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ACTIONS OF VECTOR GROUPOIDS

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ABSTRACT. In this work we deal with actions of vector groupoid which is a new concept in the literature. After we give the definition of the action of a vector groupoid on a vector space, we obtain some results related to actions of vector groupoids. We also apply some characterizations of the category and groupoid theory to vector groupoids. As the second part of the work, we define the notion of a crossed module over a vector groupoid. Finally, we show that the category \mathcal{VG} of the vector groupoids is equivalent to the category \mathcal{CModVG} of the crossed modules over a vector groupoid. Keywords: Groupoid, action, crossed module, vector groupoid. MSC(2010): Primary: 20L05; Secondary: 20L99.

1. Introduction

The concept of groupoid was first introduced by Brandt [2]. But, after the topological and differentiable versions of the groupoid were introduced by Ehresmann in 1950's, the theory of groupoids has been extensively developed and found many applications and generalisations in areas such as algebraic topology, differential topology, noncommutative geometry and theoretical physics.

The concept of action is one of the most important tools in algebraic topology. The concept plays an important role in category theory especially in the study of groupoids (in the sense of Ehresmann). The concept of a groupoid action on a set was introduced by Ehresmann [9] and is fairly well-known. Subsequently the action of groupoids has

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⁵⁶⁵

been studied extensively in algebraic and differential topology. Many mathematicians studied various aspects of groupoid actions in 1 sense of algebraic, topological and differentiable categories [5, 12, 13].

Another algebraic concept considered in this paper is a crossed module. Crossed modules have been used widely, and in various contexts, since their definition by Whitehead in his investigation of the algebraic structure of second relative homotopy groups [17]. The crossed modules have been constructed over different algebraic structures (such as groups, algebras, groupoids, etc.). The crossed modules over groupoids was given by Brown and Higgins [6]. Also many mathematicians studied crossed module over groupoids [1, 7]. We consider crossed moduless over groupoids in this work.

Specially, to show that two categories are equivalent is one of the important problems in the algebraic topology. There are certain categories which are equivalent to the categories of actions of (topological or differentiable) groupoids in the algebraic topology. For example, Gabriel and Zisman showed the equivalence of the category GdCov(G) of coverings of a groupoid G and the category GdOp(G) of actions of G on sets [10]. The topological version of my results is studied by Brown et al in [5]. Furthermore, there are also other equivalences of the categories related to (higher dimensional or structured) groupoids [8, 11].

In this work we deal with crossed modules and actions of vector groupoids which are new concepts in the literature and which has applications in geometry and other areas. It was defined by Poputa and Ivan [15, 16]. After they gave the definition of vector groupoid, they obtained some results and characterizations. We first define an action of a vector groupoid on a vector space. Also we give some new definitions related to vector groupoids that we need to study crossed modules of vector groupoids. We obtain some results related to actions of vector groupoids. We construct the categories of the category \mathcal{VG} of vector groupoids and the category \mathcal{CModVG} crossed modules of vector groupoids. Finally we show that the equivalence of the categories \mathcal{VG} and \mathcal{CModVG} .

Since actions and crossed modules of vector groupoids rely heavily on groupoid theory, we begin with a brief review of groupoid Theory in Section 2. This includes definition and elementary properties of groupoid in Section 2.1 and actions of groupoids on sets in Section 2.2. In Section 3, we present the concept of vector groupoid defined by Poputa and Ivan

and we give some results about vector groupoids. We devote Section 4 to the actions of vector groupoids. In Section 5, we describe the crossed module of vector groupoids and present the main results of the work.

2. Groupoids

In this section, we recall the definition of groupoid and some basic properties of groupoids.

2.1. Groupoids and Their Basic Properties.

Definition 2.1. A groupoid is a category in which every arrow is invertible. More precisely, a groupoid consists of two sets G and $G^{(0)}$ called the set of arrows (or morphisms) and the set of objects of groupoid respectively, together with two maps $\alpha, \beta : G \to G^{(0)}$ called source and target maps respectively, a map $\epsilon : G^{(0)} \to G$, $x \mapsto \epsilon(x) = 1_x$ called the object map, an inverse map $i : G \to G$, $a \mapsto a^{-1}$ and a composition $(b, a) \mapsto b \circ a$ defined on the pullback

$$G^{(2)} = G_{\alpha} \times_{\beta} G = \{(b, a) \mid \alpha(b) = \beta(a)\}.$$

These maps should satisfy the following conditions:

- (1) $\alpha(b \circ a) = \alpha(a)$ and $\beta(b \circ a) = \beta(b)$, for all $(b, a) \in G^{(2)}$,
- (2) $c \circ (b \circ a) = (c \circ b) \circ a$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$, for all $a, b, c \in G$,
- (3) $\alpha(1_x) = \beta(1_x) = x$, for all $x \in G^{(0)}$,
- (4) $a \circ 1_{\alpha(a)} = a$ and $1_{\beta(a)} \circ a = a$, for all $a \in G$,
- (5) $\alpha(a^{-1}) = \beta(a)$ and $\beta(a^{-1}) = \alpha(a)$, $a^{-1} \circ a = 1_{\alpha(a)}$ and $a \circ a^{-1} = 1_{\beta(a)}$ [3, 4, 12, 14].

We denote a groupoid G over $G^{(0)}$ by $(G, G^{(0)})$ or $(G, G^{(0)}, \alpha, \beta, i, \circ)$ or G. We sometimes use the notation ba instead of the composition $b \circ a$, if no confusion arises.

Let G be a groupoid. For all $x, y \in G^{(0)}$, we use the notation G(x, y)for the set of all arrows $a \in G$ such that $\alpha(a) = x$ and $\beta(a) = y$. For $x \in G^{(0)}$, we write $St_G x$ for the set of all arrows started at x, and $CoSt_G x$ for the set of all arrows ended at x. The object or vertex group at x is $G\{x\} = \{a \in G \mid \alpha(a) = \beta(a) = x\}.$

A groupoid G is called connected (or transitive) if G(x, y) is non empty for all $x, y \in G^{(0)}$. The maximal connected subgroupoids of G are called the (connected) components of G [3, 14].

Definition 2.2. Let G and H be two groupoids. A groupoid morphism from H to G is a pair (f, f_0) of maps $f : H \to G$ and $f_0 : H^{(0)} \to G^{(0)}$ such that $\alpha_G \circ f = f_0 \circ \alpha_H$, $\beta_G \circ f = f_0 \circ \beta_H$ and $f(b \circ a) = f(b) \circ f(a)$ for all $(b, a) \in H^{(2)}$ [3, 13, 14].

We denote the groupoid morphism (f, f_0) by f for brevity. If f is also bijective then it is called a groupoid isomorphism.

The left-translation (right translation) corresponding to $a \in G(x, y)$ is the map $L_a: CoSt_Gx \to CoSt_Gy, b \mapsto a \circ b$ ($R_a: St_Gy \to St_Gx, b \mapsto b \circ a$) which is an isomorphism. The inner automorphism corresponding to $a \in G(x, y)$ is the map $I_a: G(x, x) \to G(y, y), b \mapsto a \circ b \circ a^{-1}$ [3, 14].

Let us recall some basic properties of the groupoid homomorphisms in the following proposition.

Proposition 2.3. Suppose G and H are groupoids and $f: G \to H$ is a groupoid homomorphism.

i) Given $x \in G^{(0)}$ we have $f(x) \in H^{(0)}$.

ii) Given $a \in G$ we have $f(a^{-1}) = f(a)^{-1}$.

iii) For all $a \in G$ we have $\beta(f(a)) = f(\beta(a))$ and $\alpha(f(a)) = f(\alpha(a))$ [3, 14].

Thus, we can construct the category Gpd of the groupoids and their homomorphisms.

Let G be a groupoid. A subgroupoid of G is a subcategory H of G such that $a \in H \Rightarrow a^{-1} \in H$; that is, H is a subcategory which is also a groupoid. We say H is full (wide) if H is a full (wide) subcategory [3, 14].

We recall the definition of normal subgroupoid.

Let G be a groupoid. A subgroupoid N of G is called normal if N is wide in G (i.e. $N^{(0)} = G^{(0)}$) and, for any objects x, y of G and $a \in G(x, y)$, $aN(x)a^{-1} \subseteq N(y)$, from which it easily follows that

$$aN(x)a^{-1} = N(y).$$

Example 2.4. For a set X, the cartesian product $X \times X$ is a groupoid over X, called the coarse groupoid. The maps α and β are the natural projections onto the second and first factors, respectively. The object map is $x \mapsto (x, x)$ and the composition is given by (x, y)(y, z) = (x, z). The inverse of (x, y) is simply (y, x). Note that any subgroupoid of the coarse groupoid is nothing but an equivalence relation on X. If G is a groupoid over X, then the map $(\alpha, \beta) : G \to X \times X$ is a groupoid homomorphism over X and its image is a subgroupoid of the coarse

groupoid of X and this subgroupoid is given by the equivalence relation $\sim_G [3, 14].$

2.2. Actions of Groupoids. In this section we recall the definition of the action of a groupoid G on a set X. Also it will be presented some examples and properties related to the action of groupoids.

Definition 2.5. Let G be a groupoid and let M be a set. Let $J: M \to G^{(0)}$ be a map. A left action of G on M via J is a map $\phi: G_{\alpha \times J}M \to M, (a, x) \mapsto a \cdot x$ satisfying the conditions

i) $J(a \cdot x) = \beta(a)$ ii) $b \cdot (a \cdot x) = (b \circ a) \cdot x$ iii) $(1_{J(x)}) \cdot x = x$, for any $a, b \in G$, $x \in M$. In this case, we also call the set M as a left G-set. Similarly, we can define a right action of G on M [5, 10, 14].

Two simple examples for Definition 2.5 are:

(1) Any groupoid G acts on itself from both sides by the composition of G. The moments are α and β for the left and right actions, respectively [14].

(2) Any groupoid G acts on $G^{(0)}$ from both sides with moment $id_{G^{(0)}}$. The left action is $a \cdot x = \alpha(a)$ and the right action is $x \cdot a = \beta(a)$ [14].

Now we recall a concept which is called a stable subgroup.

Let G be a groupoid over $G^{(0)}$, and (M,J) a G-space. For $u \in G^{(0)}$, we call $G_u = \alpha^{-1}(u) \cap \beta^{-1}(u)$ a stable subgroup.

 G_u is a group, in fact, inheriting the operation of G, G_u has a multiplication. The unit element is u and every element $x \in G_u$ has an inverse element x^{-1} .

Example 2.6. Let X be a set with a left group action by G. We define the action groupoid $G \ltimes X$ over X to have arrows

$$\bigcup_{x \in X, a \in G} (x \xrightarrow{a} a \cdot x) = G \times X$$

The composition law is given by $(a, x)(b, a \cdot x) = (ba, x)$ [5, 14].

In the case of the action of G on $G^{(0)}$, we can define an equivalence relation " \sim ". If $x, y \in G^{(0)}$, $\exists a \in G$ such that a left action of aon x is y, then we say that x is equivalent to y. The equivalent class $\theta_x = \{y | y = a \cdot x, \exists a \in G\}$ of x is called an x-orbit of the G in $G^{(0)}$. For a right action of G on $G^{(0)}$, we can also define an x-orbit of G in $G^{(0)}$. The two kinds of definitions are the same, in fact, if $a \cdot x = \alpha(a) = y$,

then $x \cdot a^{-1} = \beta(a^{-1}) = \alpha(a) = y$, i.e., $x \sim y$ by a in the left action, meanwhile $x \sim y$ by a^{-1} in the right action.

Now we recall a concept which is called a stable subgroup.

Let G be a groupoid over $G^{(0)}$, and (M,J) a G-space. For $u \in G^{(0)}$, we call the $G_u = \alpha^{-1}(u) \cap \beta^{-1}(u)$ a stable subgroup.

 G_u is a group, in fact, inheriting the operation of G, G_u has a multiplication. The unit element is u and every element $x \in G_u$ has an inverse element x^{-1} .

3. Vector Groupoids

In this section, we will give the concept of vector groupoid defined by Poputa and Ivan [15, 16]. Also we will present some results related to vector groupoids.

Definition 3.1. A vector groupoid over a field K, is a groupoid $(V, \alpha, \beta, \odot, i, V^{(0)})$ such that

- (1) V is a vector space over K, and the set, of units $V^{(0)}$ is a subspace of V.
- (2) The source and the target maps α and β are linear maps.
- (3) The inversion $i: V \to V, a \mapsto i(a) = a^{-1}$ is a linear map and the following condition is verified:

i) $a + a^{-1} = \alpha(a) + \beta(a)$, for all $a \in V$

- (4) The map $m: V^{(2)} = \{(a,b) \in V \times V | \alpha(b) = \beta(a)\} \rightarrow V, (a,b) \mapsto m(a,b) = a \odot b$, satisfy the following conditions:
 - i) $a \odot (b+c-\beta(a)) = a \odot b + a \odot c a$, for all $a, b, c \in V$ such that $\alpha(b) = \beta(a) = \alpha(c)$.
 - ii) $a \odot (kb + (1-k)\beta(a)) = k(a \odot b) + (1-k)a$, for all $a, b \in V$ such that $\alpha(b) = \beta(a)$.
 - iii) $(b+c-\alpha(a)) \odot a = b \odot a + c \odot a a$, for all $a, b, c \in V$ such that $\alpha(a) = \beta(b) = \beta(c)$.
 - iv) $(kb+(1-k)\alpha(a)) \odot a = k(b \odot a) + (1-k)a$, for all $a, b \in V$ such that $\alpha(a) = \beta(b)$ [15, 16].

We sometimes use the notation ab instead of the composition $a \odot b$, if no confusion arises.

From Definition 3.1 we have the following corollary.

Corollary 3.2. Let $(V, \alpha, \beta, \odot, i, V^{(0)})$ be a vector groupoid. Then: i) The source and the target maps α and β are linear epimorphisms.

ii) The inversion $i: V \to V$ is a linear automorphism. iii) The fibres $\alpha^{-1}(0)$, $\beta^{-1}(0)$ and the isotropy group $V(0) = \alpha^{-1}(0) \cap \beta^{-1}(0)$ are vector subspaces of the vector space V [15, 16].

Clearly, there are some extra algebraic rules in a vector groupoid. Let us recall these rules in the following propositions.

Proposition 3.3. Let $(V, \alpha, \beta, \odot, i, V^{(0)})$ be a vector groupoid. Then the following assertions hold [15, 16]: $i) \ 0 \circ a = a, \forall a \in \alpha^{-1}(0),$ $ii) \ a \circ 0 = a, \forall a \in \beta^{-1}(0),$ $iii) \ a - \alpha(a) = b - \alpha(b) \Rightarrow a = b, \forall a, b \in \beta^{-1}(0),$ $iv) \ a - \beta(a) = b - \beta(b) \Rightarrow a = b, \forall a, b \in \alpha^{-1}(0).$

Proposition 3.4. Let $(V, V^{(0)})$ be a vector groupoid over a field K. Then:

i) $t_{\beta}: \alpha^{-1}(0) \to \beta^{-1}(0), t_{\beta}(a) = \beta(a) - a$ is a linear isomorphism, ii) $t_{\alpha}: \beta^{-1}(0) \to \alpha^{-1}(0), t_{\alpha}(a) = \alpha(a) - a$ is a linear isomorphism [15, 16].

Proposition 3.5. Let $(V, +, \cdot, \alpha, \beta, \odot, i, V^{(0)})$ be a vector groupoid over a field K and $u \in V^{(0)}$. Then the following assertions exist: i) The isotropy group $V(u) = \{a \in V \mid \alpha(a) = \beta(a) = u\}$ endowed with the laws

$$\boxplus: V \times V \to V, (a, b) \mapsto a \boxplus b = a + b - u, \quad \forall a, b \in V(u)$$

and

$$\boxtimes : K \times V \to V, (k, a) \mapsto k \boxtimes a = ka + (1 - k)u, \ \forall k \in K, \forall a \in V(u)$$

has a structure of vector space over K.

ii) The vector space $(V(u), \boxplus, \boxtimes)$ together with the restrictions of structure maps α, β, i to V(u) and the composition

$$: V(u)^{(2)} = V(u) \times V(u) \to V(u)$$

$$(3.1) \qquad (a,b) \mapsto a \boxdot b = (a-u) \odot (b-u) + u, \quad \forall a, b \in V(u)$$

has a structure of vector groupoid with a single unit over K [15, 16].

We call $(V(u), \boxplus, \boxtimes, \alpha, \beta, \boxdot, i, V(u)^{(0)} = \{u\})$ the isotropy vector groupoid at $u \in V^{(0)}$ of V [15, 16].

Let us define the concept of vector subgroupoid.

Definition 3.6. Let $(V, V^{(0)})$ be a vector groupoid over a field K. A vector subgroupoid of V is a pair of vector subspaces $V' \subset V, V^{(0)'} \subset V^{(0)}$ such that $\alpha(V') \subset V^{(0)'}, \beta(V') \subset V^{(0)'}, 1_x \in V'$ for all $x \in V^{(0)'}$, and V' is closed under the composition and the inversion in V. A vector subgroupoid $(V', V^{(0)'})$ of $(V, V^{(0)})$ is called wide if $V^{(0)'} = V^{(0)}$, and is called full if V'(x, y) = V(x, y) for all $x, y \in V^{(0)'}$.

The *identity vector subgroupoid* of $(V, V^{(0)})$ is the vector subgroupoid $\Delta_V = \{1_x \mid x \in V^{(0)}\}$. The *inner vector subgroupoid* of $(V, V^{(0)})$ is the vector subgroupoid $IV = \bigcup_{x \in V^{(0)}} V(x, x)$.

Definition 3.7. Let V be a vector groupoid on $V^{(0)}$. A normal vector subgroupoid of V is a wide vector subgroupoid N such that for any $n \in N$ and any $v \in V$ with $\alpha(v) = \alpha(n) = \beta(n)$, we have $v \odot n \odot v^{-1} \in N$.

Definition 3.8. Let $(V_1, \alpha_1, \beta_1, V_1^{(0)})$ and $(V_2, \alpha_2, \beta_2, V_2^{(0)})$ be two vector groupoids. A vector groupoid morphism (or homomorphism) is a groupoid morphism (or homomorphism) $f : V_1 \to V_2$ such that f is a linear map [15, 16].

Example 3.9. Let $(V, V^{(0)})$ and $(V', V'^{(0)})$ be vector groupoids over a field K and $f: V \to V'$ be a homomorphism of vector groupoids. Then kerf, the set of all $v \in V$ for which f(v) is an identity arrow of V', has clearly the structure of vector subspace. kerf is the wide vector subgroupoid of V, and so is a normal vector subgroupoid of V. In fact, it is obvious that kerf is wide in V, and for $n \in kerf(x), v \in V(x, y)$ the normality follows from

 $f(v \odot n \odot v^{-1}) = f(v) \boxdot f(n) \boxdot f(v^{-1}) = f(v) \boxdot f(v^{-1}) = 1.$

Therefore, we can construct the category \mathcal{VG} of vector groupoids and their homomorphisms.

Definition 3.10. Let V be a vector groupoid on $V^{(0)}$. V is transitive if $V(x,y) \neq \emptyset$ for all $x, y \in V^{(0)}$. V is totally intransitive if $V(x,y) = \emptyset$ for all $x, y \in V^{(0)}$.

As an example, it is obvious that the identity vector subgroupoid and the inner vector subgroupoid of a vector groupoid $(V, V^{(0)})$ are totally intransitive.

4. Action of Vector Groupoids

In this section we will be interested primarily in the concept of action of a vector groupoid on a vector space. Also we present some results related to the actions of vector groupoids.

Definition 4.1. Let V be a vector groupoid over $V^{(0)}$ and S be a vector space. Let $\lambda : S \to V^{(0)}$ be a linear map. If there exists a linear map

$$\phi: V_{\alpha} \times_{\lambda} S \to S, \ (v,s) \mapsto v \cdot s$$

which satisfy the following conditions, then we say that "the vector groupoid V acts on the vector space S via the linear map λ ": i) $\lambda(v \cdot s) = \beta(v)$,

 $ii) w \cdot (v \cdot s) = (w \odot v) \cdot s,$

iii) $1_{\lambda(s)} \cdot s = s$.

Then S is said to be a (left) V-space. Similarly, we can define the right action of V on S.

Example 4.2. Let $(V, \alpha, \beta, \odot, i, V^{(0)})$ be a vector groupoid over a field K. Then V acts on the vector space $S = V^{(0)}$ via the linear map $\lambda = Id : S = V^{(0)} \rightarrow V^{(0)}$. Indeed, let us define the map

$$\phi: V_{\alpha} \times_{\lambda} V^{(0)} \to V^{(0)}, \ (v, x) \mapsto \phi(v, x) = v \cdot x = \beta(v).$$

We firstly show that ϕ satisfies the conditions of action.

i) $Id(v \cdot x) = v \cdot x = \beta(v)$. Hence the first condition of action is verified. ii) We have $\phi(v, \phi(w, x)) = \phi(v, \beta(w)) = \beta(v)$ and $\phi((v \odot w), x) = \beta(v \odot w) = \beta(v)$. Hence $\phi(v, \phi(w, x)) = \phi((v \odot w), x)$. That is, the second condition is verified.

iii) It is obvious that $\phi(1_{Id(x)}, x) = \beta(1_{Id(x)}) = \beta(1_x) = x$.

Secondly we must show that the action is linear. Indeed, if we use the linearity of β , we have

$$\phi((v, x) + (w, x)) = \phi(v + w, x + x)$$
$$= \beta(v + w)$$
$$= \beta(v) + \beta(w)$$
$$= \phi(v, x) + \phi(w, x),$$

Hence, it follows that the action ϕ is linear.

It follows from the rules for an action that an element v in V(x, y) defines a linear isomorphism $v_{\sharp} : \lambda^{-1}[x] \to \lambda^{-1}[y], s \mapsto v \cdot s$. The action is said to be transitive if for all objects x, y in $V^{(0)}, s \in \lambda^{-1}[x], s' \in \lambda^{-1}[y]$, there is an element $v \in V(x, y)$ such that $v \cdot s = s'$.

If $s \in \lambda^{-1}[x]$, the stability group of s is the subgroup V(s) of V, which consists of elements v such that $v \cdot s = s$. Such an element v is said to stabilize s, and s is said to be fixed point of v.

Let us now give the concept of *action vector groupoid* by an example, which will be a useful tool in Theorems 5.4 and 5.5.

Example 4.3. Let $(V, \alpha, \beta, \odot, \epsilon, i, V^{(0)})$ be a vector groupoid acting on vector space S via $\lambda : S \to V^{(0)}$. Thus we can construct a vector groupoid with the space of objects that is the vector space S and the space of morphisms is vector space $V_{\alpha} \times_{\lambda} S$. It is called action vector groupoid and denoted by $V \times S \rightrightarrows S$ or by $(V \times S, \alpha_1, \beta_1, \boxtimes, \epsilon_1, i_1, S)$.

In the vector groupoid $(V \times S, \alpha_1, \beta_1, \boxtimes, \epsilon_1, i_1, S)$ a morphism from an object s to another object s' is a pair (v, s) such that the equality $\phi(v, s) = v \cdot s = s'$ is verified. Namely, we have the set $(V \times S)(s, s') =$ $\{(v, s) \mid v \in V(s, s') \text{ and } v \cdot s = s'\}$. Also we can list the structure maps of the vector groupoid $(V \times S, \alpha_1, \beta_1, \boxtimes, \epsilon_1, i_1, S)$ as follows:

the source map is defined by $\alpha_1(v,s) = s$,

the target map is defined by $\beta_1(v,s) = v \cdot s = \phi(v,s) = s'$,

the object map is defined by $\epsilon_1(s) = (1_{\lambda(s)}, s) = (\epsilon(\lambda(s)), s),$

the composition map is defined by $(v,s) \boxtimes (w,s') = (v \odot w,s)$ such that $s' = v \cdot s$,

and the inverse map is defined by $i_1(v,s) = (v^{-1}, v \cdot s)$, where v^{-1} denotes the inverse element of v in the vector groupoid V.

Now let us prove that the conditions (1)-(4) from Definition 3.1 hold. For the first condition, we show that the equality

$$(v,s) \boxtimes [(w,s') + (u,s'') - \epsilon_1(\beta(v,s))] = (v,s) \boxtimes (w,s') + (v,s) \boxtimes (u,s'') - (v,s'') - (v,s$$

holds, where $\alpha(w, s') = \beta(v, s) = \alpha(u, s'')$. The morphisms at the left side of the equality are as follows:

$$(v \cdot s) : s \mapsto v \cdot s,$$

$$(w, s') + (u, s'') : v \cdot s + v \cdot s \mapsto w \cdot (v \cdot s) + u \cdot (v \cdot s),$$

$$\epsilon_1(\beta(v, s)) = \epsilon_1(v \cdot s) : v \cdot s \mapsto v \cdot s.$$

Hence, it follows that the morphism at the left side is a morphism as

$$\begin{aligned} (v,s) \boxtimes \left[(w,s^{'}) + (u,s^{''}) - \epsilon_1(\beta(v,s)) \right] : s \mapsto w \cdot (v \cdot s) + u \cdot (v \cdot s) - v \cdot s. \\ \text{The morphisms at the right side of the equality are explicitly:} \\ (v,s) \boxtimes (w,s^{'}) : s \mapsto w \cdot (v \cdot s), \end{aligned}$$

$$(v,s) \boxtimes (u,s'') : s \mapsto u \cdot (v \cdot s)$$

 $-(v,s) : -s \mapsto -v \cdot s.$

If we sum these morphisms, the morphism at right side of the equality is

$$(v,s)\boxtimes(w,s^{'})+(v,s)\boxtimes(u,s^{''})-(v,s):s\mapsto w\cdot(v\cdot s)+u\cdot(v\cdot s)-v\cdot s.$$

Hence, it follows that both sides of the equality determine the same morphism. Thus, the first condition of being vector groupoid holds.

Since the source map α_1 is projection onto second factor and the action is linear, α_1 and β_1 are clearly linear maps. Hence the condition (2) of Definition 3.1 holds.

Let us now show that the inverse map i_1 of $V \times S$ is linear. Let $a, b \in V \times S$ and $k_1, k_2 \in K$ where a = (v, s) and b = (w, t). Then we have

$$\begin{split} i_1(k_1a + k_2b) &= i_1(k_1(v, s) + k_2(w, t)) \\ &= i_1((k_1v, k_1s) + (k_2w, k_2t)) \\ &= i_1(k_1v + k_2w, k_1s + k_2t) \\ &= ((k_1v + k_2w)^{-1}, (k_1v + k_2w) \cdot (k_1s + k_2t)) \\ &= (k_1v^{-1} + k_2w^{-1}, (k_1v \cdot k_1s) + (k_2w \cdot k_2t)) \\ &= (k_1v^{-1} + k_2w^{-1}, k_1(v \cdot s) + k_2(w \cdot t)) \\ &= (k_1v^{-1}, k_1(v \cdot s)) + (k_2w^{-1}, k_2(w \cdot t)) \end{split}$$

Actions of vector groupoids

$$= k_1(v^{-1}, v \cdot s)) + k_2(w^{-1}, w \cdot t)$$

= $k_1i_1(v, s) + k_2i_1(w, t)$
= $k_1i_1(a) + k_2i_1(b).$

Hence the inverse map i_1 is linear. We verify the condition (1) of 3.1.3 of Definition 3.1 in [16], i.e., for all $(v, s) \in V \times S$, such that $\alpha(v) = \lambda(s)$, $\beta(v) = \lambda(s') = \lambda(v \cdot s)$ it follows that:

$$(v,s) + (v,s)^{-1} = \epsilon_1(\alpha_1(v,s)) + \epsilon_1(\beta_1(v,s))$$

Indeed,

$$(v,s) + (v,s)^{-1} = (v,s) + (v^{-1}, v \cdot s)$$

$$= (v + v^{-1}, s + v \cdot s)$$

$$= (\epsilon(\alpha(v)) + \epsilon(\beta(v)), s + v \cdot s)$$

$$= (\epsilon(\lambda(s)) + \epsilon(\lambda(v \cdot s)), s + v \cdot s)$$

$$= (\epsilon(\lambda(s)), s) + (\epsilon(\lambda(v \cdot s)), v \cdot s)$$

$$= \epsilon_1(s) + \epsilon_1(v \cdot s)$$

$$= \epsilon_1(\alpha_1(v,s)) + \epsilon_1(\beta_1(v,s)).$$

Thus the condition (1) of 3.1.3 of Definition 3.1 in [16] holds.

Similarly, it can be easily shown that the condition (4) of Definition 3.1 holds. Consequently, $(V \times S, \alpha_1, \beta_1, \boxtimes, \epsilon_1, i_1, S)$ is a vector groupoid.

Proposition 4.4. The vector groupoid $V \times S$ is transitive if and only if the action is transitive.

Proof. The proof is straightforward. Namely, $(V \times S)(s, s')$ is non-empty if and only if there is an element v in V such that $v \cdot s = s'$.

The following definition gives the notion of the action of a vector groupoid on another vector groupoid.

Definition 4.5. Let $(V, V^{(0)})$ and $(W, W^{(0)})$ be vector groupoids over a field K and let $\lambda : W \to V^{(0)}$ be a homomorphism, where $V^{(0)}$ is considered as a vector groupoid, consists of units. Then we say that V acts on W via λ if for each $v \in V(x, y)$ and each element a in the vector groupoid $\lambda^{-1}[x]$ there is given an element $v \cdot a$ in the vector groupoid $\lambda^{-1}[y]$ such that the following conditions hold:

i) $v_1 \cdot (v_2 \cdot w) = (v_1 \odot v_2) \cdot w$, for all $v_1, v_2 \in V$ and $w \in W$ ii) $1_{\lambda(w)} \cdot w = w$, for all $w \in W$

iii) $v \cdot (w_2 \boxdot w_1) = (v \cdot w_2) \boxdot (v \cdot w_1)$, for all $v \in V$ and $w_1, w_2 \in W$.

If W is discrete, the above definition coincides with the usual action of a vector groupoid on a vector space.

Remark 4.6. For any vector groupoid V, we can regard $V^{(0)}$ as the discrete vector subgroupoid of V on $V^{(0)}$. Thus, since $V^{(0)}$ is discrete vector subgroupoid of V on $V^{(0)}$ and $\lambda : W \to V^{(0)}$ is a morphism, the vector groupoid W is the sum

$$W = \bigsqcup_{x \in V^{(0)}} W_x = \lambda^{-1}[x]$$

of the vector groupoids $W_x = \lambda^{-1}[x]$ for all $x \in V^{(0)}$.

An element v in V(x, y) defines a morphism $v_{\sharp}: W_x \to W_y$ of vector groupoids such that

$$1_{\sharp} = 1$$
 , $(v_1 \odot v_2)_{\sharp} = v_{1,\sharp} v_{2,\sharp}$

when it can be defined. Thus an action of V on W defines a functor $W': V \to \mathcal{VG}$, where \mathcal{VG} is the category of vector groupoids, by $W'(x) = W_x$ for $x \in V^{(0)}$ and $W'(v) = v_{\sharp}$ for $v \in V$. Conversely, a functor $W': V \to \mathcal{VG}$ defines an action of V on the sum of the vector groupoids $W'(x), x \in V^{(0)}$, in an obvious way.

Definition 4.7. Let V be a vector groupoid with the object space $V^{(0)}$ acting on vector spaces S and T. A linear map $\phi : S \to T$ is Vequivariant if and only if $\lambda_S(s) = \lambda_T(\phi(s))$ and $\phi(v \cdot s) = v \cdot \phi(s)$ for all $s \in S$ and $v \in V_{\lambda(s)}$.

The conditions in the definition are equivalent to the commutativity of the diagrams.



More generally, we can generalize the above definition for two vector groupoids in the following way.



Let V and V' be vector groupoids acting on vector spaces S and S', respectively. Let $f: V \to V'$ be a vector groupoid homomorphism and let $\phi: S \to S'$ be a linear map. In this case, if the diagram is commutative, then ϕ is called an equivariant map.

As with group actions, the action of a vector groupoid on a vector space defines an equivalence relation.

Definition 4.8. Let $(V, V^{(0)})$ be a vector groupoid over a field K and let S be a (left) V-space. Define the orbit equivalence relation on S determined by V to be $s \sim t$ if and only if there exists $v \in V$ such that $v \cdot s = t$. The orbit (or quotient) space with respect to this relation is denoted by S/V, the elements of S/V called the orbits of the action are denoted by $V \cdot s$, and the canonical projection is (often) denoted by π , which assigns to each s in S its orbit. When S is a right V-space, the orbit equivalence relation is defined similarly.

In some cases S will be both a left V-space and a right W-space and in these situations we will denote the orbit space with respect to the V-action by $S \setminus V$ and the orbit space with respect to the W-action by W/S and we will denote elements of the orbit space by $V \cdot s$ and $s \cdot W$, respectively.

Proposition 4.9. Let V be a vector groupoid and S be a V-space. Then the orbit equivalence relation defined in Definition 4.8 is an equivalence relation.

Proof. The proof is straightforward

Proposition 4.10. The canonical projection $\pi : S \to S/V$ is a linear map.

Proof. The proof is straightforward

Therefore, we obtain the category of actions of vector groupoids on vector spaces, which is a subcategory of \mathcal{VG} . We will denote it by \mathcal{VGOp} .

□ ear

5. Crossed module over vector groupoids

In this section we will define the concept of crossed modules over vector groupoids. Later, we will give some examples and results about crossed modules over vector groupoids.

Definition 5.1. A crossed module of vector groupoids consists of a pair of vector groupoids C and V over a common object space such that Cis totally intransitive, together with an action of V on C and a linear functor $\delta : C \to V$ which is the identity on the object space and satisfies CR1. $\delta(v \cdot c) = v^{-1} \odot \delta(c) \odot v$, CR2. $\delta(c_1) \cdot c = c_1^{-1} \odot c \odot c_1$ for $c, c_1 \in C(x, x), v \in V(x, y)$.

A crossed module of vector groupoids will be denoted by $\mathcal{C} = (C, V, \delta)$. Also we recall that the linear functor δ is called boundary map.

Let us now give a main example of a crossed module of vector groupoids.

Example 5.2. Let V be a vector groupoid and let $IV = \bigcup_{x \in V^{(0)}} V(x, x)$

be inner vector subgroupoid of V. Then, if we take the inclusion map $i: IV \to V$ as the boundary map δ , we obtain a crossed module C = (IV, V, i).

Firstly, we must define the action of V on IV. V acts on IV as follows via the partial composition of V:

:
$$V \times IV \to IV, (v, c) \mapsto v \cdot c = v^{-1} \odot c \odot v.$$

Let us now show that the conditions of action are hold.

The first condition is straightforward.

For the second condition, we take any elements $c_1, c_2 \in V(x, x) \subset IV$ and $v \in V(x, y)$. Then,

$$v \cdot (c_1 \odot c_2) = v^{-1} \odot (c_1 \odot c_2) \odot v$$

= $v^{-1} \odot c_1 \odot v \odot v^{-1} \odot c_2 \odot v$
= $v \cdot c_1 \odot v \cdot c_2.$

Thus the second condition of action is hold. Finally, the third condition is hold as follows:

$$(v_1 \odot v_2) \cdot c_1 = (v_1 \odot v_2)^{-1} \odot c_1 \odot (v_1 \odot v_2)$$

$$= v_2^{-1} \odot v_1^{-1} \odot c_1 \odot v_1 \odot v_2$$

$$= v_2^{-1} \odot (v_1 \cdot c_1) \odot v_2$$

$$= v_2 \cdot (v_1 \cdot c_1).$$

and

$$1_x \cdot c_1 = 1_x^{-1} \odot c_1 \odot 1_x = c_1$$

4

Also, it is obvious that the action is linear, since V has a structure of vector space. Therefore, V acts on IV via the composition of V.

Now let us show the conditions of crossed module. CR1.

$$\begin{split} \delta(v \cdot c) &= \delta(v^{-1} \odot c \odot v) = i(v^{-1} \odot c \odot v) \\ &= v^{-1} \odot c \odot v = v^{-1} \odot i(c) \odot v \\ &= v^{-1} \odot \delta(c) \odot v. \end{split}$$

CR2.

$$\delta(c_1) \cdot c = \delta(c_1)^{-1} \odot c \odot \delta(c_1) = i(c_1)^{-1} \odot c \odot i(c_1)$$
$$= c_1^{-1} \odot c \odot c_1.$$

Consequently, C = (IV, V, i) is a crossed module of the vector groupoids.

Definition 5.3. Let $C = (C, V, \delta)$ and $C' = (C', V', \delta')$ be two crossed modules of vector groupoids over a common field K. A linear functor $f = (f_1, f_2) : C \to C'$ is called a homomorphism of crossed modules of vector groupoids, if the linear maps $f_1 : V \to V'$ and $f_2 : C \to C'$ hold $f_1\delta = \delta' f_2$ and $f_2(v \cdot c) = f_1(v) \cdot f_2(c)$.

The conditions in the definition are equivalent to be commutative of the following diagrams:



This yields the category \mathcal{CModVG} of crossed modules of vector groupoids and their homomorphisms.

Theorem 5.4. Let V be a vector groupoid over $V^{(0)}$. Then $(V, V^{(0)})$ induces a crossed module over vector groupoids.

Proof. Let V be a vector groupoid over $V^{(0)}$. We will obtain a crossed module $\mathcal{C} = (C, V, \delta)$ of vector groupoids. For this, we will construct a totally intransitive vector groupoid C and a vector groupoid acting on C, both of which have the common object space. Also we must have a linear functor $\delta : C \to V$.

<u>The construction of C</u>: Let us consider the isotropy groups C(x) for all $x \in V^{(0)}$, i.e. $C(x) = \{v \in V \mid \alpha(v) = \beta(v) = x\}$. Then, from Proposition 3.5.(ii), C(x) has a structure of vector groupoid. Hence we determine the totally intransitive vector groupoid C as $\bigcup_{x \in V^{(0)}} C(x)$. That

is,
$$C = \bigcup_{x \in V^{(0)}} C(x)$$
.

<u>The action of V on C</u>: As shown in Example 5.2, the vector groupoid V acts on the totally intransitive vector groupoid $C = \bigcup_{x \in V^{(0)}} C(x)$.

The boundary map $\delta: C \to V$: We determine the boundary map δ as inclusion map, namely $\delta = i: C = \bigcup_{x \in V^{(0)}} C(x) \to V$ is a homomorphism

of vector groupoids.

After these determinations, we can show that the conditions of crossed module are hold.

CR1.

$$\begin{split} \delta(v \cdot c) &= \delta(v^{-1} \odot c \odot v) = i(v^{-1} \odot c \odot v) \\ &= v^{-1} \odot c \odot v = v^{-1} \odot i(c) \odot v \\ &= v^{-1} \odot \delta(c) \odot v. \end{split}$$

CR2.

$$\delta(c_1) \cdot c = \delta(c_1)^{-1} \odot c \odot \delta(c_1) = i(c_1)^{-1} \odot c \odot i(c_1)$$
$$= c_1^{-1} \odot c \odot c_1.$$

Thus C = (C, V, i) is a crossed module of vector groupoids, which we denote it by ΓV . As a result of this proposition, we observe that this crossed module is entirely contained in the vector groupoid $(V, V^{(0)})$.

Now our aim is to show how a vector groupoid V' can be recovered from the crossed module $\mathcal{C} = \Gamma V$ contained in it. We state this as a theorem.

Theorem 5.5. Let $C = (C, V, \delta)$ be a crossed module of vector groupoids. Then C induces a vector groupoid V'.

Proof. Let $C = (C, V, \delta)$ be a crossed module of vector groupoids. Then, since there exists an action of V on C due to the definition of crossed module of vector groupoids, one can construct the action vector groupoid.

Let V' be the product vector space $V \times C = \{(v, c) \mid v \in V, c \in C(\beta(v))\}.$

Then V' is a vector groupoid with object space C as follows:

We consider the totally intransitive vector groupoid C as a vector space, because C consists of the identity arrows. So we can consider C as the object space of the V'. The structure maps of the V' are as follows:

the source map is defined by $\alpha_1(v,c) = c$,

the target map is defined by $\beta_1(v,c) = v \cdot c$,

the object map is defined by $\varepsilon_1(c) = (1_c, c)$,

the partial composition is defined by $(v, c_1) \boxtimes (w, c_2) = (v \odot w, c_1)$ such that $c_2 = v \cdot c_1$,

and the inverse map is defined by $i_1(v,c) = (v^{-1}, v \cdot c)$, where v^{-1} denotes the inverse element of v in the vector groupoid V.

The conditions being a vector groupoid are easily shown as in Example 4.3. $\hfill \Box$

As a main result of this work, we state the following corollary which is the conclusion from the Theorems 5.4 and 5.5.

Corollary 5.6. The category \mathcal{CModVG} of crossed modules of vector groupoids and the category \mathcal{VG} of vector groupoids are equivalent.

Proof. Firstly, let us define a functor $\Gamma : \mathcal{VG} \to \mathcal{CModVG}$ in the following way. Let (V, V_0) be a vector groupoid. From Theorem 5.4, there exists a crossed module of vector groupoids denoted by $\Gamma V = (C, V, i)$. Thus ΓV is a crossed module of vector groupoids.

Secondly, let us define a functor $\Phi : \mathcal{CModVG} \to \mathcal{VG}$ in the following way. For any crossed module $\mathcal{C} = (C, V, \delta)$ of vector groupoids, by the Theorem 5.5, we obtain the action vector groupoid $V \times C$ with object space C. Thus $\Phi \mathcal{C}$ is a vector groupoid.

It is obvious that $\Phi\Gamma = 1_{\mathcal{VG}}$ and $\Gamma\Phi = 1_{\mathcal{CModVG}}$.

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