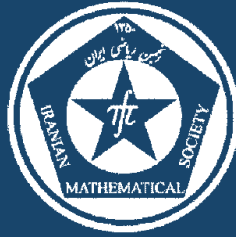


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## A FULL NESTEROV-TODD STEP INFEASIBLE INTERIOR-POINT ALGORITHM FOR SYMMETRIC CONE LINEAR COMPLEMENTARITY PROBLEM

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**ABSTRACT.** A full Nesterov-Todd (NT) step infeasible interior-point algorithm is proposed for solving monotone linear complementarity problems over symmetric cones by using Euclidean Jordan algebra. Two types of full NT-steps are used, feasibility steps and centering steps. The algorithm starts from strictly feasible iterates of a perturbed problem, and, using the central path and feasibility steps, finds strictly feasible iterates for the next perturbed problem. By using centering steps for the new perturbed problem, strictly feasible iterates are obtained to be close enough to the central path of the new perturbed problem. The starting point depends on two positive numbers  $\rho_p$  and  $\rho_d$ . The algorithm terminates either by finding an  $\epsilon$ -solution or detecting that the symmetric cone linear complementarity problem has no optimal solution with vanishing duality gap satisfying a condition in terms of  $\rho_p$  and  $\rho_d$ . The iteration bound coincides with the best known bound for infeasible interior-point methods.

**Keywords:** Monotone linear complementarity problem, interior-point algorithms, Euclidean Jordan algebra.

**MSC(2010):** Primary: 90C33; Secondary: 90C51.

### 1. Introduction

Let  $V$  be an  $n$ -dimensional vector space over  $R$ .  $(V, \circ, \langle \cdot, \cdot \rangle)$  is called an  $n$ -dimensional Euclidean Jordan algebra over  $R$  with rank  $r$  if there

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exists a bilinear map  $(x, s) \rightarrow x \circ s$  from  $V \times V$  and an inner product  $\langle x, s \rangle$  which is associative. Let  $\mathcal{K}$  be a symmetric cone such that  $\mathcal{K} := \{x^2 : x \in V\}$ , the so-called symmetric cone of squares  $\mathcal{K}$ . Here, we are concerned with monotone symmetric cone linear complementarity problem (SCLCP) in the standard form: Given an  $n$ -dimensional Euclidean Jordan algebra  $(V, \circ, \langle \cdot, \cdot \rangle)$  and its associated symmetric cone of squares  $\mathcal{K}$ , find  $(x, s) \in \mathcal{K} \times \mathcal{K}$  such that

$$s = Mx + q, \quad x \circ s = 0, \quad (\text{SCLCP})$$

where  $M \in R^{n \times n}$  and  $q \in R^n$  are given data.

It is well known that SCLCP is an NP-hard problem for a general matrix  $M$ , even in the case when  $\mathcal{K} = R_+^n$  (see [8]). Thus, we need to restrict ourselves to classes of matrices for which polynomial interior-point methods (IPMs) exist. The class of monotone matrices appears most frequently. The monotone property can be formulated as follows:

$$(1.1) \quad v = Mu \Rightarrow \langle u, v \rangle \geq 0.$$

This is equivalent to the fact that matrix  $M$  is positive semidefinite with respect to the inner product  $\langle \cdot, \cdot \rangle$  in  $(V, \circ)$ . We make the following assumption in developing our results.

**Assumption 1.** The interior-point condition (IPC), i.e., there exist  $x, s \in \text{int}\mathcal{K}$ ,  $\text{int}\mathcal{K}$  denotes the interior of  $\mathcal{K}$ , such that  $s = Mx + q$  holds for the SCLCP.

Although SCLCP is not an optimization problem, it is closely related to one. One reason is that optimality conditions of several important optimization problems can be formulated in the form of SCLCP. Nesterov and Todd [11] provided a theoretical foundation for efficient primal-dual IPMs on a special class of convex optimization, where the associated cone was self-scaled. Later on, it was observed that the self-scaled cones were precisely symmetric cones [1]. The application of the Euclidean Jordan algebra as a basic tool for analyzing complexity proofs of the IPMs for symmetric cone linear optimization (SCLO) and SCLCP was started by Faybusovich [2], who extended earlier works of Nesterov and Todd, and Kojima et al. [11, 9]. Later, Tsuchiya [18] also used Jordan algebraic techniques to analyze primal-dual IPMs for linear second-order cone optimization. Subsequently, Schmieta and Alizadeh [15, 16] studied primal-dual IPMs for symmetric cone linear optimization extensively under the framework of Euclidean Jordan algebra. In addition to Faybusovich's results [2, 3], Rangarajan [12] proposed the first infeasible interior-point method (IIPM) for SCLCP. Recently, Kheirfam

and Mahdavi-Amiri [7] also introduced a new interior-point algorithm for SCLCP by modifying the NT-step. The authors proved that the algorithm stops after at most  $O(\sqrt{r} \log \frac{r\mu^0}{\epsilon})$  iterations, being in accord with the best existing bound for IPMs. Roos [13] proposed a new IIPM for linear optimization (LO). It differs from the classical IIPMs [8] in that the new method uses only full steps which has the advantage that no line searches are needed. Gu et al. [5] extended the full-Newton step IIPM for LO to full Nesterov-Todd step (NT-step) IIPM for SCLO by using Jordan algebra.

Here, we consider a generalization of full NT-step IIPMs to SCLCP by using Euclidean Jordan algebra. The remainder of our work is organized as follows: In Section 2, we briefly recall some properties of symmetric cones and their associated Euclidean Jordan algebra. We review the notions of central path, search directions and NT-steps, where a unified proof of the quadratic convergence is given in the framework of Euclidean Jordan algebra. In Section 3, we present our full NT-step IIPM for SCLCP. The iteration bound coincides with the best known iteration bound for IIPMs.

## 2. Preliminaries

**2.1. Euclidean Jordan Algebra.** Here, we outline some needed main results on Euclidean Jordan algebra and symmetric cones. For a comprehensive study, the reader is referred to [1, 4, 19].

A Jordan algebra  $V$  is a finite dimensional vector space endowed with a bilinear map  $\circ : V \times V \rightarrow V$  satisfying the following properties for all  $x, y \in V$ :

- $x \circ y = y \circ x$ ,
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ , where  $x^2 = x \circ x$ .

Moreover, a Jordan algebra  $(V, \circ)$  is called Euclidean if there exists an inner product, denoted by " $\langle \cdot, \cdot \rangle$ ", such that

$$\langle x \circ y, z \rangle = \langle x, y \circ z \rangle,$$

for all  $x, y, z \in V$ .

A Jordan algebra has an identity element, if there exists a unique element  $e \in V$  such that  $x \circ e = e \circ x = x$ , for all  $x \in V$ . Throughout the paper, we assume that  $V$  is a Euclidean Jordan algebra with an identity element  $e$ . The set  $\mathcal{K} = \{x^2 : x \in V\}$  is called the cone of squares of Euclidean Jordan algebra  $(V, \circ, \langle \cdot, \cdot \rangle)$ . A cone is symmetric if and only if it is the cone of squares of some Euclidean Jordan algebra.

An element  $c \in V$  is idempotent if  $c \circ c = c$ . Idempotents  $x$  and  $y$  are orthogonal if  $x \circ y = 0$ . An idempotent  $c$  is primitive if it is nonzero and can not be expressed by sum of two other nonzero idempotents. A set of primitive idempotents  $\{c_1, c_2, \dots, c_k\}$  is called a Jordan frame if  $c_i \circ c_j = 0$ , for any  $i \neq j \in \{1, 2, \dots, k\}$  and  $\sum_{i=1}^k c_i = e$ . For any  $x \in V$ , let  $r$  be the smallest positive integer such that  $\{e, x, x^2, \dots, x^r\}$  is linearly dependent;  $r$  is called the degree of  $x$  and is denoted by  $\deg(x)$ . The rank of  $V$ , denoted by  $\text{rank}(V)$ , is defined as the maximum of  $\deg(x)$  over all  $x \in V$ . The importance of Jordan frame comes from the fact that any element of Euclidean Jordan algebra can be represented using some Jordan frame, as explained more precisely in the following spectral decomposition theorem.

**Theorem 2.1.** (Theorem III.1.2 in [1]) *Let  $(V, \circ, \langle \cdot, \cdot \rangle)$  be a Euclidean Jordan algebra with  $\text{rank}(V) = r$ . Then, for any  $x \in V$ , there exists a Jordan frame  $\{c_1, c_2, \dots, c_r\}$  and real numbers  $\lambda_1(x), \lambda_2(x), \dots, \lambda_r(x)$  such that*

$$x = \sum_{i=1}^r \lambda_i(x) c_i,$$

where the  $\lambda_i$ 's are the eigenvalues of  $x$ . The numbers  $\lambda_i(x)$  (with their multiplicities) are uniquely determined by  $x$ . Furthermore,

$$\text{Tr}(x) = \sum_{i=1}^r \lambda_i(x) \quad \text{and} \quad \det(x) = \prod_{i=1}^r \lambda_i(x),$$

where  $\text{Tr}(\cdot)$  and  $\det(\cdot)$  stand for the trace and determinant, respectively.

Since “ $\circ$ ” is a bilinear map, for every  $x \in V$ , there exists a matrix  $L(x)$  such that for every  $y \in V$ ,  $x \circ y = L(x)y$ . Moreover, we define

$$P(x) := 2L(x)^2 - L(x^2),$$

where  $L(x)^2 = L(x)L(x)$ . The map  $P(x)$  is called the quadratic representation of  $V$ , which is an essential concept in the theory of Jordan algebra and plays an important role in the analysis of interior-point algorithms. An element  $x \in V$  is called invertible if there exists a  $y = \sum_{i=0}^m \alpha_i x^i$  for some finite  $m < \infty$  and real numbers  $\alpha_i$  such that  $x \circ y = y \circ x = e$ , and denoted as  $x^{-1}$ . An element  $x \in V$  is invertible if and only if  $P(x)$  is invertible. In this case,  $P(x)x^{-1} = x$  and  $P(x)^{-1} = P(x^{-1})$ .

Let  $x = \sum_{i=1}^r \lambda_i(x) c_i$  be the spectral decomposition of  $x$ . It is possible to extend the definition of any real valued continuous function  $f(\cdot)$  to

elements of Jordan algebra via their eigenvalues, i.e.,  $F : V \rightarrow V$  is given by

$$F(x) = \sum_{i=1}^r f(\lambda_i(x))c_i.$$

In particular, we have the square root,  $x^{\frac{1}{2}} = \sum_{i=1}^r \sqrt{\lambda_i(x)}c_i$ , wherever  $x \in \mathcal{K}$ , and undefined otherwise, the inverse,  $x^{-1} = \sum_{i=1}^r \lambda_i(x)^{-1}c_i$ , wherever  $\lambda_i \neq 0$ , for all  $i = 1, 2, \dots, r$ , and undefined otherwise.

The next lemma contains a result of crucial importance in the design of IPMs within the framework of Jordan algebra.

**Lemma 2.2.** (Lemma 2.2 in [2]) *Let  $x, s \in \mathcal{K}$ . Then,  $Tr(x \circ s) \geq 0$ , and we have  $Tr(x \circ s) = 0$  if and only if  $x \circ s = 0$ .*

For any  $x, y \in V$ ,  $x$  and  $y$  are said to be operator commutable if  $L(x)$  and  $L(y)$  commute, i.e.,  $L(x)L(y) = L(y)L(x)$ . In other words,  $x$  and  $y$  operator commutable if for all  $z \in V$ ,  $x \circ (y \circ z) = y \circ (x \circ z)$  (see [15]).

**Theorem 2.3.** (Lemma X.2.2 in [1]) *Let  $x, y \in V$ . The elements  $x$  and  $y$  operator commutable if and only if they share a Jordan frame, that is,*

$$x = \sum_{i=1}^r \lambda_i(x)c_i \quad \text{and} \quad y = \sum_{i=1}^r \lambda_i(y)c_i,$$

for Jordan frame  $\{c_1, c_2, \dots, c_r\}$ .

For any  $x, y \in V$ , we define the canonical inner product of  $x, y \in V$  as follows

$$\langle x, y \rangle = Tr(x \circ y),$$

and the Frobenius norm of  $x$  as follows

$$\|x\|_F = \sqrt{\langle x, x \rangle} = \sqrt{Tr(x^2)}.$$

It follows that

$$\|x\|_F = \sqrt{Tr(x^2)} = \sqrt{\sum_{i=1}^r \lambda_i^2(x)} = \|\lambda(x)\|.$$

Note that  $Tr(\cdot)$  is associative, and we have

$$\langle L(x)y, z \rangle = Tr((x \circ y) \circ z) = Tr((y \circ x) \circ z) = Tr(y \circ (x \circ z)) = \langle y, L(x)z \rangle,$$

showing that  $L(x)$  is a self-adjoint operator. As the definition of  $P(x)$  depends only on  $L(x)$  and  $L(x^2)$ , both of which are self-adjoint,  $P(x)$

is also self-adjoint. Let  $\lambda_{\min}(x)$  and  $\lambda_{\max}(x)$  denote the smallest and largest eigenvalue of  $x$ , respectively. Then

$$|\lambda_{\min}(x)| \leq \|x\|_F, \quad |\lambda_{\max}(x)| \leq \|x\|_F \quad \text{and} \quad |\langle x, y \rangle| \leq \|x\|_F \|y\|_F.$$

The following lemma shows the existence and uniqueness of a scaling point  $w$  corresponding to any points  $x, s \in \text{int}\mathcal{K}$  such that  $P(w)$  takes  $s$  into  $x$ . This lemma plays a fundamental role in the design of the interior-point algorithms for SCLCP.

**Lemma 2.4.** *(Lemma 3.2 in [3]) Let  $x, s \in \text{int}\mathcal{K}$ . Then, there exists a unique  $w \in \text{int}\mathcal{K}$  such that*

$$x = P(w)s.$$

Moreover,

$$w = P(x^{\frac{1}{2}}) \left( P(x^{\frac{1}{2}})s \right)^{-\frac{1}{2}} \left[ = P(s^{-\frac{1}{2}}) \left( P(s^{\frac{1}{2}})x \right)^{\frac{1}{2}} \right].$$

The point  $w$  is called the scaling point of  $x$  and  $s$ . Hence, there exists  $\tilde{v} \in \text{int}\mathcal{K}$  such that

$$\tilde{v} = P(w)^{-\frac{1}{2}}x = P(w)^{\frac{1}{2}}s,$$

which is the so-called NT-scaling of  $R^n$ . We say that two elements  $x \in V$  and  $y \in V$  are similar, as denoted by  $x \sim y$ , if and only if  $x$  and  $y$  share the same set of eigenvalues. We say  $x \in \mathcal{K}$  if and only if  $\lambda_i \geq 0$  and  $x \in \text{int}\mathcal{K}$  if and only if  $\lambda_i > 0$ , for all  $i = 1, 2, \dots, r$ . We also say  $x$  is positive semidefinite (positive definite) if  $x \in \mathcal{K}$  ( $x \in \text{int}\mathcal{K}$ ).

In what follows, we list some results regarding similarity.

**Lemma 2.5.** *(Proposition 3.2.4 in [19]) Let  $x, s \in \text{int}\mathcal{K}$ , and  $w$  be the scaling point of  $x$  and  $s$ . Then*

$$\left( P(x^{\frac{1}{2}})s \right)^{\frac{1}{2}} \sim P(w^{\frac{1}{2}})s.$$

**Lemma 2.6.** *(Proposition 21 in [15]) Let  $x, s, u \in \text{int}\mathcal{K}$ . Then*

- (i)  $P(x^{\frac{1}{2}})s \sim P(s^{\frac{1}{2}})x$ .
- (ii)  $P\left(\left(P(u)x\right)^{\frac{1}{2}}\right)P(u^{-1})s \sim P(x^{\frac{1}{2}})s$ .

To analyze our IIPM, we need some inequalities, which are recalled in the following lemmas.

**Lemma 2.7.** *(Lemma 30 in [15]) Let  $x, s \in \text{int}\mathcal{K}$ . Then*

$$\|P(x)^{\frac{1}{2}}s - e\|_F \leq \|x \circ s - e\|_F.$$

**Lemma 2.8.** (Lemma 2.9 in [12]) Given  $x \in \text{int}\mathcal{K}$ , we have

$$\|x - x^{-1}\|_F \leq \frac{\|x^2 - e\|_F}{\lambda_{\min}(x)}.$$

**Lemma 2.9.** (Lemma 2.15 in [5]) If  $x \circ s \in \text{int}\mathcal{K}$ , then  $\det(x) \neq 0$ .

**Lemma 2.10.** (Theorem 4 in [17]) Let  $x, s \in \text{int}\mathcal{K}$ . Then

$$\lambda_{\min}(P(x)^{\frac{1}{2}}s) \geq \lambda_{\min}(x \circ s).$$

**Lemma 2.11.** (Lemma 8 in [7]) Let  $x, s \in V$  with  $\text{Tr}(x \circ s) \geq 0$ . Then

$$\|x \circ s\|_F \leq \frac{1}{2\sqrt{2}}\|x + s\|_F^2.$$

**2.2. The central path.** The basic idea of IPMs is to replace the second equation in SCLCP, the so-called complementary condition for SCLCP, by the parameterized equation  $x \circ s = \mu e$ , with  $\mu > 0$ . Thus, one may consider

$$(2.1) \quad \begin{aligned} s &= Mx + q, \quad x, s \in \mathcal{K}, \\ x \circ s &= \mu e. \end{aligned}$$

For each  $\mu > 0$ , the system (2.1) has a unique solution  $(x(\mu), s(\mu))$  (under given assumptions), and we call  $(x(\mu), s(\mu))$  the  $\mu$ -center of SCLCP. The set of  $\mu$ -centers (with  $\mu$  running through all positive real numbers) gives a homotopy path, which is called the central path of SCLCP [4]. If  $\mu \rightarrow 0$ , then the limit of the central path exists, and since the limit points satisfy the complementarity condition, the limit yields a solution of SCLCP.

**2.3. The new search directions.** IPMs follow the central path approximately and find an approximate solution of SCLCP by letting  $\mu$  go to zero. At a given feasible iterate  $(x, s)$  with  $x, s \in \text{int}\mathcal{K}$ , we are to find displacements  $\Delta x$  and  $\Delta s$  such that

$$(2.2) \quad \begin{aligned} s + \Delta s &= M(x + \Delta x) + q, \\ (x + \Delta x) \circ (s + \Delta s) &= \mu e. \end{aligned}$$

Neglecting the term  $\Delta x \circ \Delta s$  corresponding to the left-hand side of the second equation, we obtain

$$(2.3) \quad \begin{aligned} -M\Delta x + \Delta s &= 0, \\ x \circ \Delta s + s \circ \Delta x &= \mu e - x \circ s. \end{aligned}$$

Due to the fact that  $x$  and  $s$  are not operator commutable in general, i.e.,  $L(x)L(s) \neq L(s)L(x)$ , this system does not always have a unique



solution. It is well known that this difficulty can be resolved by applying a scaling scheme. This is given in the following lemma.

**Lemma 2.12.** (Lemma 28 in [15]) *Let  $u \in \text{int}\mathcal{K}$ . Then*

$$x \circ s = \mu e \Leftrightarrow P(u)x \circ P(u)^{-1}s = \mu e.$$

Now, replacing the second equation in (2.2) by  $P(u)(x+\Delta x) \circ P(u)^{-1}(s+\Delta s) = \mu e$ , and applying the Newton method and neglecting the term  $P(u)\Delta x \circ P(u)^{-1}\Delta s$ , we obtain the system

$$(2.4) \quad \begin{aligned} -M\Delta x + \Delta s &= 0, \\ P(u)^{-1}s \circ P(u)\Delta x + P(u)x \circ P(u)^{-1}\Delta s &= \mu e - P(u)x \circ P(u)^{-1}s. \end{aligned}$$

Here, we focus on the scaling point  $u = w^{-\frac{1}{2}}$ , where  $w$  is the NT-scaling point of  $x$  and  $s$  as defined in Lemma 2.4. For that case we define

$$(2.5) \quad v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}} \left[ = \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}} \right],$$

and

$$(2.6) \quad \bar{M} := P(w)^{\frac{1}{2}}MP(w)^{\frac{1}{2}}, \quad d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}}, \quad d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}.$$

The Newton system (2.4) can be rewritten as

$$(2.7) \quad \begin{aligned} -\bar{M}d_x + d_s &= 0, \\ d_x + d_s &= v^{-1} - v. \end{aligned}$$

It easily follows that the above system has a unique solution, because  $\bar{M}$  is positive semidefinite:

$$\begin{aligned} \langle u, \bar{M}u \rangle &= \langle u, P(w)^{\frac{1}{2}}MP(w)^{\frac{1}{2}}u \rangle \\ &= \langle P(w)^{\frac{1}{2}}u, MP(w)^{\frac{1}{2}}u \rangle \geq 0. \end{aligned}$$

Hence, this system uniquely defines the scaled directions  $d_x$  and  $d_s$ . To get the search directions  $\Delta x$  and  $\Delta s$  in the original space, we simply transform the scaled search directions back to the  $x$ -space and  $s$ -space by using (2.6):

$$(2.8) \quad \Delta x = \sqrt{\mu}P(w)^{\frac{1}{2}}d_x, \quad \Delta s = \sqrt{\mu}P(w)^{-\frac{1}{2}}d_s.$$

The new iterate is obtained by taking a full NT-step as follows

$$(2.9) \quad x^+ := x + \Delta x, \quad s^+ := s + \Delta s.$$

**2.4. The analysis of the NT-step.** For the analysis of the NT-step, we need to measure the distance of the iterate  $(x, s)$  to the current  $\mu$ -center  $(x(\mu), s(\mu))$ . The proximity measure that we are going to use is defined as follows

$$(2.10) \quad \delta(x, s; \mu) \equiv \delta(v) := \frac{1}{2} \|v - v^{-1}\|_F,$$

where  $v$  is defined in (2.5). Let  $x, s \in \text{int}\mathcal{K}$ ,  $\mu > 0$  and let  $w$  be the NT-scaling point of  $x$  and  $s$ . Using (2.5), (2.8) and (2.9), we obtain

$$(2.11) \quad x^+ = \sqrt{\mu}P(w)^{\frac{1}{2}}(v + d_x), \quad s^+ = \sqrt{\mu}P(w)^{-\frac{1}{2}}(v + d_s).$$

Since  $P(w)^{\frac{1}{2}}$  and its inverse  $P(w)^{-\frac{1}{2}}$  are automorphisms of  $\text{int}\mathcal{K}$  (Theorem III.2.1 in [1]),  $x^+$  and  $s^+$  will belong to  $\text{int}\mathcal{K}$  if and only if  $v + d_x$  and  $v + d_s$  belong to  $\text{int}\mathcal{K}$ . Using (2.11) and the second equation of (2.7) we have

$$(2.12) \quad \begin{aligned} (v + d_x) \circ (v + d_s) &= v^2 + v \circ (d_x + d_s) + d_x \circ d_s \\ &= v^2 + v \circ (v^{-1} - v) + d_x \circ d_s \\ &= e + d_x \circ d_s. \end{aligned}$$

**Lemma 2.13.** *The iterate  $(x^+, s^+)$  is strictly feasible if  $e + d_x \circ d_s \in \text{int}\mathcal{K}$ .*

*Proof.* For  $0 \leq \alpha \leq 1$ , define  $v_x^\alpha = v + \alpha d_x$ ,  $v_s^\alpha = v + \alpha d_s$ . Then,  $v_x^0 = v$ ,  $v_s^0 = v$ ,  $v_x^1 = v + d_x$  and  $v_s^1 = v + d_s$ . From the second equation in (2.7), it follows that

$$(2.13) \quad \begin{aligned} v_x^\alpha \circ v_s^\alpha &= (v + \alpha d_x) \circ (v + \alpha d_s) = v^2 + \alpha v \circ (d_x + d_s) + \alpha^2 d_x \circ d_s \\ &= (1 - \alpha)v^2 + \alpha(e + \alpha d_x \circ d_s). \end{aligned}$$

If  $e + d_x \circ d_s \in \text{int}\mathcal{K}$ , then we have  $d_x \circ d_s \succ_{\mathcal{K}} -e$ . Substituting this into (2.13), we get

$$v_x^\alpha \circ v_s^\alpha \succ_{\mathcal{K}} (1 - \alpha)v^2 + \alpha(1 - \alpha)e \succ_{\mathcal{K}} 0.$$

By Lemma 2.9, it follows that  $\det(v_x^\alpha) \neq 0$  and  $\det(v_s^\alpha) \neq 0$ , for  $\alpha \in [0, 1]$ . Since  $\det(v_x^0) = \det(v_s^0) = \det(v) > 0$ , by continuity,  $\det(v_x^\alpha)$  and  $\det(v_s^\alpha)$  stay positive, for all  $\alpha \in [0, 1]$ . Hence, all the eigenvalues of  $v_x^1$  and  $v_s^1$  are positive. Therefore,  $v + d_x \in \text{int}\mathcal{K}$  and  $v + d_s \in \text{int}\mathcal{K}$ , completing the proof.  $\square$

**Lemma 2.14.** *Let  $x, s \in \text{int}\mathcal{K}$  and  $\mu > 0$ . Then,  $\langle x^+, s^+ \rangle \leq \mu(r + \delta^2)$ .*

*Proof.* Due to (2.11) and the second equation in (2.7), we may write

$$\begin{aligned}
\langle x^+, s^+ \rangle &= \left\langle \sqrt{\mu}P(w)^{\frac{1}{2}}(v + d_x), \sqrt{\mu}P(w)^{-\frac{1}{2}}(v + d_s) \right\rangle \\
&= \mu \langle v + d_x, v + d_s \rangle \\
&= \mu \text{Tr}(v^2) + \mu \text{Tr}(v \circ (d_x + d_s)) + \mu \text{Tr}(d_x \circ d_s) \\
&= \mu \text{Tr}(v^2) + \mu \text{Tr}(v \circ (v^{-1} - v)) + \mu \text{Tr}(d_x \circ d_s) \\
&= \mu \text{Tr}(e) + \mu \text{Tr}(d_x \circ d_s) \\
&= \mu r + \mu \text{Tr}(d_x \circ d_s).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\text{Tr}(d_x \circ d_s) &= \frac{1}{4} \left( \|d_x + d_s\|_F^2 - \|d_x - d_s\|_F^2 \right) \\
&\leq \frac{1}{4} \|d_x + d_s\|_F^2 = \frac{1}{4} \|v^{-1} - v\|^2 = \delta^2.
\end{aligned}$$

The above relations complete the proof.  $\square$

**Lemma 2.15.** *If  $\delta := \delta(x, s; \mu) < 1$ , then the full NT-step is strictly feasible and*

$$\delta(x^+, s^+; \mu) \leq \frac{\delta^2}{\sqrt{2(1 - \delta^2)}}.$$

*Proof.* Let  $w^+$  be the NT-scaling point of  $x^+$  and  $s^+$ . According to (2.5), we have

$$\begin{aligned}
v^+ &:= \frac{P(w^+)^{\frac{1}{2}}s^+}{\sqrt{\mu}} \sim \frac{\left(P(x^+)^{\frac{1}{2}}s^+\right)^{\frac{1}{2}}}{\sqrt{\mu}} \\
&= \frac{\sqrt{\mu} \left(P(P(w)^{\frac{1}{2}}(v + d_x))^{\frac{1}{2}} P(w)^{-\frac{1}{2}}(v + d_s)\right)^{\frac{1}{2}}}{\sqrt{\mu}} \\
&\sim \left(P(v + d_x)^{\frac{1}{2}}(v + d_s)\right)^{\frac{1}{2}},
\end{aligned}$$

where the similarities follow from lemmas 2.6 and 2.5, respectively. Therefore, by applying Lemma 2.8, we obtain

$$2\delta(v^+) = \|v^+ - (v^+)^{-1}\|_F \leq \frac{\|(v^+)^2 - e\|_F}{\lambda_{\min}(v^+)}.$$

Due to lemmas 2.7, 2.10, 2.11 and (2.12), we get

$$\begin{aligned} 2\delta(v^+) &\leq \frac{\|P(v + d_x)^{\frac{1}{2}}(v + d_s) - e\|_F}{\left(\lambda_{\min}(P(v + d_x)^{\frac{1}{2}}(v + d_s))\right)^{\frac{1}{2}}} \leq \frac{\|(v + d_x) \circ (v + d_s) - e\|_F}{\lambda_{\min}((v + d_x) \circ (v + d_s))^{\frac{1}{2}}} \\ &= \frac{\|d_x \circ d_s\|_F}{(1 + \lambda_{\min}(d_x \circ d_s))^{\frac{1}{2}}} \leq \frac{\frac{1}{2\sqrt{2}}\|d_x + d_s\|_F^2}{\sqrt{1 - \delta^2}} \leq \frac{\sqrt{2}\delta^2}{\sqrt{1 - \delta^2}}, \end{aligned}$$

which completes the proof.  $\square$

As a result, the following corollary readily follows which indicates that we have quadratic convergence if the iterates are sufficient close to the  $\mu$ -center.

**Corollary 2.16.** *If  $\delta(v) \leq \frac{1}{\sqrt{2}}$ , then the full NT-step is strictly feasible and  $\delta(v^+) \leq \delta(v)^2$ .*

*Proof.* From Lemma 2.15, we have

$$\delta(v^+) \leq \frac{\delta(v)^2}{\sqrt{2(1 - \delta(v)^2)}} \leq \frac{\delta(v)^2}{\sqrt{2(1 - \frac{1}{2})}} = \delta(v)^2,$$

which proves the corollary.  $\square$

### 3. Infeasible full NT-step IPM

In the case of an infeasible method, we call the pair  $(x, s)$  an  $\epsilon$ -solution of SCLCP if the norm of the residual vector  $r_q = s - Mx - q$  does not exceed  $\epsilon$ , and also  $Tr(x \circ s) \leq \epsilon$ . In what follows, we present an infeasible-start algorithm that generates an  $\epsilon$ -solution of SCLCP, if it exists, or establishes that no such solution exists.

**3.1. The perturbed problem.** We choose a pair  $(x^0, s^0) \in \text{int}\mathcal{K} \times \text{int}\mathcal{K}$  such that  $x^0 \circ s^0 = \mu^0 e$ , for a positive  $\mu^0$ . For any  $\nu$ ,  $0 < \nu \leq 1$ , we consider the perturbed problem,

$$s - Mx - q = \nu r_q^0, \quad (x, s) \in \mathcal{K} \times \mathcal{K}, \quad (\text{SCLCP}_\nu)$$

where,  $r_q^0 = s^0 - Mx^0 - q$ . Note that if  $\nu = 1$ , then  $(x^0, s^0)$  is a strictly feasible solution  $\text{SCLCP}_\nu$ . Therefore, if  $\nu = 1$ , then  $\text{SCLCP}_\nu$  satisfies the IPC.

**Theorem 3.1.** *Let SCLCP be feasible and  $0 < \nu \leq 1$ . Then, the perturbed problem  $\text{SCLCP}_\nu$  satisfies the IPC.*

*Proof.* Let  $(\bar{x}, \bar{s})$  be a feasible solution of SCLCP. Then,  $\bar{s} = M\bar{x} + q$  with  $\bar{x} \in \mathcal{K}$  and  $\bar{s} \in \mathcal{K}$ . Now, for  $0 < \nu \leq 1$ , define

$$x = (1 - \nu)\bar{x} + \nu x^0, \quad s = (1 - \nu)\bar{s} + \nu s^0.$$

Thus,

$$s = (1 - \nu)\bar{s} + \nu s^0 = (1 - \nu)M\bar{x} + \nu Mx^0 + \nu r_q^0 + q = (Mx + q) + \nu r_q^0,$$

which shows that  $(x, s)$  is a feasible solution for  $\text{SCLCP}_\nu$ . Since  $\nu > 0$ , then  $(x, s) \in \text{int}\mathcal{K} \times \text{int}\mathcal{K}$ . This proves that  $\text{SCLCP}_\nu$  satisfies the IPC.  $\square$

Let SCLCP be feasible and  $0 < \nu \leq 1$ . Then Theorem 3.1 implies that  $\text{SCLCP}_\nu$  is strictly feasible, and therefore its central path exists. This means that the system

$$(3.1) \quad s - Mx - q = \nu r_q^0, \quad (x, s) \in \mathcal{K} \times \mathcal{K},$$

$$(3.2) \quad x \circ s = \mu e,$$

has a unique solution, for any  $\mu > 0$ . We denote this solution by  $(x(\mu, \nu), s(\mu, \nu))$ . It is the  $\mu$ -center of the perturbed problem  $\text{SCLCP}_\nu$ . In what follows, the parameters  $\mu$  and  $\nu$  will always be in a one-to-one correspondence, according to  $\mu = \nu\mu^0$ . Therefore, we feel free to omit one parameter and denote  $(x(\mu, \nu), s(\mu, \nu)) = (x(\nu), s(\nu))$ . Note that, since  $x^0 \circ s^0 = \mu^0 e$ ,  $(x^0, s^0)$  is the  $\mu^0$ -center of the perturbed problem  $\text{SCLCP}_1$ . In other words,  $(x(1), s(1)) = (x^0, s^0)$ .

**3.2. A full NT-step infeasible IPM algorithm.** We just established that if  $\nu = 1$  and  $\mu = \mu^0$ , then  $(x^0, s^0)$  is the  $\mu$ -center of the problem  $\text{SCLCP}_\nu$ . This is our initial iterate. We measure proximity to the  $\mu$ -center of the perturbed problem  $\text{SCLCP}_\nu$  by the quantity  $\delta(x, s; \mu)$  as defined by (2.10). Initially, we thus have  $\delta(x, s; \mu) = 0$ . In the sequel, we assume that at the start of each iteration, just before the  $\mu$ -update,  $\delta(x, s; \mu) \leq \tau, \tau > 0$ . This certainly holds at the start of the first iteration and also  $\langle x, s \rangle = r\mu^0$ .

Now, we describe one (main) iteration of our algorithm. Suppose that for some  $\mu \in (0, \mu^0]$ , we have  $(x, s)$  satisfying the feasibility condition (3.1), for  $\mu = \nu\mu^0$ ,  $\langle x, s \rangle \leq \mu(r + \delta^2)$  and  $\delta(x, s; \mu) \leq \tau$ . Each main iteration consists of one so-called feasibility step, a  $\mu$ -update, and a few centering steps. First, we find a new point  $(x^f, s^f)$  which is feasible for the perturbed problem with  $\nu$  replaced by  $\nu^+ := (1 - \theta)\nu$ . Then,  $\mu$  is decreased to  $\mu^+ := (1 - \theta)\mu$ . Generally, there is no guarantee that  $\delta(x^f, s^f; \mu^+) \leq \tau$ . So, a limited number of centering steps are applied

to produce a new point  $(x^+, s^+)$  such that  $\langle x^+, s^+ \rangle \leq \mu^+(r + \delta^2)$  and  $\delta(x^+, s^+; \mu^+) < \tau$ , where  $\mu^+ = \nu^+ \mu^0$ .

A formal description of the algorithm is given in Algorithm 1. Recall that after each iteration the residual vector  $r_q$  and  $Tr(x \circ s)$  are reduced by the factor  $(1 - \theta)$ . The algorithm stops if the norms of the residual vector and the duality gap are less than the accuracy parameter  $\epsilon$ .

**3.3. Analysis of the feasibility step.** Here, we define and analyze the feasibility step. Suppose that we have a strictly feasible iterate  $(x, s)$  for  $SCLCP_\nu$ . This means that  $(x, s)$  satisfies

$$s - Mx - q = \nu r_q^0, \quad (x, s) \in \mathcal{K} \times \mathcal{K},$$

with  $\mu = \nu \mu^0$ . We need the displacements  $\Delta^f x$  and  $\Delta^f s$  such that

$$(3.3) \quad x^f := x + \Delta^f x, \quad s^f := s + \Delta^f s,$$

are feasible for  $SCLCP_\nu$ . One may easily verify that  $(x^f, s^f)$  satisfies  $SCLCP_\nu$ , with  $\nu$  replaced by  $\nu^+$  and  $\mu$  by  $\mu^+ = \nu^+ \mu^0 = (1 - \theta)\mu$ , only if the first equation in the following system is satisfied

$$\begin{aligned} M\Delta^f x - \Delta^f s &= \theta \nu r_q^0, \\ P(u)^{-1} s \circ P(u) \Delta^f x + P(u) x \circ P(u)^{-1} \Delta^f s \\ &= (1 - \theta) \mu e - P(u) x \circ P(u)^{-1} s. \end{aligned}$$

The second equation above is inspired by the second equation of the system (2.4) that we used to define the search directions for the feasible case, except that we target at the  $\mu^+$ -center of  $SCLCP_{\nu^+}$ . As in the feasible case, we use the NT-scaling scheme to guarantee that the above system has a unique solution. So, we take  $u = w^{-\frac{1}{2}}$ , where  $w$  is the NT-scaling point of  $x$  and  $s$ . Then, the above system turns to

$$\begin{aligned} M\Delta^f x - \Delta^f s &= \theta \nu r_q^0, \\ P(w)^{\frac{1}{2}} s \circ P(w)^{-\frac{1}{2}} \Delta^f x + P(w)^{-\frac{1}{2}} x \circ P(w)^{\frac{1}{2}} \Delta^f s \\ &= (1 - \theta) \mu e - P(w)^{-\frac{1}{2}} x \circ P(w)^{\frac{1}{2}} s. \end{aligned}$$

---

*Algorithm 1 : Afull Nesterov – Todd step IIPM for SCLCP.*

---

**Input :**

Accuracy parameter  $\epsilon > 0$ ;  
 barrier update parameter  $\theta$ ,  $0 < \theta < 1$ ;  
 threshold parameter  $\tau > 0$ .

**begin**

$x := x^0 \in \text{int}\mathcal{K}; s := s^0 \in \text{int}\mathcal{K};$   
 $\langle x^0, s^0 \rangle = \mu^0 e; \mu := \mu^0;$

**while**  $\max(r\mu, \|r_q\|) > \epsilon$ ;

feasibility step :

$(x, s) := (x, s) + (\Delta^f x, \Delta^f s);$

$\mu$  – update :

$\mu := (1 - \theta)\mu;$

centering steps :

**while**  $\delta(x, s; \mu) \geq \tau$ ;

$(x, s) := (x, s) + (\Delta x, \Delta s);$

**end while**

**end while**

**end.**

---

We conclude that after the feasibility step, the iterates satisfy the affine equation (3.1) with  $\nu = \nu^+$ . The hard part in the analysis will be to guarantee that  $x^f, s^f \in \text{int}\mathcal{K}$  and that the new iterate satisfies  $\delta(x^f, s^f; \mu) \leq \frac{1}{\sqrt{2}}$ .

Let  $(x, s)$  denote the iterate at the start of an iteration with  $Tr(x \circ s) \leq \mu(r + \delta^2)$  and  $\delta(x, s; \mu) \leq \tau$ . Recall that at the start of the first iteration this is certainly true, because then  $Tr(x^0 \circ s^0) = \mu^0 r$  and  $\delta(x^0, s^0; \mu^0) = 0$ . We scale the search directions, just as we did in the feasible case as (2.6), by defining

$$(3.4) \quad d_x^f := \frac{P(w)^{-\frac{1}{2}} \Delta^f x}{\sqrt{\mu}}, \quad d_s^f := \frac{P(w)^{\frac{1}{2}} \Delta^f s}{\sqrt{\mu}},$$

where  $w$  denotes the NT-scaling point of  $x$  and  $s$  as defined in Lemma 2.4. With the vector  $v$  as defined by (2.5), the above system can be restated as

$$(3.5) \quad \begin{aligned} \overline{M} d_x^f - d_s^f &= \frac{P(w)^{\frac{1}{2}}}{\sqrt{\mu}} \theta \nu r_q^0, \\ d_x^f + d_s^f &= (1 - \theta) v^{-1} - v, \end{aligned}$$

where  $\overline{M} = P(w)^{\frac{1}{2}}MP(w)^{\frac{1}{2}}$ . To get the search directions  $\Delta^f x$  and  $\Delta^f s$  in the original  $x$ -space and  $s$ -space we use (3.4), which gives

$$\Delta^f x = \sqrt{\mu}P(w)^{\frac{1}{2}}d_x^f, \quad \Delta^f s = \sqrt{\mu}P(w)^{-\frac{1}{2}}d_s^f.$$

The new iterates are obtained by taking a full step, as given by (3.3). Hence, we have

$$(3.6) \quad x^f = \sqrt{\mu}P(w)^{\frac{1}{2}}(v + d_x^f), \quad s^f = \sqrt{\mu}P(w)^{-\frac{1}{2}}(v + d_s^f).$$

Using the second equation in (3.5) and (3.6), we derive that

$$(3.7) \quad \begin{aligned} (v + d_x^f) \circ (v + d_s^f) &= v^2 + v \circ [(1 - \theta)v^{-1} - v] + d_x^f \circ d_s^f \\ &= (1 - \theta)e + d_x^f \circ d_s^f. \end{aligned}$$

Using the same arguments as in Subsection 2.4 it follows from (3.6) that  $x^f$  and  $s^f$  will belong to  $\text{int}\mathcal{K}$  if and only if  $v + d_x^f$  and  $v + d_s^f$  belong to  $\text{int}\mathcal{K}$ . The proof of the following lemma is identical to the proof of Lemma 4.2 in [5].

**Lemma 3.2.** *The iterate  $(x^f, s^f)$  is strictly feasible if  $(1 - \theta)e + d_x^f \circ d_s^f \in \text{int}\mathcal{K}$ .*

Now, we proceed by deriving an upper bound for  $\delta(x^f, s^f; \mu^+)$ . Let  $w^f$  be the NT-scaling point of  $x^f$  and  $s^f$ . Let  $v^f$  be the vector after the feasibility step with respect to the  $\mu^+$ -center. According to (2.5), define

$$(3.8) \quad v^f := \frac{P(w^f)^{-\frac{1}{2}}x^f}{\sqrt{\mu(1 - \theta)}} \left[ = \frac{P(w^f)^{\frac{1}{2}}s^f}{\sqrt{\mu(1 - \theta)}} \right].$$

**Lemma 3.3.** *If  $\|\lambda(d_x^f \circ d_s^f)\|_\infty \leq 1 - \theta$ , then*

$$2\delta(v^f) \leq \frac{\frac{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}{2(1 - \theta)}}{\left(1 - \frac{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}{2(1 - \theta)}\right)^{\frac{1}{2}}}.$$

*Proof.* Since  $\|\lambda(d_x^f \circ d_s^f)\|_\infty \leq 1 - \theta$ , from Lemma 3.2 and (3.7) follows that  $v + d_x^f, v + d_s^f$  and  $(v + d_x^f) \circ (v + d_s^f)$  belong to  $\text{int}\mathcal{K}$ . Applying



lemmas 2.6 and 2.5, we get

$$\begin{aligned} v^f &:= \frac{P(w^f)^{\frac{1}{2}} s^f}{\sqrt{\mu(1-\theta)}} \sim \frac{\left(P(x^f)^{\frac{1}{2}} s^f\right)^{\frac{1}{2}}}{\sqrt{\mu(1-\theta)}} \\ &= \frac{\sqrt{\mu} \left(P(P(w)^{\frac{1}{2}}(v+d_x^f))^{\frac{1}{2}} P(w)^{-\frac{1}{2}}(v+d_s^f)\right)^{\frac{1}{2}}}{\sqrt{\mu(1-\theta)}} \\ &\sim \frac{\left(P(v+d_x^f)^{\frac{1}{2}}(v+d_s^f)\right)^{\frac{1}{2}}}{\sqrt{(1-\theta)}}. \end{aligned}$$

Applying Lemma 2.8 and the above relation, we obtain

$$2\delta(v^f) = \|v^f - (v^f)^{-1}\|_F = \frac{\|(v^f)^2 - e\|_F}{\lambda_{\min}(v^f)} = \frac{\|P(\frac{v+d_x^f}{\sqrt{1-\theta}})^{\frac{1}{2}}(\frac{v+d_s^f}{\sqrt{1-\theta}}) - e\|_F}{\lambda_{\min}\left(P(\frac{v+d_x^f}{\sqrt{1-\theta}})^{\frac{1}{2}}(\frac{v+d_s^f}{\sqrt{1-\theta}})\right)^{\frac{1}{2}}}.$$

Due to lemmas 2.7, 2.10 and the relation (3.7), we get

$$\begin{aligned} 2\delta(v^f) &\leq \frac{\left\|\left(\frac{v+d_x^f}{\sqrt{1-\theta}}\right) \circ \left(\frac{v+d_s^f}{\sqrt{1-\theta}}\right) - e\right\|_F}{\lambda_{\min}\left(\left(\frac{v+d_x^f}{\sqrt{1-\theta}}\right) \circ \left(\frac{v+d_s^f}{\sqrt{1-\theta}}\right)\right)^{\frac{1}{2}}} = \frac{\left\|\frac{d_x^f \circ d_s^f}{1-\theta}\right\|_F}{\lambda_{\min}\left(e + \frac{d_x^f \circ d_s^f}{1-\theta}\right)^{\frac{1}{2}}} \\ &\leq \frac{\left\|\frac{d_x^f \circ d_s^f}{1-\theta}\right\|_F}{\left(1 - \frac{\|d_x^f \circ d_s^f\|_F}{1-\theta}\right)^{\frac{1}{2}}} \leq \frac{\frac{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}{2(1-\theta)}}{\left(1 - \frac{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}{2(1-\theta)}\right)^{\frac{1}{2}}}, \end{aligned}$$

which completes the proof.  $\square$

**3.4. An upper bound for  $\|d_x^f\|_F^2 + \|d_s^f\|_F^2$ .** We have derived an upper bound for  $\delta(v^f)$  in terms of  $\|d_x^f\|_F^2 + \|d_s^f\|_F^2$ . Therefore, to obtain  $\theta, 0 < \theta < 1$ , as large as possible, such that  $\delta(v^f) \leq \frac{1}{\sqrt{2}}$ , we need an upper bound for  $\|d_x^f\|_F^2 + \|d_s^f\|_F^2$ . We consider the system (3.5). By eliminating  $d_s^f$ , we obtain

$$d_x^f = (\overline{M} + I)^{-1} \left( (1-\theta)v^{-1} - v + \frac{P(w)^{\frac{1}{2}} \theta \nu r_q^0}{\sqrt{\mu}} \right).$$

Since  $\overline{M}$  is semidefinite, it follows that

$$\begin{aligned}
 \|d_x^f\|_F &= \left\| (\overline{M} + I)^{-1} \left( (1 - \theta)v^{-1} - v + \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right) \right\|_F \\
 &\leq \lambda_{\max}((\overline{M} + I)^{-1}) \left\| (1 - \theta)v^{-1} - v + \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\|_F \\
 &= \frac{1}{\lambda_{\min}(\overline{M} + I)} \left\| (1 - \theta)v^{-1} - v + \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\|_F \\
 (3.9) \quad &\leq \left\| (1 - \theta)v^{-1} - v + \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\|_F.
 \end{aligned}$$

Hence, using the first equation of (3.5), the Cauchy-Schwartz inequality and positive semidefiniteness of  $\overline{M}$ , we get

$$\begin{aligned}
 \|d_x^f\|_F^2 + \|d_s^f\|_F^2 &= \|d_x^f + d_s^f\|_F^2 - 2\langle d_x^f, d_s^f \rangle \\
 &= \|(1 - \theta)v^{-1} - v\|_F^2 - 2\left\langle d_x^f, \overline{M}d_x^f - \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\rangle \\
 &= \|(1 - \theta)v^{-1} - v\|_F^2 - 2\langle d_x^f, \overline{M}d_x^f \rangle + 2\left\langle d_x^f, \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\rangle \\
 (3.10) \quad &\leq \|(1 - \theta)v^{-1} - v\|_F^2 + 2\|d_x^f\|_F \left\| \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\|_F.
 \end{aligned}$$

Substituting (3.9) in (3.10) gives

$$\begin{aligned}
 \|d_x^f\|_F^2 + \|d_s^f\|_F^2 &\leq \|(1 - \theta)v^{-1} - v\|_F^2 \\
 &\quad + 2\left\| (1 - \theta)v^{-1} - v + \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\|_F \left\| \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\|_F \\
 &\leq \|(1 - \theta)v^{-1} - v\|_F^2 \\
 (3.11) \quad &+ 2\left( \|(1 - \theta)v^{-1} - v\|_F + \left\| \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\|_F \right) \left\| \frac{P(w)^{\frac{1}{2}}\theta\nu r_q^0}{\sqrt{\mu}} \right\|_F.
 \end{aligned}$$

From (2.10) and  $\|v\|_F^2 \leq r + \delta^2$ , we have

$$\begin{aligned}
\|(1-\theta)v^{-1} - v\|_F^2 &= \|(1-\theta)(v^{-1} - v) - \theta v\|_F^2 \\
&= (1-\theta)^2 \|v^{-1} - v\|_F^2 - 2\theta(1-\theta)\langle v, v^{-1} - v \rangle + \theta^2 \|v\|_F^2 \\
&= (1-\theta)^2 \|v^{-1} - v\|_F^2 + 2\theta(1-\theta)(\|v\|_F^2 - r) + \theta^2 \|v\|_F^2 \\
&\leq 4(1-\theta)^2 \delta^2 + 2\theta(1-\theta)\delta^2 + \theta^2(r + \delta^2) \\
(3.12) \qquad &= (1 + 3(1-\theta)^2)\delta^2 + r\theta^2.
\end{aligned}$$

Let  $(x^*, s^*)$  be the optimal solution of SCLCP such that

$$(3.13) \qquad \|x^*\|_\infty \leq \rho_p, \quad \max\{\|s^*\|_\infty, \rho_p \|M\|\} \leq \rho_d,$$

and as usual we start the algorithm with

$$(3.14) \qquad (x^0, s^0) = (\rho_p e, \rho_d e), \quad \mu^0 = \rho_p \rho_d.$$

For such starting points we have

$$\begin{aligned}
&\left\| \frac{P(w)^{\frac{1}{2}} \theta \nu r_q^0}{\sqrt{\mu}} \right\|_F = \frac{\theta \nu}{\sqrt{\mu}} \left\| P(w)^{\frac{1}{2}} (s^0 - s^* - M(x^0 - x^*)) \right\|_F \\
(3.15) \qquad &\leq \frac{\theta \nu}{\sqrt{\mu}} \left( \|P(w)^{\frac{1}{2}} (s^0 - s^*)\|_F + \|M\| \|P(w)^{\frac{1}{2}} (x^0 - x^*)\|_F \right).
\end{aligned}$$

Now, we obtain an upper bound for  $\|P(w)^{\frac{1}{2}} (x^0 - x^*)\|_F$ . Using that  $P(w)^{\frac{1}{2}}$  is self-adjoint with respect to the inner product and  $P(w)e = w^2$  [19], we have

$$\begin{aligned}
\|P(w)^{\frac{1}{2}} (x^0 - x^*)\|_F^2 &= \langle P(w)^1 (x^0 - x^*), x^0 - x^* \rangle \\
&= \langle P(w)^1 (x^0 - x^*), \rho_p e \rangle - \langle P(w)^1 (x^0 - x^*), \rho_p e - (x^0 - x^*) \rangle \\
&\leq \langle P(w)^1 (x^0 - x^*), \rho_p e \rangle = \rho_p \langle P(w)^1 e, x^0 - x^* \rangle \\
&\leq \rho_p \langle P(w)^1 e, \rho_p e \rangle - \rho_p \langle P(w)^1 e, \rho_p e - (x^0 - x^*) \rangle \\
&\leq \rho_p^2 Tr(w^2).
\end{aligned}$$

Similarly, it follows that  $\|P(w)^{\frac{1}{2}} (s^0 - s^*)\|_F^2 \leq \rho_d^2 Tr(w^2)$ . Substitution of the last two inequalities into (3.15) gives

$$\begin{aligned}
(3.16) \qquad \left\| \frac{P(w)^{\frac{1}{2}} \theta \nu r_q^0}{\sqrt{\mu}} \right\|_F &\leq \frac{\theta \nu}{\sqrt{\mu}} (\rho_d + \rho_p \|M\|) \sqrt{Tr(w^2)} \\
&\leq \frac{2\theta \nu}{\sqrt{\mu}} \rho_d \sqrt{Tr(w^2)}.
\end{aligned}$$

**Lemma 3.4.** (Lemma 4.5 in [5]) One has

$$\text{Tr}(w^2) \leq \frac{\text{Tr}(x^2)}{\mu \lambda_{\min}(v)^2}.$$

Using (3.12), (3.16) and Lemma 3.4 in (3.11), we get

$$\begin{aligned} \|d_x^f\|_F^2 + \|d_s^f\|_F^2 &\leq (1 + 3(1 - \theta)^2)\delta^2 + r\theta^2 \\ &+ 2\left(\sqrt{(1 + 3(1 - \theta)^2)\delta^2 + r\theta^2} + \frac{2\theta\text{Tr}(x)}{\rho_p\lambda_{\min}(v)}\right) \frac{2\theta\text{Tr}(x)}{\rho_p\lambda_{\min}(v)}. \end{aligned}$$

The proof of the next lemma is exactly the same as Lemma II.60 in [14].

**Lemma 3.5.** If  $\delta := \delta(v)$  is defined by (2.10), then

$$\frac{1}{\varrho(\delta)} \leq \lambda_{\min}(v) \leq \lambda_{\max}(v) \leq \varrho(\delta),$$

where  $\varrho(\delta) := \delta + \sqrt{1 + \delta^2}$ .

**Lemma 3.6.** Let  $(x, s)$  be feasible for the perturbed problem  $\text{SCLCP}_\nu$  and  $(x^0, s^0) = (\rho_p e, \rho_d e)$ . Then, for any  $(x^*, s^*) \in \mathcal{K} \times \mathcal{K}$  with  $s^* = Mx^* + q$  and  $x^* \circ s^* = 0$ , we have

$$\begin{aligned} \nu(\langle x^0, s \rangle + \langle s^0, x \rangle) &\leq \nu^2 \langle x^0, s^0 \rangle + \nu(1 - \nu)(\langle x^0, s^* \rangle \\ &+ \langle s^0, x^* \rangle) - (1 - \nu)(\langle s, x^* \rangle + \langle x, s^* \rangle) + \langle x, s \rangle. \end{aligned}$$

*Proof.* From  $r_q^0 = s^0 - Mx^0 - q$  and the definition of the perturbed problem  $\text{SCLCP}_\nu$ , it is easily seen that

$$\begin{aligned} \nu s^0 + (1 - \nu)s^* - s &= \nu(r_q^0 + Mx^0 + q) + (1 - \nu)s^* - (\nu r_q^0 + Mx + q) \\ &= \nu(r_q^0 + Mx^0 + s^* - Mx^*) + (1 - \nu)s^* - (\nu r_q^0 + Mx - Mx^* + s^*) \\ &= M(\nu x^0 + (1 - \nu)x^* - x). \end{aligned}$$

Since  $M$  is positive semidefinite, we obtain

$$\begin{aligned} 0 &\leq \langle \nu x^0 + (1 - \nu)x^* - x, M(\nu x^0 + (1 - \nu)x^* - x) \rangle \\ &= \langle \nu x^0 + (1 - \nu)x^* - x, \nu s^0 + (1 - \nu)s^* - s \rangle \\ &= \nu^2 \langle x^0, s^0 \rangle + \nu(1 - \nu)(\langle x^0, s^* \rangle + \langle s^0, x^* \rangle) - (1 - \nu)(\langle s, x^* \rangle + \langle x, s^* \rangle) \\ &\quad + \langle x, s \rangle - \nu(\langle x^0, s \rangle + \langle s^0, x \rangle) + (1 - \nu)^2 \langle s^*, x^* \rangle. \end{aligned}$$

Using Lemma 2.2 and assumption  $x^* \circ s^* = 0$ , we have  $\langle x^*, s^* \rangle = 0$ . This completes the proof.  $\square$

**Lemma 3.7.** *Let  $(x, s)$  be feasible for the perturbed problem SCLCP $_{\nu}$ . With  $(x^0, s^0) = (\rho_p e, \rho_d e)$ , we then have*

$$\text{Tr}(x) \leq r\rho_p(2 + \varrho(\delta)^2), \quad \text{Tr}(s) \leq r\rho_d(2 + \varrho(\delta)^2),$$

where  $\varrho(\delta)$  is defined as in Lemma 3.5.

*Proof.* Since  $x, s, x^*$  and  $s^*$  belong to  $\mathcal{K}$ , it implies that  $\langle s, x^* \rangle + \langle x, s^* \rangle \geq 0$ . Therefore, Lemma 3.6 implies

$$\langle x^0, s \rangle + \langle s^0, x \rangle \leq \nu \langle x^0, s^0 \rangle + (1 - \nu)(\langle x^0, s^* \rangle + \langle s^0, x^* \rangle) + \frac{1}{\nu} \langle x, s \rangle.$$

Since  $x^0 = \rho_p e, s^0 = \rho_d e, \|x^*\|_{\infty} \leq \rho_p$  and  $\|s^*\|_{\infty} \leq \rho_d$ , we have

$$\begin{aligned} \langle x^0, s^* \rangle + \langle s^0, x^* \rangle &\leq \rho_d \langle e, x^0 \rangle + \rho_p \langle e, s^0 \rangle = 2r\rho_p\rho_d \\ \langle s^0, x^0 \rangle &= \rho_p\rho_d \langle e, e \rangle = r\rho_p\rho_d. \end{aligned}$$

Hence, we get

$$\begin{aligned} \langle x^0, s \rangle + \langle s^0, x \rangle &\leq \nu r\rho_p\rho_d + (1 - \nu)2r\rho_p\rho_d + \frac{\mu\langle v, v \rangle}{\nu} \\ &= 2r\rho_p\rho_d - \nu r\rho_p\rho_d + \frac{\mu\langle v, v \rangle}{\nu} \\ &\leq 2r\rho_p\rho_d + \frac{\mu\langle v, v \rangle}{\nu} \\ &= 2r\rho_p\rho_d + \mu^0 \langle v, v \rangle \\ &= 2r\rho_p\rho_d + \rho_p\rho_d \sum_{i=1}^r \lambda_i^2(v) \\ &\leq r\rho_p\rho_d(2 + \varrho(\delta)^2), \end{aligned}$$

where the last inequality follows from Lemma 3.5. Since  $\langle x^0, s \rangle \geq 0$  and  $\langle s^0, x \rangle \geq 0$ , we get

$$\langle s^0, x \rangle \leq r\rho_p\rho_d(2 + \varrho(\delta)^2), \quad \langle x^0, s \rangle \leq r\rho_p\rho_d(2 + \varrho(\delta)^2).$$

Moreover, since  $s^0 = \rho_d e$  and  $x^0 = \rho_p e$ , we have

$$\langle s^0, x \rangle = \rho_d \text{Tr}(x), \quad \langle x^0, s \rangle = \rho_p \text{Tr}(s).$$

Therefore,

$$\text{Tr}(x) \leq r\rho_p(2 + \varrho(\delta)^2), \quad \text{Tr}(s) \leq r\rho_d(2 + \varrho(\delta)^2),$$

which proves the lemma.  $\square$

By using lemmas 3.5 and 3.7, we obtain

$$(3.17) \quad \begin{aligned} \|d_x^f\|_F^2 + \|d_s^f\|_F^2 &\leq (1 + 3(1 - \theta)^2)\delta^2 + r\theta^2 \\ &\quad + 2\left(\sqrt{(1 + 3(1 - \theta)^2)\delta^2 + r\theta^2} \right. \\ &\quad \left. + 2r\theta\varrho(\delta)(2 + \varrho(\delta)^2)\right)2r\theta\varrho(\delta)(2 + \varrho(\delta)^2). \end{aligned}$$

**3.5. Value for  $\theta$ .** We would like to choose  $\theta, 0 < \theta < 1$ , as large as possible, and such that  $(x^f, s^f)$  lies in the quadratic convergence neighborhood with respect to the  $\mu^+$ -center of the perturbed problem SCLCP $_{\nu}$ , i.e.,  $\delta(v^f) \leq \frac{1}{\sqrt{2}}$ . By Lemma 3.3, we derive that this is the case when

$$\frac{\frac{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}{2(1 - \theta)}}{\left(1 - \frac{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}{2(1 - \theta)}\right)^{\frac{1}{2}}} \leq \sqrt{2}.$$

Considering  $\frac{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}{1 - \theta}$  as a single term, and performing some elementary calculations, we obtain that

$$(3.18) \quad \frac{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}{1 - \theta} \leq 2\sqrt{3} - 2 \approx 1.4641.$$

By (3.17), the above inequality holds if

$$(1 + 3(1 - \theta)^2)\delta^2 + r\theta^2 + 2\left(\sqrt{(1 + 3(1 - \theta)^2)\delta^2 + r\theta^2} + 2r\theta\varrho(\delta)(2 + \varrho(\delta)^2)\right)2r\theta\varrho(\delta)(2 + \varrho(\delta)^2) \leq (2\sqrt{3} - 2)(1 - \theta).$$

Choosing  $\tau = \frac{1}{16}$ , one may easily verify that the above inequality is satisfied if

$$(3.19) \quad \theta = \frac{1}{10r}.$$

Moreover,

$$\|\lambda(d_x^f \circ d_s^f)\|_{\infty} \leq \frac{1}{2}(\|d_x^f\|_F^2 + \|d_s^f\|_F^2) \leq (\sqrt{3} - 1)(1 - \theta) < 1 - \theta,$$

which, by Lemma 3.3, means that  $(x^f, s^f)$  is strictly feasible. Thus, we have obtained a desired update parameter  $\theta$ .

**3.6. Complexity.** We have seen that if at the start of an iteration the iterate satisfies  $\delta(x, s; \mu) \leq \tau$ , with  $\tau = \frac{1}{16}$ , then after the feasibility step, with  $\theta$  as defined by (3.19), the iterate is strictly feasible and satisfies  $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt{2}}$ , i.e.,  $(x^f, s^f)$  lies in the quadratic convergence neighborhood with respect to the  $\mu^+$ -center of the perturbed problem  $SCLCP_\nu$ .

After the feasibility step, we perform a few centering steps in order to get the iterates  $(x^+, s^+)$  to satisfy  $\langle x^+, s^+ \rangle \leq \mu(r + \delta^2)$  and  $\delta(x^+, s^+; \mu^+) \leq \tau$ . By Corollary 2.16, after  $k$  centering steps we will have the iterate  $(x^+, s^+)$  that is still feasible for  $SCLCP_{\nu^+}$  and such that

$$\delta(x^+, s^+; \mu^+) \leq \left(\frac{1}{\sqrt{2}}\right)^{2^k}.$$

From this, one easily deduces that  $\delta(x^+, s^+; \mu^+) \leq \tau$  will hold after at most

$$(3.20) \quad 1 + \left\lceil \log_2 \left( \log_2 \frac{1}{\tau} \right) \right\rceil,$$

centering steps. According to (3.20), and since  $\tau = \frac{1}{16}$ , at most three centering steps suffice to get the iterate  $(x^+, s^+)$  that satisfies  $\delta(x^+, s^+; \mu^+) \leq \tau$  again. So, each main iteration consists of at most four so-called inner iterations.

In each main iteration both the value of  $r\mu$  and the norm of the residual are reduced by the factor  $1 - \theta$ . Hence, the total number of the main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\{Tr(x^0 \circ s^0), \|r_q^0\|_F\}}{\epsilon}.$$

Due to (3.19) and the fact that we need at most four inner iterations per main iteration, we may state the main result of the paper.

**Theorem 3.8.** *If SCLCP has an optimal solution  $(x^*, s^*)$  such that  $\|x^*\|_\infty \leq \rho_p$  and  $\|s^*\|_\infty \leq \rho_d$ , then after at most*

$$40r \log \frac{\max\{Tr(x^0 \circ s^0), \|r_q^0\|_F\}}{\epsilon}$$

*iterations the algorithm finds a solution of SCLCP.*

## REFERENCES

- [1] J. Faraut and A. Korényi, *Analysis on Symmetric Cones*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1994.
- [2] L. Faybusovich, Linear systems in Jordan algebras and primal-dual interior-point algorithms, *J. Comput. Appl. Math.* **86** (1997), no. 1, 149–175.
- [3] L. Faybusovich, A Jordan-algebraic approach to potential-reduction algorithms, *Math. Z.* **239** (2002), no. 1, 117–129.
- [4] L. Faybusovich, Euclidean Jordan algebras and interior-point algorithms, *Positivity* **1** (1997), no. 4, 331–357.
- [5] G. Gu, M. Zangiabadi and C. Roos, Full Nesterov-Todd step infeasible interior-point method for symmetric optimization, *European J. Oper. Res.* **214** (2011), no. 3, 473–484.
- [6] C. Helemberg, F. Rendl, R. J. Vanderbei and H. Wolkowicz, An interior-point method for semidefinite programming, *SIAM J. Optim.* **6** (1996), no. 2, 342–361.
- [7] B. Kheirfam and N. Mahdavi-Amiri, A new interior-point algorithm based on modified Nesterov-Todd direction for symmetric cone linear complementarity problem, *Optim. Let.* **8** (2014), no. 3, 1017–1029.
- [8] M. Kojima, N. Megiddo, T. Noma and A. Yoshise, A unified approach to interior point algorithms for linear complementarity problems, *Lecture Notes in Computer Science*, 538, Springer-Verlag, Berlin, 1991.
- [9] M. Kojima, S. Shindoh and S. Hara, Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices, *SIAM J. Optim.* **7** (1997), no. 1, 86–125.
- [10] R. D. C. Monteiro, Primal-dual path-following algorithms for semidefinite programming, *SIAM J. Optim.* **7** (1997), no. 3, 663–678.
- [11] Y. E. Nesterov and M. J. Todd, Primal-dual interior-point methods for self-scaled cones, *SIAM J. Optim.* **8** (1998), no. 2, 324–364.
- [12] B. K. Rangarajan, Polynomial convergence of infeasible interior-point methods over symmetric cones, *SIAM J. Optim.* **16** (2006), no. 4, 1211–1229.
- [13] C. Roos, A full-Newton step  $O(n)$  infeasible interior-point algorithm for linear optimization, *SIAM J. Optim.* **16** (2006), no. 4, 1110–1136.
- [14] C. Roos, T. Terlaky and J. Ph. Vial, *Theory and algorithms for linear optimization, An interior point approach*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Ltd, Chichester, 1997.
- [15] S. H. Schmieta and F. Alizadeh, Extension of primal-dual interior-point algorithms to symmetric cones, *Math. Program.* **96** (2003), no. 3, 409–438.
- [16] S. H. Schmieta and F. Alizadeh, Associative and Jordan algebras and polynomial time interior-point algorithms for symmetric cones, *Math. Oper. Res.* **26** (2001), no. 3, 543–564.
- [17] J. F. Sturm, Similarity and other spectral relations for symmetric cones, *Algebra Appl.* **312** (2000), no. 1-3, 135–154.
- [18] T. Tsuchiya, A convergence analysis of the scaling-invariant primal-dual path-following algorithms for second-order cone programming, *Optim. Methods Softw.* **11/12** (1999), no. 1-4, 141–182.



- [19] M. V. C. Vieira, Jordan algebraic approach to symmetric optimization, Ph.D. Thesis, Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, The Netherlands.

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