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ON GENERALISATIONS OF ALMOST PRIME AND WEAKLY PRIME IDEALS

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Abstract. Let R be a commutative ring with identity.

A proper ideal P of R is an $(n-1,n)$-Φ_m-prime ($(n-1,n)$-weakly prime) ideal if $a_1,\ldots,a_n \in R$, $a_1\cdots a_n \in P\backslash P^m$ ($a_1\cdots a_n \in P\backslash\{0\}$) implies $a_1\cdots a_{i-1}a_{i+1}\cdots a_n \in P$, for some $i \in \{1,\ldots,n\}$; $(m,n \geq 2)$.

In this paper several results concerning $(n-1,n)$-Φ_m-prime and $(n-1,n)$-weakly prime ideals are proved. We show that in a Noetherian domain a Φ_m-prime ideal is primary and we show that in some well known rings $(n-1,n)$-Φ_m-prime ideals and $(n-1,n)$-prime ideals coincide.

Keywords: Quasi-local ring, prime ideal, almost prime ideal, $(n-1,n)$-weakly prime ideal, $(n-1,n)$-Φ_m-prime ideal.
MSC(2010): Primary: 13A15.

1. Introduction

Let R be a commutative ring with $1 \neq 0$. We recall that a prime ideal P of R is a proper ideal with the property that for $a,b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$. Prime ideals play a central role in commutative ring theory.

Authors [1], defined the notion of a weakly prime ideal, i.e., a proper ideal P of R with the property that for $a,b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. So a prime ideal is weakly prime. However, since 0 is

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always weakly prime (by definition), a weakly prime ideal need not be prime.

The notion of a weakly prime element (i.e., an element $p \in R$ such that (p) is a weakly prime ideal) was introduced by Galovich [9] in his study of unique factorization rings with zero divisors. He called such elements "prime". It is hoped that weakly prime elements and weakly prime ideals will prove useful in the study of commutative rings with zero divisors and in particular, factorization in such rings.

The concept of an 2-absorbing ideal (i.e., a proper ideal P of R with the property that for $a, b, c \in R$, $abc \in P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$) was introduced and investigated by Badawi in [5]. He also introduced the concept of a n-absorbing ideal (i.e., a proper ideal P of R with the property that for $a_1, \ldots, a_{n+1} \in R$, $a_1 \cdots a_{n+1} \in P$ implies $a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n+1} \in P$ for some $i \in \{1, \ldots, n+1\}$). Thus a 1-absorbing ideal is just a prime ideal.

Recently, Anderson and Badawi in [2] studied n-absorbing ideals. In this paper, if $n \geq 2$, an $(n-1)$-absorbing ideal P of R is called an $(n-1, n)$-prime. For principal ideals in an integral domain, this concept has been studied with respect to nonunique factorization in [4].

Badawi and Darani in [6] introduced the concept of a weakly 2-absorbing ideal, i.e., a proper ideal P of R with the property that for $a, b, c \in R$, $0 \neq abc \in P$ implies $ab \in P$ or $ac \in P$ or $bc \in P$. In [8], for $n \geq 2$, we defined a proper ideal P of R to be $(n-1, n)$-weakly prime if $a_1, \cdots, a_n \in R$, $0 \neq a_1 \cdots a_n \in P$ implies $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in P$ for some $i \in \{1, \ldots, n\}$. So a $(1, 2)$-weakly prime ideal is just weakly prime. For example every proper ideal of a quasi-local ring (R, M) with $M^n = 0$ is $(n-1, n)$-weakly prime.

In studying unique factorization domains, Bhatwadekar and Sharma [7] defined the notion of almost prime ideals, i.e., proper ideals P of R with the property that if for $a, b \in R$, $ab \in P \backslash P^2$, then $a \in P$ or $b \in P$. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is also almost prime.

Anderson and Bataineh in [3] extended the concepts of prime, weakly prime and almost prime ideals to Φ-prime ideals as follows:

Let R be a commutative ring and $S(R)$ be the set of all ideals of R. Let $\Phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. Then a proper ideal P of R is called Φ-prime if for $a, b \in R$, $ab \in P \backslash \Phi(P)$ implies $a \in P$ or $b \in P$. They defined $\Phi_m : S(R) \to S(R) \cup \{\emptyset\}$ with $\Phi_m(J) = J^m$, for all $J \in S(R)$ $(m \geq 2)$.

A proper ideal P of R is $(n-1,n)$-Φ_m-prime if $a_1,\ldots,a_n \in R$, $a_1\cdots a_n \in P\backslash P^m$ implies $a_1\cdots a_{i-1}a_{i+1}\cdots a_n \in P$, for some $i \in \{1,\ldots,n\}$ $(m,n \geq 2)$. [8]

So a $(1,2)$-Φ_2-prime ideal is just almost prime. We shall call $(1,2)$-Φ_m-prime ideals "Φ_m-prime".

In this paper we study $(n-1,n)$-weakly prime and $(n-1,n)$-Φ_m-prime ideals, which are generalizations of weakly prime and almost prime ideals, respectively $(n,m \geq 2)$.

In Theorem 2.7, we show that in a Noetherian domain a Φ_n-prime ideal is primary $(n \geq 2)$. Furthermore, in Theorem 2.7, we show that in a one-dimensional quasi-local domain R if every proper principal ideal of R is a product of Φ_n-prime ideals, then R is a DVR $(n \geq 2)$. Then in Theorem 2.8, we show that in a valuation domain the concepts of $(n-1,n)$-Φ_n-prime and $(n-1,n)$-prime coincide $(n \geq 2)$.

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring and I be an ideal of R. Then rad I will denote the radical of I. We say that I is a radical ideal of R if rad $I = I$.

An integral domain R is said to be a valuation domain if $x|y$ or $y|x$, for every nonzero $x, y \in R$.

We denote the total quotient ring of R by $T(R)$. If I is a nonzero ideal of a ring R, then $I^{-1} = \{x \in T(R) | xI \subseteq R\}$. An integral domain R is called a *Prüfer* domain if $II^{-1} = R$, for every nonzero finitely generated ideal I of R.

Some of our results use the $R(+)M$ construction. Let R be a ring and M be an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1,0)$ under addition defined by $(r,m)+(s,n) = (r+s, m+n)$ and multiplication defined by $(r,m)(s,n) = (rs, rn+sm)$.

2. Results

Example 2.1. *Let (R,M) be a quasi-local ring and P be a proper ideal of R. If $P \cap M^n = 0$, then P is an $(n-1,n)$-weakly prime ideal of R. For if $0 \neq a_1\cdots a_n \in P$, then $a_1\cdots a_n \notin M^n$. So there exists an $i \in \{1,\ldots,n\}$ such that a_i is a unit of R. So $a_1\cdots a_{i-1}a_{i+1}\cdots a_n \in P$. Similarly, if $P \cap M^n \subseteq P^m$, then P is a $(n-1,n)$-Φ_m-prime ideal of R $(m,n \geq 2)$.*

Example 2.2. *Let $R = \dfrac{k[|x,y|]}{(x)(x,y)}$, where k is a field and suppose $M = (\overline{x},\overline{y})$ is the unique maximal ideal of R. Let $P = (\overline{x})$. Then $P \cap M^n = 0$*

$n \geq 2$. *Therefore, P is $(n-1,n)$-weakly prime and so $(n-1,n)$-Φ_m-prime by Example 1 ($m \geq 2$).*

Now we show that in a Noetherian domain a Φ_n-prime ideal is primary ($n \geq 2$).

Lemma 2.3. *Let R be an integral domain and J, K be two nonzero finitely generated ideals of R such that $K \nsubseteq J$ and $J \nsubseteq K$. If $J \cap K$ is Φ_n-prime, then the radical of J equals the radical of K ($n \geq 2$).*

Proof. We show that $K \subseteq \text{rad } J$. If not, then there exists a prime ideal P of R minimal over J such that $K \nsubseteq P$. Choose an element $x \in K \backslash P$. Clearly, for $J^{(n)} = J^n R_P \cap R$, $J^n R_P = J^{(n)} R_P$. Since J is finitely generated, $JR_P \neq J^n R_P$. Thus $JR_P \neq J^{(n)} R_P$, and consequently $J \nsubseteq J^{(n)}$. Since $J \nsubseteq K$ and $J \nsubseteq J^{(n)}$, we have $J \nsubseteq K \cup J^{(n)}$. Choose an element $y \in J \backslash (K \cup J^{(n)})$. Then $x, y \notin J \cap K$, but $xy \in J \cap K$. We claim that $xy \notin (J \cap K)^n$. Since otherwise $xy \in J^n \subseteq J^{(n)}$. However, $x \in R \backslash P$ and $xy \in J^{(n)}$ implies $y \in J^{(n)}$, which is a contradiction. So we have $xy \in J \cap K \backslash (J \cap K)^n$ and $x, y \notin J \cap K$. This contradicts the fact that $J \cap K$ is Φ_n-prime.

Thus $K \subseteq \text{rad } J$. Similarly, $radJ \subseteq \text{rad } K$. So we have $radK = \text{rad } J$. □

Theorem 2.4. *Let R be a Noetherian domain and P be a Φ_n-prime ideal of R. Then P is primary ($n \geq 2$).*

Proof. If P is not primary, then every minimal primary decomposition of P must have at least two components. Take a minimal primary decomposition of P and let Q be a primary component of this decomposition. If K is the intersection of all other primary components in the decomposition, then $P = Q \cap K$, where $Q \nsubseteq K$ and $K \nsubseteq Q$ and $radQ \neq \text{rad } K$. So by Lemma 2.3, we have a contradiction. □

Next we show that in an integral domain for a proper principal ideal the concepts $(n-1,n)$-Φ_m-prime and $(n-1,n)$-prime are the same ($m, n \geq 2$).

Lemma 2.5. *Let c be a nonzero nounit element in an integral domain R. Let $m, n \geq 2$ be two integers. If (c) is not a $(n-1,n)$-prime ideal, then there exist $a_1, \ldots, a_n \in R$ such that $a_1 \cdots a_{i-1} a_{i+1} \cdots a_n \notin (c)$, for all $i \in \{1, \ldots, n\}$ and $a_1 \cdots a_n \in (c)$ but $a_1 \cdots a_n \notin (c^m)$.*

Proof. Since (c) is not $(n-1,n)$-prime, there exist $a_1, \ldots, a_n \in R$ with $a_1 \cdots a_n \in (c)$ but $a_1 \cdots a_{i-1} a_{i+1} \cdots a_n \notin (c)$, for $i \in \{1, \ldots, n\}$. If

$a_1 \cdots a_n \notin (c^m)$, we are done. So we assume that $a_1 \cdots a_n \in (c^m)$. Let $a'_n = a_n + c^{m-1}$.

Then $a_1 \cdots a_{n-1}a'_n \in (c)$ and $a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n-1}a'_n \notin (c)$ for all $i \in \{1, \ldots, n-1\}$ and $a_1 \cdots a_{n-1} \notin (c)$. If $a_1 \cdots a_{n-1}a'_n \in (c^m)$, then $a_1 \cdots , a_{n-1}c^{m-1} \in (c^m)$. So there exists $r \in R$ such that $a_1 \cdots a_{n-1}c^{m-1} = rc^m$ and so $a_1 \cdots a_{n-1} \in (c)$, which is a contradiction. Therefore, $a_1 \cdots a_{n-1}a'_n \notin (c^m)$. □

Corollary 2.6. *Let R be an integral domain and c an element of R. Then (c) is an $(n-1,n)$-Φ_m-prime ideal of R if and only if (c) is $(n-1,n)$-prime $(n, m \geq 2)$.*

Proof. ($\Leftarrow$) If (c) is an $(n-1,n)$-prime ideal of R, then it is clear that (c) is $(n-1,n)$-Φ_m-prime.

($\Rightarrow$) If $c = 0$, then (c) is a prime ideal of R and so (c) is an $(n-1,n)$-prime ideal. So we assume that $c \neq 0$. If (c) is not $(n-1,n)$-prime, then there exist $a_1, \ldots, a_n \in (c)\backslash(c^m)$ such that $a_1 \ldots a_{i-1}a_{i+1} \cdots a_n \notin (c)$ for all $i \in \{1, \ldots, n\}$, by Lemma 2.5, which is a contradiction □

Theorem 2.7. *Let R be a quasi-local domain and $dim R = 1$. If every proper principal ideal of R is a product of Φ_n-prime ideals, then R is a PID $(n \geq 2)$.*

Proof. Let I be a proper principal ideal which is a product of Φ_n-prime ideals. Now in a domain R every nonzero principal ideal is invertible and a factor of an invertible ideal is also invertible. Since R is quasi-local, an invertible ideal is principal. So I is a product of principal Φ_n-prime ideals. So by Corollary 2.6, I is a product of principal prime ideals. Thus R is a UFD. But a one-dimensional quasi-local UFD is a PID. □

Next we show that in a valuation domain the concepts $(n-1,n)$-Φ_n-prime and $(n-1,n)$-prime are the same $(n \geq 2)$.

Theorem 2.8. *Let V be a valuation domain. Then an ideal P of V is $(n-1,n)$-Φ_n-prime if and only if it is $(n-1,n)$-prime $(n \geq 2)$.*

Proof. ($\Rightarrow$) Let $a_1, \cdots , a_n \in R$ and $a_1 \cdots a_n \in P$.

Assume $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \notin P$, for all $i \in \{1, \ldots, n\}$. So $(a_i) \nsubseteq P$, for all $i \in \{1, \ldots, n\}$. Since V is a valuation domain, we have $P \subseteq (a_i)$ for all $i \in \{1, \ldots, n\}$. So $P^n \subseteq (a_1 \cdots a_n)$. If $P^n \neq (a_1 \cdots a_n)$ then $a_1 \cdots a_n \in P\backslash P^n$. Since P is $(n-1,n)$-Φ_n-prime, this implies that $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in P$ for some $i \in \{1, \ldots, n\}$, which is a contradiction. So we have $(a_1 \cdots a_n) = P^n$. Then P being a factor of a principal ideal is principal. Thus by Corollary 2.6, P is $(n-1,n)$-prime.

($\Leftarrow$) This holds for any ring. $\square$

The next example shows that in a $Prüfer$ domain R, the above result is not necessarily true.

Example 2.9. *Let R be the ring of all algebraic integers. Then every radical ideal of R is idempotent. So let $M \neq N$ be two maximal ideals of R. Then $MN = M \cap N$ and $(MN)^2 = MN$. So MN is $(n-1,n)$-Φ_n-prime but not a prime ideal ($n \geq 2$).*

Let R be a commutative ring. Two elements $a, b \in R$ are associates, denoted $a \sim b$, if $a|b$ and $b|a$. A nonzero nounit $a \in R$ is n-irreducible if $a = a_1 \cdots a_n$ implies $a \sim a_1 \cdots a_{i-1}a_{i+1} \cdots a_n$ for some $i \in \{1, \ldots, n\}$ ($n \geq 2$).

Theorem 2.10. *Let R be a ring and P be a proper ideal of R. Suppose that every nonzero element of P is n-irreducible. Then P is $(n-1,n)$-weakly prime and hence $(n-1,n)$-Φ_m-prime ($n, m \geq 2$).*

Proof. Let $a_1 \cdots a_n \in P \setminus \{0\}$, where $a_1, \ldots, a_n \in R$. So $a_1 \cdots a_n$ is n-irreducible and hence $(a_1 \cdots a_n) = (a_1 \cdots a_{i-1}a_{i+1} \cdots a_n)$ for some $i \in \{1, \ldots, n\}$. So $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in P$. $\square$

Corollary 2.11. *Let (R, M) be a quasi-local ring and $x \in M$. If x is n-irreducible and $xM = 0$, then (x) is $(n-1,n)$-weakly prime ($n \geq 2$).*

Proof. Suppose that x is n-irreducible and $xM = 0$. Then every nonzero element of (x) is an associate of x and hence n-irreducible. So, by Theorem 2.10, (x) is $(n-1,n)$-weakly prime. $\square$

Let R be a commutative ring and A be an R-module and $D = R(+)A$. We observe that for an ideal I of R and a positive integer $n \geq 1$, $(I(+)A)^n = I^n(+)I^{n-1}A$.

Theorem 2.12. *Let R be a ring and A be an R-module. Let P be a proper ideal of R. Then $P' = P(+)A$ is an $(n-1,n)$-Φ_n-prime ideal of $D = R(+)A$ if and only if P is $(n-1,n)$-Φ_n-prime and for all $i \in \{1, \ldots, n\}$, $a_1, \ldots, a_n \in R$ and $m_1, \ldots, m_n \in A$, with $a_1 \cdots a_n \in P^n$ and $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \notin P$, the second component of $(a_1, m_1) \cdots (a_n, m_n)$ is in $P^{n-1}A$; ($n \geq 2$).*

Proof. ($\Leftarrow$) Suppose that $(a_1, m_1) \cdots (a_n, m_n) \in (P(+)A) \backslash (P(+)A)^n$. So $(a_1, m_1) \cdots (a_n, m_n) \in (P(+)A) \backslash (P^n(+)P^{n-1}A)$.

So $a_1 \cdots a_n \in P$. If $a_1 \cdots a_n \notin P^n$, then $(n-1,n)$-Φ_n-primeness of P gives $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in P$ for some $i \in \{1, \ldots, n\}$. Hence

$$(a_1, m_1) \cdots (a_{i-1}, m_{i-1})(a_{i+1}, m_{i+1}) \cdots (a_n, m_n) \in P'.$$

Now assume that $a_1 \cdots a_n \in P^n$.

If $a_1 \ldots a_{i-1}a_{i+1} \cdots a_n \notin P$ for all $i \in \{1, \ldots, n\}$, then by hypothesis the second component of $(a_1, m_1) \cdots (a_n, m_n)$ is in $P^{n-1}A$.

So $(a_1, m_1) \cdots (a_n, m_n) \in P^n(+)P^{n-1}A = (P(+)A)^n$, a contradiction.

($\Rightarrow$) Suppose that $P' = P(+)A$ is $(n-1,n)$-Φ_n-prime. Let $a_1 \cdots a_n \in P \backslash P^n$, where $a_1, \ldots, a_n \in R$. Then

$$(a_1, 0) \cdots (a_n, 0) = (a_1 \cdots a_n, 0) \in (P(+)A) \backslash (P(+)A)^n.$$

So $(a_1 \cdots a_{i-1}a_{i+1} \cdots a_n, 0) \in P(+)A$, for some $i \in \{1, \ldots, n\}$. Hence $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in P$. Thus P is $(n-1,n)$-Φ_n-prime.

Next suppose that $a_1 \cdots a_n \in P^n$, and $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \notin P$ for all $i \in \{1, \ldots, n\}$ and that the second component of $(a_1, m_1) \cdots (a_n, m_n)$ is not in $P^{n-1}A$. Then

$$(a_1, m_1) \cdots (a_{i-1}, m_{i-1})(a_{i+1}, m_{i+1}) \cdots (a_n, m_n) \notin P',$$

for all $i \in \{1, \ldots, n\}$ and $(a_1, m_1) \cdots (a_n, m_n) \in P' \backslash P'^n$, a contradiction. □

Corollary 2.13. *Let R be a ring and P be a proper ideal of R such that $PA = A$. Then $P(+)A$ is an $(n-1,n)$-Φ_n-prime ideal of $R(+)A$ if and only if P is an $(n-1,n)$-Φ_n-prime ideal of R ($n \geq 2$).*

Proof. Since $PA = A$ we have $P^{n-1}A = A$. So by the Theorem 2.12 we have the corollary. □

Let R be a ring and A be an R-module. We know from [10, Theorem 25.1 (3)] that every prime ideal of $D = R(+)A$ is of the form $P(+)A$, for some prime ideal P of R.

Next we show that unlike the case for prime ideals, an $(n-1,n)$-weakly prime or $(n-1,n)$-Φ_m-prime ideal of $D = R(+)A$ need not have the form $I(+)A$ ($n \geq 2$).

Example 2.14. *If F is a field and V is a F-vector space, then $D = F(+)V$ is a quasi-local ring with unique maximal ideal $M = 0(+)V$ and $M^2 = 0$.*

If V' is a proper subspace of V, then $0(+)V'$ is $(n-1,n)$-weakly prime and so $(n-1,n)$-Φ_m-prime ideal of D ($n, m \geq 2$).

If P is an $(n-1,n)$-prime ideal of a ring R, then there are $n-1$ almost prime ideals of R that are minimal over P (see[2, Theorem 2.5]).

In Theorem 2.15 we show that for every $m \geq 2$, there is a ring R and a nonzero $(n-1,n)$-weakly prime ideal P of R such that there are exactly m prime ideals of R that are minimal over P. Also, we show that there exists a ring R and a nonzero $(n-1,n)$-weakly prime ideal P of R such that P is not $(n-1,n)$-prime $(n \geq 2)$.

Theorem 2.15. *Let $m \geq 2$ be an integer. Then there is a ring R and a nonzero $(n-1,n)$-weakly prime ideal P of R such that there are exactly m prime ideals of R that are minimal over P; $(n \geq 2)$.*

Proof. Let $m \geq 2$ and $R = \mathbf{Z}_8 \times \cdots \times \mathbf{Z}_8$ (m times). Let $A = \{0,4\}$ be an ideal of $\mathbf{Z}_8$. For every $x = (a_1, \ldots, a_m) \in R$, define $xA = a_1A$. Then A is an R-module.

Now consider the ring $D = R(+)A$ and $P = \{(0,\ldots,0)\}(+)A$. It is easy to show that P is nonzero weakly prime and so an $(n-1,n)$-weakly prime ideal of D $(n \geq 2)$.

Since every prime ideal of D is of the form $P'(+)A$ for some prime ideal P' of R, by [10, Theorem 25.1 (3)], we conclude that there are exactly m prime ideals of D that are minimal over P. $\square$

Let R be a ring and S be a subset of R. S is a multiplicatively closed subset of R if $a, b \in S$ implies $ab \in S$, where $a, b \in R$.

We say that S is an $(n-1,n)$-weakly multiplicatively closed subset of R if $a_1, \ldots, a_n \in R$ and $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in S$, for all $i \in \{1, \ldots, n\}$ imply $a_1 \cdots a_n \in S \cup \{0\}$.

Moreover, we say that S is an $(n-1,n)$-Φ_m-multiplicatively closed subset of R if $a_1, \ldots, a_n \in R$ and $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in S$ for all $i \in \{1, \ldots, n\}$ imply $a_1 \cdots a_n \in S \cup (R \backslash S)^m$ $(n, m \geq 2)$.

It is well-known that P is a prime ideal of R if and only if $R \setminus P$ is a multiplicatively closed subset of R. In the next theorems we show that this result is true for $(n-1,n)$-weakly prime ($(n-1,n)-\phi_m$-prime) ideals and $(n-1,n)$-weakly multiplicatively closed ($(n-1,n)-\phi_m$-multiplicatively closed) subsets of R.

Theorem 2.16. *Let R be a ring and P be a proper ideal of R. P is an $(n-1,n)$-weakly prime ideal of R if and only if $R \backslash P$ is an $(n-1,n)$-weakly multiplicatively closed subset of R $(n \geq 2)$.*

Proof. $(\Rightarrow)$ Let P be a $(n-1,n)$-weakly prime ideal of R and $a_1, \ldots, a_n \in R$ with $a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in R \backslash P$ for all $i \in \{1, \ldots, n\}$. If $a_1, \ldots, a_n \in$

$R\backslash P$, then we are done. So assume that $a_1,\dots,a_n\in P$. If $a_1\cdots a_n\neq 0$, then we have $a_1\cdots a_{i-1}a_{i+1}\cdots a_n\in P$ for some $i\in\{1,\dots,n\}$, a contradiction. Thus $a_1\cdots a_n\in R\backslash P\cup\{0\}$.

Therefore, $R\backslash P$ is an $(n-1,n)$-weakly multiplicatively closed subset of R.

($\Leftarrow$) Let $R\backslash P$ be an $(n-1,n)$-weakly multiplicatively closed subset of R and $a_1,\dots,a_n\in R$ with $a_1\cdots a_n\in P\backslash\{0\}$. If $a_1\cdots a_{i-1}a_{i+1}\cdots a_n\notin P$ for all $i\in\{1,\dots,n\}$, then $a_1\cdots a_n\in(R\backslash P)\cup\{0\}$, which is a contradiction. So there exists an $i\in\{1,\dots,n\}$ such that $a_1\cdots a_{i-1}a_{i+1}\cdots a_n\in P$. $\square$

Theorem 2.17. *Let R be a ring and P be a proper ideal of R. P is an $(n-1,n)$-Φ_m-prime ideal of R if and only if $R\backslash P$ is an $(n-1,n)$-Φ_m-multiplicatively closed subset of R ($n,m\geq 2$).*

Proof. ($\Rightarrow$) Let P be an $(n-1,n)$-Φ_m-prime ideal of R and $a_1,\dots,a_n\in R$ with $a_1\cdots a_{i-1}a_{i+1}\cdots a_n\in R\backslash P$, for all $i\in\{1,\dots,n\}$. If $a_1\cdots a_n\in R\backslash P$, then we are done. So assume that $a_1\cdots a_n\in P$. If $a_1\cdots a_n\notin P^m$, then we have $a_1\cdots a_{i-1}a_{i+1}\cdots a_n\in P$ for some $i\in\{1,\dots,n\}$, which is a contradiction.

So $a_1\cdots a_n\in(R\backslash P)\cup P^m$. Therefore, $R\backslash P$ is an $(n-1,n)$-Φ_m-multiplicatively closed subset of R.

($\Leftarrow$) Let $R\backslash P$ be an $(n-1,n)$-Φ_m-multiplicatively closed subset of R and $a_1,\dots,a_n\in R$ with $a_1\cdots a_n\in P\backslash P^m$. If $a_1\cdots a_{i-1}a_{i+1}\cdots a_n\notin P$ for all $i\in\{1,\dots,n\}$, then $a_1\cdots a_n\in(R\backslash P)\cup P^m$, which is a contradiction. So there exists an $i\in\{1,\dots,n\}$ such that $a_1\cdots a_{i-1}a_{i+1}\cdots a_n\in P$. Thus P is an $(n-1,n)$-Φ_m-prime ideal of R. $\square$

A well-known result of Krull states that if S is a multiplicatively closed subset of R and I is an ideal of R maximal with respect to $I\cap S=\emptyset$, then I is an prime ideal of R.

A similar result does not hold for $(n-1,n)$-weakly multiplicatively closed and $(n-1,n)$-Φ_n-multiplicatively closed sets ($n\geq 2$).

Example 2.18. *Let $R=\dfrac{\boldsymbol{Z}}{2^{n+2}\boldsymbol{Z}}$, $I=\dfrac{2^{n+1}\boldsymbol{Z}}{2^{n+2}\boldsymbol{Z}}$ and $S=\{\overline{1},\overline{2}^n\}$. Then S is both an $(n-1,n)$-weakly multiplicatively closed and an $(n-1,n)$-Φ_n-multiplicatively closed subset of R and I is maximal with respect to $I\cap S=\emptyset$, but I is neither $(n-1,n)$-Φ_n-prime nor $(n-1,n)$-weakly prime since $\overline{2}^{n-1}\times\overline{4}\in I\backslash I^n$ but $\overline{2}^{n-1}\notin I$ and $\overline{2}^n\notin I$ ($n\geq 2$).*

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