Some Results on \( p \)-Best Approximation in Vector Spaces

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Abstract. The purpose of this paper is to introduce and to discuss the concept of \( p \)-approximation and \( p \)-orthogonality in vector spaces, and to obtain some results on \( p \)-orthogonality in vector spaces similar to some well known results on the orthogonality in normed spaces. We also discuss the concept of \( p \)-extension of linear functionals on a vector space, and give a characterization of linear functionals on a subspace having a unique \( p \)-extension Hahn-Banach to the whole vector space.

1. Introduction

Here, all normed spaces under consideration are real. A seminorm is a function \( p : X \rightarrow [0, \infty) \) such that \( p(x + y) \leq p(x) + p(y) \) and \( p(\alpha x) = |\alpha|p(x) \), for all \( x, y \in X \) and \( \alpha \in \mathbb{R} \). It is clear that for every seminorm \( p \), \( p(0) = 0 \). Also, the seminorm \( p \) is a norm, if \( p(x) = 0 \) implies \( x = 0 \).

Many authors have introduced the concept of orthogonality in different ways (see [1-4], [6]). In [1], Birkhoff modified the concept of orthogonality. By his definition, if \( X \) is a normed linear space and \( x, y \in X \), \( x \) is said to be orthogonal to \( y \) and is denoted by \( x \perp y \) if and only if


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\[ \|x\| \leq \|x + \alpha y\|, \text{ for all scalars } \alpha. \] Note that this orthogonality is not symmetric in general [2].

Let \( X \) be such a vector space, \( x, y \in X \) and \( p \) be a fixed seminorm. We say that \( x \) is \( p \)-orthogonal to \( y \) if \( x = 0 \) or else,

\[ p(x) \neq 0, \quad p(x) = \inf \alpha p(x + \alpha y), \]

in which case we write \( x \perp^p y \). If \( M_1 \) and \( M_2 \) are subsets of \( X \), then we say that \( M_1 \) is \( p \)-orthogonal to \( M_2 \) if \( g_1 \perp^p g_2 \), for all \( g_1 \in M_1, \ g_2 \in M_2 \).

First we state the following lemma of Hahn- Banach which is needed in the proof of the main results.

**Lemma 1.1.** [5] Let \( M \) be a subspace of a vector space \( X \), \( p \) be a seminorm on \( X \), and let \( f \) be a linear functional on \( M \) such that

\[ |f(x)| \leq p(x) \quad (x \in M). \]

Then, \( f \) extends to a linear functional \( \Lambda \) on \( X \) which satisfies:

\[ |\Lambda(x)| \leq p(x) \quad (x \in X). \]

Suppose \( p \) is a seminorm on \( X \). For \( x \in X \), let

\[ M^p_x = \{ \Lambda : X \rightarrow \mathbb{R} : \Lambda(x) = p(x), \ |\Lambda(z)| \leq p(z), \forall z \in X \}. \]

For \( x \in X \), if we let \( M = \langle x \rangle \) (\( \langle x \rangle \) is the subspace of \( X \) generated by \( x \)) and define \( f(\alpha x) = \alpha p(x) \), then by Lemma 1.1, the linear functional \( f \) extends to a linear functional \( \Lambda \in M^p_x \). Therefore, \( M^p_x \) is nonempty.

Here, we are concerned with the concepts of \( p \)-best approximation and \( p \)-orthogonality in a vector space. The concept of approximation in normed linear spaces was defined by I. Singer [6].

### 2. Orthogonality in vector spaces

In this section, we state and prove our main results for vector spaces. Also, we obtain results related to \( p \)-orthogonality on vector spaces.

**Theorem 2.1.** Let \( X \) be a vector space, \( G \) be a subspace of \( X \), \( p \) be a seminorm on \( X \), \( x \in X \setminus G \) and \( p(x) \neq 0 \). Then, the following statements are equivalent:

a) \( x \perp^p G \).
b) There is a linear functional $\Lambda$ on $X$ such that $\Lambda \in M^p_x$ and $\Lambda |_G = 0$.

**Proof.** $a) \Rightarrow b)$. Suppose $x \perp^p G$. Consider $M = \langle x \rangle \bigoplus G$. We define a linear functional $f$ on $M$ by $f(\alpha x + y) = \alpha p(x)$, where $y \in G$ and $\alpha \in \mathbb{R}$. It is clear that $f(y) = 0$, for every $y \in G$, and $f(x) = p(x)$. Now, suppose $z = \alpha x + y \in M$. Then,

\[
|f(z)| = |f(\alpha x + y)| \\
= |\alpha| p(x) \\
\leq |\alpha| p(x + \frac{1}{\alpha} y) \\
= p(\alpha x + y) \\
= p(z).
\]

From Lemma 1.1, there exists a linear functional $\Lambda$ on $X$ such that $\Lambda(x) = p(x)$, $\Lambda |_G = 0$, $|\Lambda(z)| \leq p(z)$ for all $z \in X$.

$b) \Rightarrow a)$. Suppose that there exists a linear functional $\Lambda$ on $X$ such that $\Lambda \in M^p_x$ and $\Lambda |_G = 0$. For every $\alpha \in \mathbb{R}$ and $y \in G$, we have,

\[
p(x + \alpha y) \geq |\Lambda(x + \alpha y)| = |\Lambda(x)| = p(x).
\]

Therefore, $\inf_{\alpha} p(x + \alpha y) = p(x)$, and hence $x \perp^p y$. Thus, $x \perp^p G$. \(\square\)

Now, we shall obtain from Theorem 2.1 various corollaries on $p$-orthogonality.

**Corollary 2.2.** Let $X$ be a vector space, $p$ be a seminorm on $X$ and $x, y \in X$. If $x \perp^p y$, then $\langle x \rangle \cap \langle y \rangle = \{0\}$.

**Corollary 2.3.** Let $X$ be a vector space, $A$ be a nonempty subset of $X$, $p$ be a seminorm on $X$ such that $p(y) \neq 0$, for all $y \in A$ and $x \in X \setminus < A >$. Then, the following two statements are equivalent:

a) $A \perp^p x$.

b) For every $y \in A$, there exists a linear functional $\Lambda$ on $X$ with $\Lambda \in M^p_y$ and $\Lambda(x) = 0$.

**Definition 2.4.** Let $X$ be a vector space and $p$ be a seminorm on $X$. The element $x \in X$ is called a $p$-normal element if there exists only one linear functional $\Lambda_x$ on $X$ such that $\Lambda_x \in M^p_x$; i.e., $M^p_x$ is a singleton.
Corollary 2.5. Let $X$ be a vector space, $G$ be a linear subspace of $X$, $p$ be a seminorm on $X$ and $p(x) \neq 0$. If $x \in X$ is a $p$-normal element associated with $p$ on $X$, then the following statements are equivalent:

a) $x \perp p^G$.

b) There exists a unique linear functional $\Lambda$ on $X$ such that $\Lambda \in M^p_x$ and $\Lambda|_G = 0$.

Let $G$ be a subspace of the space $X$ equipped with a seminorm $p$. Define,

$$\hat{G}_p = \{x \in X : x \perp p^G\},$$

and

$$\hat{G}_p = \{x \in X : G \perp p^x\}.$$

Corollary 2.6. Let $X$ be a vector space, $G$ be a subspace of $X$ and $p$ be a seminorm on $X$. Then,

a) $G \cap \hat{G}_p = \{0\}$

b) $G \cap \hat{G}_p = \{0\}$.

c) $\alpha x \in \hat{G}_p$, if $x \in \hat{G}_p$ and $\alpha \in \mathbb{R}$.

d) $\alpha x \in \hat{G}_p$, if $x \in \hat{G}_p$ and $\alpha \in \mathbb{R}$.

Proof. The statements (c) and (d) are consequences of the definition of $p$-orthogonality. Suppose $x \in G \cap \hat{G}_p$ (resp. $x \in G \cap \hat{G}_p$). Then, $x \perp G$ w.r.t. $p$ (resp. $G \perp x$ w.r.t. $p$) and $x \in G$. Therefore, $x \perp p^x$, and form Corollary 2.2, $x = 0$. \hfill \square

3. $p$-Best approximation and linear functional

Here, we shall introduce and discuss the concept of $p$-extension of linear functionals on a vector space, and show that a linear functional on a subspace has a unique $p$-extension to the whole vector space if and only if $G^\perp$ has some properties.

Let $X$ be a vector space and $p$ be a seminorm on $X$. A point $g_0 \in G$ is said to be a $p$-best approximation for $x \in X$ if and only if $p(x - g_0) \neq 0$ and for all $g \in G$, $p(x - g) \leq p(x - g_0)$. The set of all $p$-best approximations of $x \in X$ in $G$ is denoted by $P^p_G(x)$. In other words,

$$P^p_G(x) = \{g_0 \in G : p(x - g_0) \neq 0, p(x - g_0) \leq p(x - g) \forall g \in G\}.$$
If $P^p_G(x)$ is non-empty for every $x \in X$, then $G$ is called a $p$-proximinal set. The set $G$ is $p$-Chebyshev if $P^p_G(x)$ is a singleton for every $x \in X$.

**Theorem 3.1.** Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$, $g_0 \in G$, $x \in X \setminus G$ and $p(x - g_0) \neq 0$. Then, the following statements are equivalent:

a) $g_0 \in P^p_G(x)$
b) There exists a linear functional $\Lambda$ on $X$ such that $\Lambda \in M^p_{x - g_0}$ and $\Lambda|_G = 0$.

**Proof.** We know that $g_0 \in P^p_G(x)$ if and only if $x - g_0 \perp_p G$. Now, apply Theorem 2.1.

**Theorem 3.2.** Let $X$ be a vector space, $p$ be a seminorm on $X$ and $G$ be a $p$-proximinal subspace of $X$. If $G_p$ is a convex set, then $G$ is $p$-Chebyshev.

**Proof.** If $x \in X$ and $g_1, g_2 \in P^p_G(x)$, then $x - g_1, x - g_2 \in \hat{G}_p$. Since $\hat{G}_p$ is convex, then it follows that $\frac{1}{2}(g_1 - g_2) \in \hat{G}_p$. Since $\frac{1}{2}(g_1 - g_2) \in G$, then Lemma 2.6 shows that $g_1 = g_2$.

**Theorem 3.3.** Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$, $g_0 \in G$, and $x \in X \setminus G$ $p(x - g_0) \neq 0$. Then,

$$g_0 \in P^p_G(x) \iff p((x - g_0)|_{G^\perp}) = p(x - g_0),$$

where,

$$p((x - g_0)|_{G^\perp}) = \sup\{|\Lambda(x - g_0)| : \Lambda \in G^\perp, |\Lambda(z)| \leq p(z), \forall z \in X\},$$

and the annihilator of $G$ is the set,

$$G^\perp = \{f : X \text{ linear} \rightarrow R : f(x) = 0 \text{ for all } x \in G\}.$$

**Proof.** Let $g_0 \in P^p_G(x)$. It follows from $p(x - g_0) \neq 0$ and Theorem 3.1 that there exists a linear functional $\Lambda$ on $X$ such that for all $z \in X$, $|\Lambda(z)| \leq p(z)$, $\Lambda(x - g_0) = p(x - g_0)$ and $\Lambda|_G = 0$. Therefore, $p(x - g_0) = |\Lambda(x - g_0)| \leq p((x - g_0)|_{G^\perp})$. Now, suppose $\Lambda \in G^\perp$ and $|\Lambda(z)| \leq p(z)$ for all $z \in X$. Then, $|\Lambda(x - g_0)| \leq p(x - g_0)$, and thus $p((x - g_0)|_{G^\perp}) \leq p(x - g_0)$. 

Conversely, suppose \( p((x - g_0) |_{G^\perp}) = p(x - g_0) \). Since \( p((x - g) |_{G^\perp}) \leq p(x - g) \), then similarly we have,

\[
p(x - g_0) = p((x - g_0) |_{G^\perp}) = p((x - g) |_{G^\perp}) \leq p(x - g).
\]

That is, \( g_0 \in P^p_G(x) \). \( \square \)

Let \( X \) be a vector space and \( p \) be a seminorm on \( X \). The dual space of \( X \) with respect to \( p \) is denoted by:

\[
X^*_p = \{ \Lambda : X \rightarrow \mathbb{R} : p'(\Lambda) < \infty \}
\]

where,

\[
p'(\Lambda) = \sup\{ |\Lambda(x)| : p(x) \leq 1, \ x \in X \}.
\]

It is clear that \( p' \) is a seminorm on \( X^*_p \). Similarly, we can define \( p'' \) on \( X^*_p \) and \( p''' \) on \( X^{**}_p \) (see [5]).

It is clear that if \( X \) is a vector, \( p \) is a seminorm on \( X \), \( \Lambda \in X^*_p \), \( x \in X \) and \( p(x) \neq 0 \), then,

\[
p(x) = \sup\{ |\Lambda(x)| : p'(\Lambda) \leq 1, \ \Lambda \in X^*_p \}.
\]

**Lemma 3.4.** Let \( X \) be a vector space, and \( p \) be a seminorm on \( X \). For each \( x \in X \), define the linear functional \( \hat{x} \) on \( X^*_p \) by \( \hat{x}(\varphi) = \varphi(x) \), for \( \varphi \in X^*_p \). Then, for all \( x \in X \),

\[
p''(\hat{x}) = p(x).
\]

**Proof.** We have,

\[
p''(\hat{x}) = \sup\{ |\hat{x}(\varphi)| : p'(\varphi) \leq 1, \ \varphi \in X^*_p \}
\]

\[
= \sup\{ |\varphi(x)| : p'(\varphi) \leq 1, \ \varphi \in X^*_p \}
\]

\[
= p(x).
\]

\( \square \)

**Definition 3.5.** Let \( X \) be a vector space, \( p \) be a seminorm on \( X \), \( \Lambda \in X^*_p \) and \( G \) be a subspace of \( X \). \( \Lambda \) is called a \( p \)-extension of the linear functional \( f : G \rightarrow \mathbb{R} \), if \( \Lambda|_G = f \) and \( p'(\Lambda) = p'(f) \). \( p'(\Lambda) \) and \( p'(f) \) are computed relative to the domains of \( \Lambda \) and \( f \), explicitly as:

\[
p'(\Lambda) = \sup\{ |\Lambda(x)| : p(x) \leq 1, \ x \in X \}
\]
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and

$$p'(f) = \sup\{|f(x)| : p(x) \leq 1, \ x \in G\}.$$ 

**Theorem 3.6.** Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$ and $f \in X_p^* \setminus G^\perp$. Then, there is an one-to-one correspondence between the set $P_{G^\perp}^p(f)$ and the set of all $g \in X_p^*$ such that $\hat{g}$ is a $p$-extension of $\hat{f}|_{G^\perp}$, given by $g \mapsto f - g$.

**Proof.** If $g \in P_{G^\perp}^p(f)$, then it is clear that $\hat{f}|_{G^\perp} = \hat{f} - g|_{G^\perp}$, and from Theorem 3.2 and Corollary 3.4 we have,

$$p''((f - g)|_{G^\perp}) = p'(f - g) = p''((\hat{f} - g)|_{G^\perp}) = p''(\hat{f}|_{G^\perp}).$$

Therefore, $\hat{f} - g$ is a $p$-extension of $\hat{f}|_{G^\perp}$. We show that this map is onto. Suppose $\hat{h}$ is a $p$-extension of $\hat{f}|_{G^\perp}$. Let $g = f - h$. We show that $g \in P_{G^\perp}^p(f)$. For this, since $\hat{f} - g$ is a $p$-extension of $\hat{f}|_{G^\perp}$, then $\hat{f}|_{G^\perp} = \hat{f} - g|_{G^\perp}$ and $p''(\hat{f}|_{G^\perp}) = p''(\hat{f} - g)$. Now, we have,

$$p'((f - g)|_{G^\perp}) = p''((\hat{f} - g)|_{G^\perp}) = p''(\hat{f}|_{G^\perp}) = p''(f - g) = p'(f - g).$$

Therefore, by Theorem 3.2, $g \in P_{G^\perp}^p(f)$. □

**Theorem 3.7.** Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$, $f \in G_p^*$ and let $\tilde{f} \in X_p^*$ be a $p$-extension of $f$. Then, there is a one-to-one correspondence between the set of all $p$-extensions of $f$ to $X$ and the set $P_{G^\perp}^p(\tilde{f})$, given by $g \mapsto \tilde{f} - g$.

**Proof.** Suppose $g \in X_p^*$ is a $p$-extension of $f$. Then, $g|_G = f$ and $p'(f) = p'(g)$. For all $\varphi \in G^\perp$,

$$\hat{g}(\varphi) = \varphi(g) = \varphi(\tilde{f}) = \tilde{f}(\varphi),$$

$$\hat{g}(\varphi) = \varphi(g) = \varphi(\tilde{f}) = \tilde{f}(\varphi),$$

and

$$p'(f) = \sup\{|f(x)| : p(x) \leq 1, \ x \in G\}.$$ 

**Theorem 3.6.** Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$ and $f \in X_p^* \setminus G^\perp$. Then, there is an one-to-one correspondence between the set $P_{G^\perp}^p(f)$ and the set of all $g \in X_p^*$ such that $\hat{g}$ is a $p$-extension of $\hat{f}|_{G^\perp}$, given by $g \mapsto f - g$.

**Proof.** If $g \in P_{G^\perp}^p(f)$, then it is clear that $\hat{f}|_{G^\perp} = \hat{f} - g|_{G^\perp}$, and from Theorem 3.2 and Corollary 3.4 we have,

$$p''((f - g)|_{G^\perp}) = p'(f - g) = p''((\hat{f} - g)|_{G^\perp}) = p''(\hat{f}|_{G^\perp}).$$

Therefore, $\hat{f} - g$ is a $p$-extension of $\hat{f}|_{G^\perp}$. We show that this map is onto. Suppose $\hat{h}$ is a $p$-extension of $\hat{f}|_{G^\perp}$. Let $g = f - h$. We show that $g \in P_{G^\perp}^p(f)$. For this, since $\hat{f} - g$ is a $p$-extension of $\hat{f}|_{G^\perp}$, then $\hat{f}|_{G^\perp} = \hat{f} - g|_{G^\perp}$ and $p''(\hat{f}|_{G^\perp}) = p''(\hat{f} - g)$. Now, we have,

$$p'((f - g)|_{G^\perp}) = p''((\hat{f} - g)|_{G^\perp}) = p''(\hat{f}|_{G^\perp}) = p''(f - g) = p'(f - g).$$

Therefore, by Theorem 3.2, $g \in P_{G^\perp}^p(f)$. □

**Theorem 3.7.** Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$, $f \in G_p^*$ and let $\tilde{f} \in X_p^*$ be a $p$-extension of $f$. Then, there is a one-to-one correspondence between the set of all $p$-extensions of $f$ to $X$ and the set $P_{G^\perp}^p(\tilde{f})$, given by $g \mapsto \tilde{f} - g$.

**Proof.** Suppose $g \in X_p^*$ is a $p$-extension of $f$. Then, $g|_G = f$ and $p'(f) = p'(g)$. For all $\varphi \in G^\perp$,

$$\hat{g}(\varphi) = \varphi(g) = \varphi(\tilde{f}) = \tilde{f}(\varphi),$$

$$\hat{g}(\varphi) = \varphi(g) = \varphi(\tilde{f}) = \tilde{f}(\varphi),$$

and

$$p'(f) = \sup\{|f(x)| : p(x) \leq 1, \ x \in G\}. $$
since \( g|_G = \tilde{f}|_G \). Also \( \tilde{f}|_G = f \), and thus \( \tilde{f}|_{G^\perp} = \hat{f} \) and

\[
p''(\tilde{g}) = p'(g) = p'(f) = p''(\hat{f}) = p''(\tilde{f}|_{G^\perp}).
\]

So by Theorem 3.6, \( \tilde{f} - g \in P_{G^\perp}^p(\hat{f}) \).

Now, suppose that \( g \in P_{G^\perp}^p(\tilde{f}) \). Then, \( \hat{\tilde{f}} - g \) is a \( p \)-extension of \( \tilde{f}|_{G^\perp} \).

We know,

\[
(\tilde{f} - g)(x) = f(x),
\]

for all \( x \in G \), and

\[
p'(\tilde{f} - g) = p''(\tilde{f} - g) = p''(\tilde{f}|_{G^\perp}) = p''(\hat{f}) = p'(f).
\]

That is, \( \tilde{f} - g \) is a \( p \)-extension of \( f \). \( \square \)

**Corollary 3.8.** Let \( X \) be a vector space, \( G \) be a subspace of \( X \), and \( p \) be a seminorm on \( X \). Then, the following statements are equivalent:

a) Every non-zero \( f \in G^* \) has a unique \( p \)-extension of \( X \).

b) For each \( f \in G^* \setminus G^\perp \), there is a unique \( g \in X^* \) such that \( \hat{g} \) is a \( p \)-extension of \( \tilde{f}|_{G^\perp} \).

c) \( G^\perp \) is a \( p' \)-Chebyshev subspace of \( X^* \).

**Example 3.9.** Let \( f \) be a linear functional on a real vector space \( X \). Then, \( p(x) = |f(x)| \) gives a seminorm on \( X \). If \( f \) is nonzero and \( \dim(X) > 1 \), then \( p \) is seminorm that is not a norm. Now, let \( G_1 = \ker f \) and \( G_2 = X \setminus \ker f \). Then, \( G_1 \) is a subspace of \( X \). For \( y \in G_1 \), we have \( f(y) = 0 \), and therefore for all \( x \in X \setminus G_1, x \perp^p G_1 \). Also, since for every \( y \in G_2, f(y) \neq 0 \), then,

\[
\inf_\alpha p(x + \alpha y) \leq |f(x) + (-\frac{f(x)}{f(y)})f(y)| = 0.
\]

Therefore, for all \( x \in X \setminus G_2, x \perp^p G_2 \).

**Example 3.10.** The elements of \( X = L^2 \) are the real Lebesgue measurable functions \( f \) on \([0,1]\). We can define a seminorm \( p \) on \( X \) by:

\[
p(f) = \{ \int_0^1 |f(t)|^2 dt \}^{\frac{1}{2}}, \; f \in X.
\]

It is clear that \( p \) is a seminorm on \( X \) and is not a norm. Now, we can apply Theorem 3.7 and Theorem 3.6 to this example.
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