Title:
Cohen-Macaulay $r$-partite graphs with minimal clique cover

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ABSTRACT. In this paper, we give some necessary conditions for an r-partite graph such that the edge ring of the graph is Cohen-Macaulay. It is proved that if there exists a cover of an r-partite Cohen-Macaulay graph by disjoint cliques of size r, then such a cover is unique.

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1. Introduction

Mainly, after using the notion of simplicial complexes and its algebraic interpretation by Stanley in 1970s to prove the upper bound conjecture for number of simplicial spheres [10], this notion has been one of the main streams of research in commutative algebra. In this stream, characterization and classification of Cohen-Macaulay simplicial complexes have been extensively studied in last decades. It is known that the Cohen-Macaulay property of a simplicial complex and complement of its comparability graph coincide [8]. Therefore, to characterize all simplicial complexes which are Cohen-Macaulay, it is enough to characterize all graphs with this property [10].

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To examine special classes of graphs, Estrada and Villarreal in [3] found some necessary conditions for bipartite graphs to be Cohen-Macaulay. Finally, Herzog and Hibi in [5] presented a combinatorial characterization for bipartite graphs equivalent to the Cohen-Macaulay property of these graphs. This purely combinatorial method can not be generalized for $r$-partite graphs in general. Because, as shown in Example 2.3, the Cohen-Macaulay property may depend on characteristics of the base field. In this paper, we consider $r$-partite graphs with a minimal clique cover and find a necessary condition for Cohen-Macaulay property of these graphs. More precisely, we prove that in a Cohen-Macaulay $r$-partite graph with a minimal clique cover, there is a vertex of degree $r - 1$ and the cover is unique.

2. Preliminaries

A simple graph is an undirected graph with no loop or multiple edge. A finite graph is denoted by $G = (V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. Let $|V(G)| = n$. We use $[n] = \{1, 2, \ldots, n\}$ as vertices of $G$. The complementary graph of $G$ is the graph $\overline{G}$ on $[n]$ whose edge set $\overline{E}(G)$ consists of those edges $\{i, j\}$ which are not in $E(G)$. An independent set of vertices is a set of pairwise nonadjacent vertices. An $r$-partite graph is a graph whose set of vertices can be partitioned into $r$ disjoint subsets such that each set is independent. A subset $A \subset [n]$ is called a minimal vertex cover of $G$ if (i) each edge of $G$ is incident with at least one vertex in $A$, and (ii) there is no proper subset of $A$ with property (i). It is easy to check that any minimal vertex cover of a graph is the complement set of a maximal independent set of the graph. A graph $G$ is called unmixed (well-covered) if any two minimal vertex covers of $G$ have the same cardinality. A clique in a graph is a set of pairwise adjacent vertices, and by an $r$-clique we mean a clique of size $r$. An $r$-matching in $G$ is a set of pairwise disjoint $r$-cliques in $G$ and a perfect $r$-matching is an $r$-matching which covers all vertices of $G$.

Let $\omega(G)$ denote the maximum size of cliques in $G$, which is called the clique number of $G$. Let $f : V(G) \to [k]$ be a map such that if $v_1$ is adjacent to $v_2$ then $f(v_1) \neq f(v_2)$. If such a map exists, we say that $G$ is colorable by $k$ colors. The smallest such $k$ is called the chromatic number of the graph and is denoted by $\chi(G)$. A graph $G$ is called perfect if $\omega(H) = \chi(H)$ for each induced subgraph $H$ of $G$. The class
of perfect graphs plays an important role in graph theory and most of computations in this class can be done by fast algorithms. L. Lovász in [9] has proved that a graph is perfect if and only if its complement is perfect. M. Chudnovsky et al in [2] have proved that a necessary and sufficient condition for a graph $G$ to be perfect is that $G$ does not have an odd hole (a cycle of odd length greater than 3) or an odd anti-hole (complement of an odd hole) as induced subgraph.

Let $G$ be a graph on $[n]$. Let $S = K[x_1, \ldots, x_n]$, the polynomial ring over a field $K$. The edge ideal $I(G)$ of $G$ is defined to be the ideal of $S$ generated by all square-free monomials $x_i x_j$ provided that $i$ is adjacent to $j$ in $G$. The quotient ring $R(G) = S/I(G)$ is called the edge ring of $G$.

Let $R$ be a commutative ring with an identity. The depth of $R$, denoted by $\text{depth}(R)$, is the largest integer $r$ such that there is a sequence $f_1, \ldots, f_r$ of elements of $R$ such that $f_i$ is not a zero-divisor in $R/(f_1, \ldots, f_{i-1})$ for all $1 \leq i \leq r$, and $(f_1, \ldots, f_r) \neq R$. Such a sequence is called a regular sequence. The depth is an important invariant of a ring. It is bounded by another important invariant, the Krull dimension, the length of the longest chain of prime ideals in the ring. A ring $R$ is called Cohen-Macaulay if $\text{depth}(R) = \text{dim}(R)$. A graph $G$ is called Cohen-Macaulay if the ring $R(G)$ is Cohen-Macaulay.

**Theorem 2.1.** [11, Proposition 6.1.21] If $G$ is a Cohen-Macaulay graph, then $G$ is unmixed.

A simplicial complex $\Delta$ on $n$ vertices is a collection of subsets of $[n]$ such that the following conditions hold:

(i) $\{i\} \in \Delta$ for each $i \in [n]$,
(ii) if $E \in \Delta$ and $F \subseteq E$ then $F \in \Delta$.

An element of $\Delta$ is called a face and a maximal face with respect to inclusion is called a facet. The set of all facets of $\Delta$ is denoted by $\mathcal{F}(\Delta)$. The dimension of a face $F \in \Delta$ is defined to be $|F| - 1$ and dimension of $\Delta$ is the maximum of dimensions of its faces. A simplicial complex is called pure if all of its facets have the same dimension. For more details on simplicial complexes see [10].

The clique complex of a finite graph $G$ on $[n]$ is the simplicial complex $\Delta(G)$ on $[n]$ whose faces are cliques of $G$. Let $\Delta$ be a simplicial complex on $[n]$. We say that $\Delta$ is shellable if its facets can be ordered as $F_1, F_2, \ldots, F_m$ such that for all $j \geq 2$ the subcomplex $(F_1, \ldots, F_{j-1}) \cap F_j$ is pure of dimension $\text{dim} F_j - 1$. An order of the facets satisfying this condition is called a shelling order. To say that $F_1, F_2, \ldots, F_m$ is a shelling
order of $\Delta$ is equivalent to say that for all $i$, $2 \leq i \leq m$ and all $j < i$, there exists $l \in F_i \setminus F_j$ and $k < i$ such that $F_i \setminus F_k = \{l\}$. A graph $G$ is called shellable if $\Delta(G)$ is a shellable simplicial complex.

Let $\Delta$ be a simplicial complex on $[n]$ and $I_\Delta$ be the ideal of $S = K[x_1, \ldots, x_n]$ generated by all square-free monomials $x_{i_1} \cdots x_{i_t}$, provided that $\{i_1, \ldots, i_t\}$ is not a face of $\Delta$. The ring $S/I_\Delta$ is called the Stanley-Reisner ring of $\Delta$. A simplicial complex is called Cohen-Macaulay if its Stanley-Reisner ring is Cohen-Macaulay.

**Theorem 2.2.** [6, Theorem 8.2.6] If $\Delta$ is a pure and shellable simplicial complex, then $\Delta$ is Cohen-Macaulay.

Estrada and Villarreal in [3] have proved that for a bipartite graph $G$ the Cohen-Macaulay property and pure shellability are equivalent. This is not true in general for $r$-partite graphs when $r > 2$ (Example 2.3).

Also in bipartite graphs, Cohen-Macaulay property does not depend on characteristic of the ground field. But again, this is not true in general as shown in the following example.

**Example 2.3.** Let $G$ be the graph in Figure 1. Then, $R(G)$ is Cohen-Macaulay when the characteristic of the ground field $K$ is zero but it is not Cohen-Macaulay in characteristic 2. Therefore the graph $G$ is not shellable ([7]).
3. The Cohen-Macaulay property and uniqueness of perfect $r$-matching

M. Estrada and R. H. Villarreal in [3] have proved that if $G$ is a Cohen-Macaulay bipartite graph and has at least one vertex of positive degree, then there is a vertex $v$ such that $\text{deg}(v) = 1$. By $\text{deg}(v)$ we mean the number of vertices adjacent to $v$. J. Herzog and T. Hibi in [5] have proved that a bipartite graph $G$ with parts $V$ and $W$ is Cohen-Macaulay if and only if, $|V| = |W|$ and there is an order on the vertices of $V$ and $W$ as $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ respectively, such that:
1) $v_i \sim w_i$ for $i = 1, \ldots, n$,
2) if $v_i \sim w_j$, then $i \leq j$,
3) for each $1 \leq i < j < k \leq n$ if $v_i \sim w_j$ and $v_j \sim w_k$, then $v_i \sim w_k$.

R. Zaare-Nahandi in [12] has proved that a well-covered bipartite graph $G$ is Cohen-Macaulay if and only if there is a unique perfect 2-matching in $G$.

Let $(G)$ denote the maximum cardinality of independent sets of vertices of $G$. Let $\mathcal{G}$ be the class of graphs such that for each $G \in \mathcal{G}$ there are $k = \alpha(G)$ cliques in $G$ covering all its vertices. For each $G \in \mathcal{G}$ and cliques $Q_1, \ldots, Q_k$ such that $V(Q_1) \cup \cdots \cup V(Q_k) = V(G)$, we may take $Q'_1 = Q_1$ and for $i = 2, \ldots, k$, $Q'_i$ the induced subgraph on the vertices $V(Q_i) \setminus (V(Q_1) \cup \cdots \cup V(Q_{i-1}))$. Then $Q'_1, \ldots, Q'_k$ are $k$ disjoint cliques covering all vertices of $G$. We call such a set of cliques, a basic clique cover of the graph $G$. Therefore any graph in the class $\mathcal{G}$ has a basic clique cover.

**Proposition 3.1.** Let $G$ be an $r$-partite, unmixed and perfect graph such that all maximal cliques are of size $r$. Then $G$ is in the class $\mathcal{G}$.

**Proof.** Let $V_1, \ldots, V_r$ be parts of $G$. By [13], $|V_1| = |V_2| = \cdots = |V_r| = \alpha(G)$. Also by [9], the complement graph $\bar{G}$ is perfect. On the other hand, $V_i$ is a clique of maximal size in $\bar{G}$ for each $1 \leq i \leq r$. Therefore, $\chi(\bar{G}) = \omega(\bar{G}) = \alpha(G)$. This implies that $\bar{G}$ is $\alpha(G)$-partite. Therefore there are $\alpha(G)$ disjoint maximal cliques in $\bar{G}$ covering all vertices. □

The converse of the above proposition is not true as the following example shows.

**Example 3.2.** Let $G$ be the graph in Figure 2. Then $G$ is a graph in class $\mathcal{G}$ which is 4-partite, unmixed and all maximal cliques are of size...
4. But the induced subgraph on \{A, B, C, D, E\} is a cycle of length 5 and therefore, by [2], the graph \(G\) is not perfect.

An easy computation by Singular [4] shows that the dimension and the depth of the edge ring of \(G\) are both 4 and therefore, \(G\) is Cohen-Macaulay.

Let \(H\) be a graph and \(v\) be a vertex of \(H\). Let \(N(v)\) be the set of all vertices of \(H\) adjacent to \(v\).

**Theorem 3.3.** [11, Proposition 6.2.4] If \(H\) is Cohen-Macaulay and \(v\) is a vertex of \(H\), then \(H \setminus (v, N(v))\) is Cohen-Macaulay.

**Theorem 3.4.** [13] Let \(G\) be an \(r\)-partite unmixed graph such that all maximal cliques are of size \(r\). Then all parts have the same cardinality and there is a perfect 2-matching between each two parts.

Now, we present the main theorem of this paper which is a generalization of [3, Theorem 2.4].

**Theorem 3.5.** Let \(G\) be an \(r\)-partite graph in the class \(G\) such that each maximal clique is of size \(r\). If \(G\) is Cohen-Macaulay then there is a vertex of degree \(r - 1\) in \(G\).

**Proof.** By Theorem 3.4 all parts have the same cardinality. So there is a positive integer \(n\) such that \(|V| = rn\). Assume that for all vertices \(v\) in \(G\) we have \(\deg(v) \geq r\). Let \(Q_i = \{x_{1i}, x_{2i}, \ldots, x_{ri}\}\) for \(i = 1, \ldots, n\) are cliques in a basic clique cover of \(G\). Without loss of generality, assume that \(v_{11}\) be a vertex of the minimal degree. If \(\deg(v_{11}) = (r - 1)n\) then
$G = K_{n,n,...,n}$ is a complete $r$–partite graph. Thus $G$ is not Cohen-Macaulay by [1, Exercise 5.1.26] and we get a contradiction. Therefore, $r \leq \deg(v_{11}) \leq (r-1)n - 1$.

Let $N(v_{11}) = \{v_{21}, \ldots, v_{2l_2}, v_{31}, \ldots, v_{3l_3}, \ldots, v_{rl_r}\}$. We have $\deg(v_{11}) = l_2 + \cdots + l_r$. Without loss of generality, we may assume that $l_2 \leq l_i$ for $i = 3, \ldots, r$. Set $G' = G \setminus \{v_{11}, N(v_{11})\}$. The graph $G'$ is Cohen-Macaulay by Theorem 3.3. If $l_2 = 1$, then, there exists $3 \leq i \leq r$ such that $l_i \geq 2$. The sets

$$\{v_{12}, \ldots, v_{1n}, v_{22}, \ldots, v_{2n}, v_{3l_3+1}, \ldots, v_{3n}, \ldots, \overbrace{(v_{i(l_i+1)}, \ldots, v_{in})}^{v_{r(l_r+1)}, \ldots, v_{rn}} \}$$

and

$$\{v_{12}, \ldots, v_{1n}, v_{3(l_3+1)}, \ldots, v_{3n}, \ldots, v_{i(l_i+1)}, \ldots, v_{in}, \ldots, v_{r(l_r+1)}, \ldots, v_{rn} \}$$

are two minimal vertex covers for $G'$ and their cardinalities are not equal. Here, by $(v_{i(l_i+1)}, \ldots, v_{in})$ we mean the vertices $v_{i(l_i+1)}, \ldots, v_{in}$ are removed from the set. This contradicts to Cohen-Macaulay property of $G'$. Therefore, $l_2 \geq 2$. We claim that

$$\deg(v_{1i}) = l_2 + l_3 + \cdots + l_r = \deg(v_{11}), \quad i = 1, \ldots, l_2.$$ 

It is enough to show that $\deg(v_{12}) = l_2 + l_3 + \cdots + l_r$ and analogous argument proves the claim. If $\deg(v_{12}) > l_2 + l_3 + \cdots + l_r$, then there is a $j_t$, $l_t + 1 \leq j_t \leq n$ for some $2 \leq t \leq r$, such that $v_{12} \sim v_{1j_t}$. Without loss of generality we assume that $t = 2$.

If there is $j_2$, $l_2 + 1 \leq j_2 \leq n$, such that $v_{12} \sim v_{2j_2}$ then there is a minimal vertex cover for $G'$ containing the set

$$\{v_{12}, v_{1(l_2+1)}, \ldots, v_{1n}, v_{3(l_3+1)}, \ldots, v_{3n}, \ldots, v_{r(l_r+1)}, \ldots, v_{rn}\}.$$ 

On the other hand, $\{v_{2(l_2+1)}, \ldots, v_{2n}, \ldots, v_{r(l_r+1)}, \ldots, v_{rn}\}$ is a minimal vertex cover of $G'$. By $l_2 \geq 2$ and Theorem 2.1, this contradicts the Cohen-Macaulay property of $G'$. Therefore $\deg(v_{12}) = l_2 + l_3 + \cdots + l_r$.

Thus, for all $1 \leq i \leq l_2$ we have $N(v_{1i}) = \{v_{21}, \ldots, v_{2l_2}, v_{31}, \ldots, v_{3l_3}, \ldots, v_{r1}, \ldots, v_{rl_r}\}$. Consider the graph $H = G \setminus \{v_{2(l_2+1)}, \ldots, v_{2n}, \ldots, v_{r(l_r+1)}, \ldots, v_{rn} \} \cup N(v_{2(l_2+1)}) \cup \cdots \cup N(v_{2n}) \cup \cdots \cup N(v_{r(l_r+1)}) \cup \cdots \cup N(v_{rn})$.

By Theorem 3.3, $H$ is Cohen-Macaulay but the complement of $H$ is not connected. This is a contradiction by [1, Exercise 5.1.26].

Theorem 3.5 implies that the perfect $r$-matching in a Cohen-Macaulay $r$-partite graph is unique.
Corollary 3.6. Let $G$ be an $r$-partite graph in the class $G$ such that all maximal cliques are of size $r$. If $G$ is Cohen-Macaulay, then there is a unique perfect $r$-matching in $G$.

Proof. Since $G$ is in the class $G$, there is a perfect $r$-matching in $G$. By Theorem 3.5, there is a vertex $v \in V(G)$ of degree $r - 1$. Therefore, the $r$-clique in the $r$-matching which contains $v$, must be in all perfect $r$-matchings of $G$. The graph $G \setminus \{v, N(v)\}$ is again an $r$-partite graph in the class $G$ which is Cohen-Macaulay by Theorem 3.3. Continuing this process, we find that the chosen perfect $r$-matching is the unique perfect $r$-matching in $G$. □

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