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COHEN-MACAULAY r-PARTITE GRAPHS WITH MINIMAL CLIQUE COVER

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Abstract. In this paper, we give some necessary conditions for an r-partite graph such that the edge ring of the graph is Cohen-Macaulay. It is proved that if there exists a cover of an r-partite Cohen-Macaulay graph by disjoint cliques of size r, then such a cover is unique.
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MSC(2010): Cohen-Macaulay graph, r-partite, clique cover, perfect r-matching.

1. Introduction

Mainly, after using the notion of simplicial complexes and its algebraic interpretation by Stanley in 1970s to prove the upper bound conjecture for number of simplicial spheres [10], this notion has been one of the main streams of research in commutative algebra. In this stream, characterization and classification of Cohen-Macaulay simplicial complexes have been extensively studied in last decades. It is known that the Cohen-Macaulay property of a simplicial complex and complement of its comparability graph coincide [8]. Therefore, to characterize all simplicial complexes which are Cohen-Macaulay, it is enough to characterize all graphs with this property [10].

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To examine special classes of graphs, Estrada and Villarreal in [3] found some necessary conditions for bipartite graphs to be Cohen-Macaulay. Finally, Herzog and Hibi in [5] presented a combinatorial characterization for bipartite graphs equivalent to the Cohen-Macaulay property of these graphs. This purely combinatorial method can not be generalized for r-partite graphs in general. Because, as shown in Example 2.3, the Cohen-Macaulay property may depend on characteristics of the base field. In this paper, we consider r-partite graphs with a minimal clique cover and find a necessary condition for Cohen-Macaualay property of these graphs. More precisely, we prove that in a Cohen-Macaulay r-partite graph with a minimal clique cover, there is a vertex of degree $r-1$ and the cover is unique.

2. **Preliminaries**

A simple graph is an undirected graph with no loop or multiple edge. A finite graph is denoted by $G = (V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. Let $|V(G)| = n$. We use $[n] = \{1, 2, \ldots, n\}$ as vertices of G. The complementary graph of G is the graph $\bar{G}$ on $[n]$ whose edge set $E(\bar{G})$ consists of those edges $\{i, j\}$ which are not in $E(G)$. An independent set of vertices is a set of pairwise nonadjacent vertices. An r-partite graph is a graph whose set of vertices can be partitioned into r disjoint subsets such that each set is independent. A subset $A \subset [n]$ is called a minimal vertex cover of G if (i) each edge of G is incident with at least one vertex in A, and (ii) there is no proper subset of A with property (i). It is easy to check that any minimal vertex cover of a graph is the complement set of a maximal independent set of the graph. A graph G is called unmixed (well-covered) if any two minimal vertex covers of G have the same cardinality. A clique in a graph is a set of pairwise adjacent vertices, and by an r-clique we mean a clique of size r. An r-matching in G is a set of pairwise disjoint r-cliques in G and a perfect r-matching is an r-matching which covers all vertices of G.

Let $\omega(G)$ denote the maximum size of cliques in G, which is called the clique number of G. Let $f : V(G) \to [k]$ be a map such that if v_1 is adjacent to v_2 then $f(v_1) \neq f(v_2)$. If such a map exists, we say that G is colorable by k colors. The smallest such k is called the chromatic number of the graph and is denoted by $\chi(G)$. A graph G is called perfect if $\omega(H) = \chi(H)$ for each induced subgraph H of G. The class of perfect graphs plays an important role in graph theory and most of computations in this class can be done by fast algorithms. L. Lovász in [9] has proved that a graph is perfect if and only if its complement is perfect. M. Chudnovsky et al in [2] have proved that a necessary and sufficient condition for a graph G to be perfect is that G does not have an odd hole (a cycle of odd length greater than 3) or an odd anti-hole (complement of an odd hole) as induced subgraph.

Let G be a graph on $[n]$. Let $S = K[x_1, \ldots, x_n]$, the polynomial ring over a field K. The edge ideal $I(G)$ of G is defined to be the ideal of S generated by all square-free monomials $x_i x_j$ provided that i is adjacent to j in G. The quotient ring $R(G) = S/I(G)$ is called the edge ring of G.

Let R be a commutative ring with an identity. The depth of R, denoted by $\text{depth}(R)$, is the largest integer r such that there is a sequence $f_1, \ldots, f_r$ of elements of R such that f_i is not a zero-divisor in $R/(f_1, \ldots, f_{i-1})$ for all $1 \leq i \leq r$, and $(f_1, \ldots, f_r) \neq R$. Such a sequence is called a regular sequence. The depth is an important invariant of a ring. It is bounded by another important invariant, the Krull dimension, the length of the longest chain of prime ideals in the ring. A ring R is called Cohen-Macaulay if $\text{depth}(R) = \dim(R)$. A graph G is called Cohen-Macaulay if the ring $R(G)$ is Cohen-Macaulay.

Theorem 2.1. [11, Proposition 6.1.21] *If G is a Cohen-Macaulay graph, then G is unmixed.*

A simplicial complex Δ on n vertices is a collection of subsets of $[n]$ such that the following conditions hold:
(i) $\{i\} \in \Delta$ for each $i \in [n]$,
(ii) if $E \in \Delta$ and $F \subseteq E$ then $F \in \Delta$.
An element of Δ is called a face and a maximal face with respect to inclusion is called a facet. The set of all facets of Δ is denoted by $\mathcal{F}(\Delta)$. The dimension of a face $F \in \Delta$ is defined to be $|F| - 1$ and dimension of Δ is the maximum of dimensions of its faces. A simplicial complex is called pure if all of its facets have the same dimension. For more details on simplicial complexes see [10].

The clique complex of a finite graph G on $[n]$ is the simplicial complex $\Delta(G)$ on $[n]$ whose faces are cliques of G. Let Δ be a simplicial complex on $[n]$. We say that Δ is shellable if its facets can be ordered as $F_1, F_2, \ldots, F_m$ such that for all $j \geq 2$ the subcomplex $(F_1, \ldots, F_{j-1}) \cap F_j$ is pure of dimension $\dim F_j - 1$. An order of the facets satisfying this condition is called a shelling order. To say that $F_1, F_2, \ldots, F_m$ is a shelling

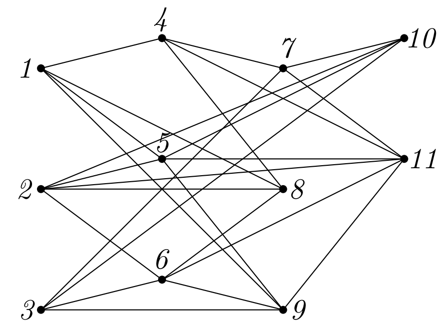

FIGURE 1. Cohen-Macaulay property depends on characteristic

order of Δ is equivalent to say that for all i, $2 \leq i \leq m$ and all $j < i$, there exists $l \in F_i \setminus F_j$ and $k < i$ such that $F_i \setminus F_k = \{l\}$. A graph G is called shellable if $\Delta(G)$ is a shellable simplicial Complex.

Let Δ be a simplicial complex on $[n]$ and I_Δ be the ideal of $S = K[x_1, \dots, x_n]$ generated by all square-free monomials $x_{i_1} \cdots x_{i_t}$, provided that $\{i_1, \dots, i_t\}$ is not a face of Δ. The ring S/I_Δ is called the Stanley-Reisner ring of Δ. A simplicial complex is called Cohen-Macaulay if its Stanley-Reisner ring is Cohen-Macaulay.

Theorem 2.2. [6, Theorem 8.2.6] *If Δ is a pure and shellable simplicial complex, then Δ is Cohen-Macaulay.*

Estrada and Villarreal in [3] have proved that for a bipartite graph G the Cohen-Macaulay property and pure shellability are equivalent. This is not true in general for r-partite graphs when $r > 2$ (Example 2.3).

Also in bipartite graphs, Cohen-Macaulay property does not depend on characteristic of the ground field. But again, this is not true in general as shown in the following example.

Example 2.3. *Let G be the graph in Figure 1. Then, $R(G)$ is Cohen-Macaulay when the characteristic of the ground field K is zero but it is not Cohen-Macaulay in characteristic 2. Therefore the graph G is not shellable ([7]).*

3. **The Cohen-Macaulay property and uniqueness of perfect r-matching**

M. Estrada and R. H. Villarreal in [3] have proved that if G is a Cohen-Macaulay bipartite graph and has at least one vertex of positive degree, then there is a vertex v such that $\deg(v) = 1$. By $\deg(v)$ we mean the number of vertices adjacent to v. J. Herzog and T. Hibi in [5] have proved that a bipartite graph G with parts V and W is Cohen-Macaulay if and only if, $|V| = |W|$ and there is an order on the vertices of V and W as $v_1, \dots, v_n$ and $w_1, \dots, w_n$ respectively, such that:
1) $v_i \sim w_i$ for $i = 1, \dots, n$,
2) if $v_i \sim w_j$, then $i \leq j$,
3) for each $1 \leq i < j < k \leq n$ if $v_i \sim w_j$ and $v_j \sim w_k$, then $v_i \sim w_k$.

R. Zaare-Nahandi in [12] has proved that a well-covered bipartite graph G is Cohen-Macaulay if and only if there is a unique perfect 2-matching in G.

Let $\alpha(G)$ denote the maximum cardinality of independent sets of vertices of G. Let $\mathcal{G}$ be the class of graphs such that for each $G \in \mathcal{G}$ there are $k = \alpha(G)$ cliques in G covering all its vertices. For each $G \in \mathcal{G}$ and cliques $Q_1, \dots, Q_k$ such that $V(Q_1) \cup \cdots \cup V(Q_k) = V(G)$, we may take ${Q'}_1 = Q_1$ and for $i = 2, \dots, k$, ${Q'}_i$ the induced subgraph on the vertices $V(Q_i) \setminus (V(Q_1) \cup \cdots \cup V(Q_{i-1}))$. Then ${Q'}_1, \dots, {Q'}_k$ are k disjoint cliques covering all vertices of G. We call such a set of cliques, a basic clique cover of the graph G. Therefore any graph in the class $\mathcal{G}$ has a basic clique cover.

Proposition 3.1. *Let G be an r-partite, unmixed and perfect graph such that all maximal cliques are of size r. Then G is in the class $\mathcal{G}$.*

Proof. Let $V_1, \dots, V_r$ be parts of G. By [13], $|V_1| = |V_2| = \cdots = |V_r| = \alpha(G)$. Also by [9], the complement graph $\bar{G}$ is perfect. On the other hand, V_i is a clique of maximal size in $\bar{G}$ for each $1 \leq i \leq r$. Therefore, $\chi(\bar{G}) = \omega(\bar{G}) = \alpha(G)$. This implies that $\bar{G}$ is $\alpha(G)$-partite. Therefore there are $\alpha(G)$ disjoint maximal cliques in G covering all vertices. $\square$

The converse of the above proposition is not true as the following example shows.

Example 3.2. Let G be the graph in Figure 2. Then G is a graph in class $\mathcal{G}$ which is 4-partite, unmixed and all maximal cliques are of size

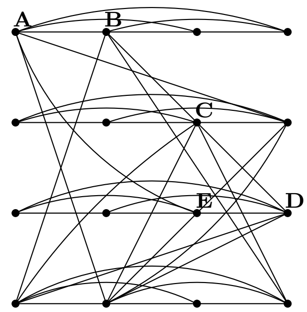

FIGURE 2. A graph in class $\mathcal{G}$ which is not perfect

4. But the induced subgraph on $\{A, B, C, D, E\}$ is a cycle of length 5 and therefore, by [2], the graph G is not perfect.

An easy computation by Singular [4] shows that the dimension and the depth of the edge ring of G are both 4 and therefore, G is Cohen-Macaulay.

Let H be a graph and v be a vertex of H. Let $N(v)$ be the set of all vertices of H adjacent to v.

Theorem 3.3. [11, Proposition 6.2.4] *If H is Cohen-Macaulay and v is a vertex of H, then $H \setminus (v, N(v))$ is Cohen-Macaulay.*

Theorem 3.4. [13] *Let G be an r-partite unmixed graph such that all maximal cliques are of size r. Then all parts have the same cardinality and there is a perfect 2-matching between each two parts.*

Now, we present the main theorem of this paper which is a generalization of [3, Theorem 2.4].

Theorem 3.5. *Let G be an r-partite graph in the class $\mathcal{G}$ such that each maximal clique is of size r. If G is Cohen-Macaulay then there is a vertex of degree $r-1$ in G.*

Proof. By Theorem 3.4 all parts have the same cardinality. So there is a positive integer n such that $|V| = rn$. Assume that for all vertices v in G we have $\deg(v) \geq r$. Let $Q_i = \{x_{1i}, x_{2i}, \ldots, x_{ri}\}$ for $i = 1, \ldots, n$ are cliques in a basic clique cover of G. Without loss of generality, assume that v_{11} be a vertex of the minimal degree. If $\deg(v_{11}) = (r-1)n$ then

$G = K_{n,n,\dots,n}$ is a complete $r-$partite graph. Thus G is not Cohen-Macaulay by [1, Exercise 5.1.26] and we get a contradiction. Therefore, $r \le \deg(v_{11}) \le (r-1)n - 1$.

Let $N(v_{11}) = \{v_{21}, \dots, v_{2l_2}, v_{31}, \dots, v_{3l_3}, \dots, v_{r1}, \dots, v_{rl_r}\}$. We have $\deg(v_{11}) = l_2 + \cdots + l_r$. Without loss of generality, we may assume that $l_2 \le l_i$ for $i = 3, \dots, r$. Set $G' = G \setminus (\{v_{11}\}, N(v_{11}))$. The graph G' is Cohen-Macaulay by Theorem 3.3. If $l_2 = 1$, then, there exists $3 \le i \le r$ such that $l_i \ge 2$. The sets

$$\{v_{12}, \dots, v_{1n}, v_{22}, \dots, v_{2n}, v_{3(l_3+1)}, \dots, v_{3n}, \dots, \widehat{(v_{i(l_i+1)}, \dots, v_{in})}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\}$$

and

$$\{v_{12}, \dots, v_{1n}, v_{3(l_3+1)}, \dots, v_{3n}, \dots, v_{i(l_i+1)}, \dots, v_{in}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\}$$

are two minimal vertex covers for G' and their cardinalities are not equal. Here, by $\widehat{(v_{i(l_i+1)}, \dots, v_{in})}$ we mean the vertices $v_{i(l_i+1)}, \dots, v_{in}$ are removed from the set. This contradicts to Cohen-Macaulay property of G'. Therefore, $l_2 \ge 2$. We claim that

$$\deg(v_{1i}) = l_2 + l_3 + \cdots + l_r = \deg(v_{11}), \qquad i = 1, \dots, l_2.$$

It is enough to show that $\deg(v_{12}) = l_2 + l_3 + \cdots + l_r$ and analogous argument proves the claim. If $\deg(v_{12}) > l_2 + l_3 + \cdots + l_r$, then there is a j_t, $l_t + 1 \le j_t \le n$ for some $2 \le t \le r$, such that $v_{12} \sim v_{tj_t}$. Without loss of generality we assume that $t = 2$.

If there is j_2, $l_2 + 1 \le j_2 \le n$, such that $v_{12} \sim v_{2j_2}$ then there is a minimal vertex cover for G' containing the set

$$\{v_{12}, v_{1(l_2+1)}, \dots, v_{1n}, v_{3(l_3+1)}, \dots, v_{3n}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\}.$$

On the other hand, $\{v_{2(l_2+1)}, \dots, v_{2n}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\}$ is a minimal vertex cover of G'. By $l_2 \ge 2$ and Theorem 2.1, this contradicts the Cohen-Macaulay property of G'. Therefore $\deg(v_{12}) = l_2 + l_3 + \cdots + l_r$. Thus, for all $1 \le i \le l_2$ we have $N(v_{1i}) = \{v_{21}, \dots, v_{2l_2}, v_{31}, \dots, v_{3l_3}, \dots, v_{r1}, \dots, v_{rl_r}\}$. Consider the graph $H = G \setminus \big(\{v_{2(l_2+1)}, \dots, v_{2n}, \dots, v_{r(l_r+1)}, \dots, v_{rn}\} \cup N(v_{2(l_2+1)}) \cup \cdots \cup N(v_{2n}) \cup \cdots \cup N(v_{r(l_r+1)}) \cup \cdots \cup N(v_{rn})\big)$. By Theorem 3.3, H is Cohen-Macaulay but the complement of H is not connected. This is a contradiction by [1, Exercise 5.1.26]. □

Theorem 3.5 implies that the perfect r-matching in a Cohen-Macaulay r-partite graph is unique.

Corollary 3.6. *Let G be an r-partite graph in the class $\mathcal{G}$ such that all maximal cliques are of size r. If G is Cohen-Macaulay, then there is a unique perfect r-matching in G.*

Proof. Since G is in the class $\mathcal{G}$, there is a perfect r-matching in G. By Theorem 3.5, there is a vertex $v \in V(G)$ of degree $r-1$. Therefore, the r-clique in the r-matching which contains v, must be in all perfect r-matchings of G. The graph $G \setminus (\{v, N(v)\})$ is again an r-partite graph in the class $\mathcal{G}$ which is Cohen-Macaulay by Theorem 3.3. Continuing this process, we find that the chosen perfect r-matching is the unique perfect r-matching in G. □

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