pi_r^t)$) respectively, as follows*

$$\begin{aligned} Z_{a''}^t(B^{**}) &= Z_{a''}^t(\pi_\ell^t) = \{b'' \in B^{**} : \pi_\ell^{t***t}(a'', b'') = \pi_\ell^{***}(a'', b'')\}, \\ Z_{a''}(B^{**}) &= Z_{a''}(\pi_r^t) = \{b'' \in B^{**} : \pi_r^{t***t}(b'', a'') = \pi_r^{***}(b'', a'')\}. \end{aligned}$$

It is clear that

$$\bigcap_{a'' \in A^{**}} Z_{a''}^t(B^{**}) = Z_{A^{**}}^t(B^{**}) = Z(\pi_\ell^t),$$

$$\bigcap_{a'' \in A^{**}} Z_{a''}(B^{**}) = Z_{A^{**}}(B^{**}) = Z(\pi_r).$$

Theorem 3.6. *Assume that A is a Banach algebra and A^{**} has a mixed unit e'' . Then we have the following assertions.*

- i) *Let B be a left Banach A -module. Then, B^* factors on the left if and only if B^{**} has a left unit e'' .*
- ii) *Let B be a right Banach A -module and $Z_{e''}(\pi_r^t) = B^{**}$. Then, B^* factors on the right if and only if B^{**} has a right unit e'' .*
- iii) *Let B be a Banach A -bimodule and $Z_{e''}(\pi_r^t) = B^{**}$. Then, B^* factors on both sides if and only if B^{**} has a unit e'' .*

Proof. i) Let $(e_\alpha)_\alpha \subseteq A$ be a BAI for A such that $e_\alpha \xrightarrow{w^*} e''$. Suppose that $B^*A = B^*$. Thus for all $b' \in B^*$, there are $a \in A$ and $x' \in B^*$ such that $x'a = b'$. Then for all $b'' \in B^{**}$, we have

$$\begin{aligned} \langle \pi_\ell^{***}(e'', b''), b' \rangle &= \langle e'', \pi_\ell^{**}(b'', b') \rangle = \lim_\alpha \langle \pi_\ell^{**}(b'', b'), e_\alpha \rangle \\ &= \lim_\alpha \langle b'', \pi_\ell^*(b', e_\alpha) \rangle = \lim_\alpha \langle b'', \pi_\ell^*(x'a, e_\alpha) \rangle \\ &= \lim_\alpha \langle b'', \pi_\ell^*(x', ae_\alpha) \rangle = \lim_\alpha \langle \pi_\ell^{**}(b'', x'), ae_\alpha \rangle \\ &= \langle \pi_\ell^{**}(b'', x'), a \rangle = \langle b'', \pi_\ell^*(x', a) \rangle = \langle b'', b' \rangle. \end{aligned}$$

Thus $\pi_\ell^{***}(e'', b'') = b''$. Consequently B^{**} has left unit. Conversely, Let e'' be a left unit for B^{**} and $b' \in B^*$. Then for all $b'' \in B^{**}$, we have

$$\begin{aligned} \langle b'', b' \rangle &= \langle \pi_\ell^{***}(e'', b''), b' \rangle = \langle e'', \pi_\ell^{**}(b'', b') \rangle \\ &= \lim_\alpha \langle \pi_\ell^{**}(b'', b'), e_\alpha \rangle = \lim_\alpha \langle b'', \pi_\ell^*(b', e_\alpha) \rangle. \end{aligned}$$

Thus we conclude that $\text{weak-}\lim_\alpha \pi_\ell^*(b', e_\alpha) = b'$. So by the Cohen factorization Theorem, the proof is immediate.

- ii) Suppose that $AB^* = B^*$. Thus for all $b' \in B^*$, there are $a \in A$ and $x' \in B^*$ such that $ax' = b'$. Assume $(e_\alpha)_\alpha \subseteq A$ is a BAI for A such that

$e_\alpha \xrightarrow{w^*} e''$. Let $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$. Then

$$\begin{aligned} \langle \pi_r^{***}(b'', e''), b' \rangle &= \lim_\beta \langle \pi_r^{***}(b_\beta, e''), b' \rangle = \lim_\beta \lim_\alpha \langle b', \pi_r(b_\beta, e_\alpha) \rangle \\ &= \lim_\beta \lim_\alpha \langle ax', \pi_r(b_\beta, e_\alpha) \rangle = \lim_\beta \lim_\alpha \langle x', \pi_r(b_\beta, e_\alpha)a \rangle \\ &= \lim_\beta \lim_\alpha \langle x', \pi_r(b_\beta, e_\alpha a) \rangle = \lim_\beta \lim_\alpha \langle \pi_r^*(x', b_\beta), e_\alpha a \rangle \\ &= \lim_\beta \langle \pi_r^*(x', b_\beta), a \rangle = \langle b'', b' \rangle. \end{aligned}$$

We conclude that

$$\pi_r^{***}(b'', e'') = b''$$

for all $b'' \in B^{**}$.

Conversely, suppose that $\pi_r^{***}(b'', e'') = b''$ where $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$. Let $(e_\alpha)_\alpha \subseteq A$ be a BAI for A such that $e_\alpha \xrightarrow{w^*} e''$. Since $Z_{e''}(\pi_r^t) = B^{**}$, for all $b' \in B^*$, we have

$$\begin{aligned} \langle b'', b' \rangle &= \langle \pi_r^{***}(b'', e''), b' \rangle = \langle b'', \pi_r^{**}(e'', b') \rangle = \lim_\beta \langle \pi_r^{**}(e'', b'), b_\beta \rangle \\ &= \lim_\beta \langle e'', \pi_r^*(b', b_\beta) \rangle = \lim_\beta \lim_\alpha \langle \pi_r^*(b', b_\beta), e_\alpha \rangle \\ &= \lim_\beta \lim_\alpha \langle \pi_r^*(b', b_\beta), e_\alpha \rangle = \lim_\beta \lim_\alpha \langle b', \pi_r(b_\beta, e_\alpha) \rangle \\ &= \lim_\alpha \lim_\beta \langle b', \pi_r(b_\beta, e_\alpha) \rangle = \lim_\alpha \lim_\beta \langle \pi_r^{***}(b_\beta, e_\alpha), b' \rangle \\ &= \lim_\alpha \lim_\beta \langle b_\beta, \pi_r^{**}(e_\alpha, b') \rangle = \lim_\alpha \langle b'', \pi_r^{**}(e_\alpha, b') \rangle. \end{aligned}$$

It follows that $\text{weak-}\lim_\alpha \pi_r^{**}(e_\alpha, b') = b'$. So by the Cohen factorization theorem, we are done.

iii) This is clear. \square

In Theorem 3.6, if we set $B = A$, then we obtain Proposition 2.10 from [11].

Corollary 3.7. *Let B be a Banach A -bimodule and A^{**} has a mixed unit e'' .*

- a) *Let $Z_{e''}(\pi_r^t) = B^{**}$. Then we have the following assertions*
 - i) *If B^* factors on the left but not on the right, then $\pi_\ell \neq \pi_r^t$.*
 - ii) *If B^* factors on the left and $\pi_\ell = \pi_r^t$, then B^* factors on the right.*
 - iii) *e'' is a right unit for B^{**} if and only if $(e_\alpha)_\alpha$ is a W^* RAI for B^{**} whenever $e_\alpha \xrightarrow{w^*} e''$.*
- b) *Let $Z_{e''}^t(\pi_\ell^t) = B^{**}$. Then we have the following assertions*

- i) If B^* factors on the right but not on the left, then $\pi_r \neq \pi_\ell^t$.
- ii) If B^* factors on the right and $\pi_r = \pi_\ell^t$, then B^* factors on the left.
- iii) e'' is a left unit for B^{**} if and only if $(e_\alpha)_\alpha$ is a W^*LAI for B^{**} whenever $e_\alpha \xrightarrow{w^*} e''$.
- c) Let $Z_{e''}^{\ell}(\pi_\ell) = Z_{e''}^r(\pi_r) = B^{**}$. Then we have the following assertions
- i) If B^* factors on the one side, but not on the other side, then $\pi_r \neq \pi_\ell^t$ and $\pi_\ell \neq \pi_r^t$.
- ii) e'' is a unit for B^{**} if and only if $(e_\alpha)_\alpha$ is a W^*AI for B^{**} whenever $e_\alpha \xrightarrow{w^*} e''$.

Proof. a) i) Let B^* factor on the right but not on the left. By Theorem 3.5, e'' is a right unit for B^{**} . Thus we have $\pi_r^{***}(b'', e'') = b''$ for all $b'' \in B^{**}$. If we set $\pi_\ell = \pi_r^t$, then $\pi_\ell^{***}(e'', b'') = \pi_r^{t***}(e'', b'') = \pi_r^{t***t}(b'', e'') = \pi_r^{***}(b'', e'') = b''$ for all $b'' \in B^{**}$. Consequently, e'' is left unit for B^{**} . Then by Theorem 3.5, B^* factors on the left which is impossible.

ii) The proof similar to that of (i).

iii) Since $e_\alpha \xrightarrow{w^*} e''$, $weak^*-\lim_\alpha \pi_r^{***}(b'', e_\alpha) = \pi_r^{***}(b'', e'')$ for all $b'' \in B^{**}$. Hence the proof is complete.

The proofs of (b) and (c) are the same and are immediately followed. \square

Assume that $Z_{e''}^{\ell}(\pi_\ell^t) = Z_{e''}^r(\pi_r) = B^{**}$. Let $\pi_r = \pi_\ell^t$ and $\pi_\ell = \pi_r^t$. Using Corollary 3.7, we know that if B^* factors on the one side, then B^* factors on the other side. On the other hand, if we set $\pi_\ell = \pi_r^t$ and $Z_{e''}^r(\pi_r) = B^{**}$ where e'' is a mixed unit for A^{**} , then, by using this corollary, B^* factors on the right if and only if B factors on the left.

Questions.

Suppose that B is a Banach A -bimodule. Under which conditions the following results hold?

- i) B factors on the left if and only if B^{**} has a left unit.
- ii) B factors on the one side if and only if B^* factors on the same side.

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