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# SOME RESULTS ON $p$-BEST APPROXIMATION IN VECTOR SPACES

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Abstract. The purpose of this paper is to introduce and to discuss the concept of $p$-approximation and $p$-orthogonality in vector spaces, and to obtain some results on $p$- orthogonality in vector spaces similar to some well known results on the orthogonality in normed spaces. We also discuss the concept of $p$-extension of linear functionals on a vector space, and give a characterization of linear functionals on a subspace having a unique $p$-extension Hahn-Banach to the whole vector space.

## 1. **Introduction**

Here, all normed spaces under consideration are real. A seminorm is a function $p: X \to [0, \infty)$ such that $p(x+y) \leq p(x)+p(y)$ and $p(\alpha x) = |\alpha| p(x)$, for all $x, y \in X$ and $\alpha \in \mathbb{R}$. It is clear that for every seminorm $p$, $p(0) = 0$. Also, the seminorm $p$ is a norm, if $p(x) = 0$ implies $x = 0$.

Many authors have introduced the concept of orthogonality in different ways (see [1-4], [6]). In [1], Birkhoff modified the concept of orthogonality. By his definition, if $X$ is a normed linear space and $x, y \in X$, $x$ is said to be orthogonal to $y$ and is denoted by $x \perp y$ if and only if

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$\|x\| \leq \|x + \alpha y\|$, for all scalars $\alpha$. Note that this orthogonality is not symmetric in general [2].

Let $X$ be such a vector space, $x, y \in X$ and $p$ be a fixed seminorm. We say that $x$ is $p$-orthogonal to $y$ if $x = 0$ or else,

$$p(x) \neq 0, \; p(x) = \inf{}_{\alpha} p(x + \alpha y),$$

in which case we write $x \perp^p y$. If $M_1$ and $M_2$ are subsets of $X$, then we say that $M_1$ is $p$-orthogonal to $M_2$ $p$ if $g_1 \perp^p g_2$, for all $g_1 \in M_1, \; g_2 \in M_2$. If $M_1$ is $p$-orthogonal to $M_2$, then we write $M_1 \perp^p M_2$.

First we state the following lemma of Hahn- Banach which is needed in the proof of the main results.

**Lemma 1.1.** *[5] Let $M$ be a subspace of a vector space $X$, $p$ be a seminorm on $X$, and let $f$ be a linear functional on $M$ such that*

$$|f(x)| \leq p(x) \; (x \in M).$$

*Then, $f$ extends to a linear functional $\Lambda$ on $X$ which satisfies:*

$$|\Lambda(x)| \leq p(x) \; (x \in X).$$

Suppose $p$ is a seminorm on $X$. For $x \in X$, let

$$M_x^p = \{\Lambda : \; X \overset{linear}{\rightarrow} \mathbb{R} : \; \Lambda(x) = p(x), \; |\Lambda(z)| \leq p(z), \forall z \in X\}.$$

For $x \in X$, if we let $M =< x >$ ( $< x >$ is the subspace of $X$ generated by $x$) and define $f(\alpha x) = \alpha p(x)$, then by Lemma 1.1, the linear functional $f$ extends to a linear functional $\Lambda \in M_x^p$. Therefore, $M_x^p$ is nonempty.

Here, we are concerned with the concepts of $p$-best approximation and $p$-orthogonality in a vector space. The concept of approximation in normed linear spaces was defined by I. Singer [6].

## 2. **Orthogonality in vector spaces**

In this section, we state and prove our main results for vector spaces. Also, we obtain results related to $p$-orthogonality on vector spaces.

**Theorem 2.1.** *Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$, $x \in X \backslash G$ and $p(x) \neq 0$. Then, the following statements are equivalent:*
*a) $x \perp^p G$.*

*b) There is a linear functional* $\Lambda$ *on* $X$ *such that* $\Lambda \in M^p{}_x$ *and* $\Lambda|_G = 0$.

**Proof.** $a) \Rightarrow b)$. Suppose $x \perp^p G$. Consider $M = < x > \bigoplus G$. We define a linear functional $f$ on $M$ by $f(\alpha x + y) = \alpha p(x)$, where $y \in G$ and $\alpha \in \mathbb{R}$. It is clear that $f(y) = 0$, for every $y \in G$, and $f(x) = p(x)$. Now, suppose $z = \alpha x + y \in M$. Then,

$$\begin{aligned} |f(z)| &= |f(\alpha x + y)| \\ &= |\alpha| p(x) \\ &\leq |\alpha| p(x + \frac{1}{\alpha} y) \\ &= p(\alpha x + y) \\ &= p(z). \end{aligned}$$

From Lemma 1.1, there exists a linear functional $\Lambda$ on $X$ such that

$$\Lambda(x) = p(x),\ \Lambda|_G = 0,\ |\Lambda(z)| \leq p(z)\ for\ all\ z \in X.$$

$b) \Rightarrow a)$. Suppose that there exists a linear functional $\Lambda$ on $X$ such that $\Lambda \in M^P{}_x$ and $\Lambda|_G = 0$. For every $\alpha \in \mathbb{R}$ and $y \in G$, we have,

$$p(x + \alpha y) \geq |\Lambda(x + \alpha y)| = |\Lambda(x)| = p(x).$$

Therefore, $\inf_\alpha p(x + \alpha y) = p(x)$, and hence $x \perp^p y$. Thus, $x \perp^p G$. □

Now, we shall obtain from Theorem 2.1 various corollaries on $p$-orthogonality.

**Corollary 2.2.** *Let* $X$ *be a vector space,* $p$ *be a seminorm on* $X$ *and* $x, y \in X$. *If* $x \perp^p y$, *then* $< x > \cap < y >= \{0\}$.

**Corollary 2.3.** *Let* $X$ *be a vector space,* $A$ *be a nonempty subset of* $X$, $p$ *be a seminorm on* $X$ *such that* $p(y) \neq 0$, *for all* $y \in A$ *and* $x \in X \setminus < A >$. *Then, the following two statements are equivalent:*
*a)* $A \perp^p x$.
*b) For every* $y \in A$, *there exists a linear functional* $\Lambda$ *on* $X$ *with* $\Lambda \in M^p_y$ *and* $\Lambda(x) = 0$.

**Definition 2.4.** Let $X$ be a vector space and $p$ be a seminorm on $X$. The element $x \in X$ is called a $p$-normal element if there exists only one linear functional $\Lambda_x$ on $X$ such that $\Lambda_x \in M^p_x$; i.e., $M^p_x$ is a singleton.

**Corollary 2.5.** *Let $X$ be a vector space, $G$ be a linear subspace of $X$, $p$ be a seminorm on $X$ and $p(x) \neq 0$. If $x \in X$ is a $p$-normal element associated with $p$ on $X$, then the following statements are equivalent:*
*a)* $x \perp^p G$.
*b) There exists a unique linear functional $\Lambda$ on $X$ such that $\Lambda \in {M^p}_x$ and $\Lambda|_G = 0$.*

Let $G$ be a subspace of the space $X$ equipped with a seminorm $p$. Define,

$$\hat{G}_p = \{x \in X : \ x \perp^p G\},$$

and

$$\breve{G}_p = \{x \in X : \ G \perp^p x\}.$$

**Corollary 2.6.** *Let $X$ be a vector space, $G$ be a subspace of $X$ and $p$ be a seminorm on $X$. Then,*
*a)* $G \cap \hat{G}_p = \{0\}$
*b)* $G \cap \breve{G}_p = \{0\}$.
*c)* $\alpha x \in \hat{G}_p$, *if* $x \in \hat{G}_p$ *and* $\alpha \in \mathbb{R}$.
*d)* $\alpha x \in \breve{G}_p$, *if* $x \in \breve{G}_p$ *and* $\alpha \in \mathbb{R}$.

**Proof.** The statements (c) and (d) are consequences of the definition of $p$-orthogonality. Suppose $x \in G \cap \hat{G}_p$ (resp. $x \in G \cap \breve{G}_p$). Then, $x \perp G$ w.r.t. $p$ (resp. $G \perp x$ w.r.t. $p$) and $x \in G$. Therefore, $x \perp^p x$, and form Corollary 2.2, $x = 0$. □

## 3. p-Best approximation and linear functional

Here, we shall introduce and discuss the concept of $p$-extension of linear functionals on a vector space, and show that a linear functional on a subspace has a unique $p$-extension to the whole vector space if and only if $G^\perp$ has some properties.

Let $X$ be a vector space and $p$ be a seminorm on $X$. A point $g_0 \in G$ is said to be a $p$-best approximation for $x \in X$ if and only if $p(x - g_0) \neq 0$ and for all $g \in G$, $p(x - g_0) \leq p(x - g)$. The set of all $p$-best approximations of $x \in X$ in $G$ is denoted by $P_G^p(x)$. In other words,

$$P_G^p(x) = \{g_0 \in G \ : \ p(x - g_0) \neq 0, \ p(x - g_0) \leq p(x - g) \ \forall g \in G\}.$$

If $P^p_G(x)$ is non-empty for every $x \in X$, then $G$ is called a $p$-proximinal set. The set $G$ is $p$-Chebyshev if $P^p_G(x)$ is a singleton for every $x \in X$.

**Theorem 3.1.** *Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$, $g_0 \in G$, $x \in X \backslash G$ and $p(x - g_0) \neq 0$. Then, the following statements are equivalent:*
*a) $g_0 \in P^p_G(x)$*
*b) There exists a linear functional $\Lambda$ on $X$ such that $\Lambda \in M^p_{x-g_0}$ and $\Lambda|_G = 0$.*

**Proof.** We know that $g_0 \in P^p_G(x)$ if and only if $x - g_0 \perp^p G$. Now, Apply Theorem 2.1.

**Theorem 3.2.** *Let $X$ be a vector space, $p$ be a seminorm on $X$ and $G$ be a $p$-proximinal subspace of $X$. If $G_p$ is a convex set, then $G$ is $p$-Chebyshev.*

**Proof.** If $x \in X$ and $g_1, g_2 \in P^p{}_G(x)$, then $x - g_1, x - g_2 \in \hat{G}_p$. Since $\hat{G}_p$ is convex, then it follows that $\frac{1}{2}(g_1 - g_2) \in \hat{G}_p$. Since $\frac{1}{2}(g_1 - g_2) \in G$, then Lemma 2.6 shows that $g_1 = g_2$. □

**Theorem 3.3.** *Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$, $g_0 \in G$, and $x \in X \backslash G$ $p(x - g_0) \neq 0$. Then,*

$$g_0 \in P^p_G(x) \Leftrightarrow p((x - g_0)|_{G^\perp}) = p(x - g_0),$$

*where,*

$$p((x - g_0)|_{G^\perp}) = \sup\{|\Lambda(x - g_0)| : \ \Lambda \in G^\perp, \ |\Lambda(z)| \leq p(z), \ \forall z \in X\},$$

*and the annihilator of $G$ is the set,*

$$G^\perp = \{f : \ X \stackrel{linear}{\to} \boldsymbol{R} : \ f(x) = 0 \ for \ all \ x \in G\}.$$

**Proof.** Let $g_0 \in P^p_G(x)$. It follows from $p(x - g_0) \neq 0$ and Theorem 3.1 that there exists a linear functional $\Lambda$ on $X$ such that for all $z \in X$, $|\Lambda(z)| \leq p(z)$, $\Lambda(x - g_0) = p(x - g_0)$ and $\Lambda|_G = 0$. Therefore, $p(x - g_0) = |\Lambda(x - g_0)| \leq p((x - g_0)|_{G^\perp})$. Now, suppose $\Lambda \in G^\perp$ and $|\Lambda(z)| \leq p(z)$ for all $z \in X$. Then, $|\Lambda(x - g_0)| \leq p(x - g_0)$, and thus $p((x - g_0)|_{G^\perp}) \leq p(x - g_0)$.

Conversely, suppose $p((x-g_0)|_{G^\perp}) = p(x-g_0)$. Since $p((x-g)|_{G^\perp}) \leq p(x-g)$, then similarly we have,

$$\begin{aligned} p(x-g_0) &= p((x-g_0)|_{G^\perp}) \\ &= p((x-g)|_{G^\perp}) \\ &\leq p(x-g). \end{aligned}$$

That is, $g_0 \in P_G^p(x)$. □

Let $X$ be a vector space and $p$ be a seminorm on $X$. The dual space of $X$ with respect to $p$ is denoted by:

$$X_p^* = \{\Lambda : \ X \overset{linear}{\to} \mathbf{R} : \ p'(\Lambda) < \infty\},$$

where,

$$p'(\Lambda) = \sup\{|\Lambda(x)| : \ p(x) \leq 1, \ x \in X\}.$$

It is clear that $p'$ is a seminorm on $X_p^*$. Similarly, we can define $p''$ on $X_p^{**}$ and $p'''$ on $X_p^{***}$(see [5]).

It is clear that if $X$ is a vector, $p$ is a seminorm on $X$, $\Lambda \in X_p^*$, $x \in X$ and $p(x) \neq 0$, then,

$$p(x) = \sup\{|\Lambda(x)| : \ p'(\Lambda) \leq 1, \ \Lambda \in X_p^*\}.$$

**Lemma 3.4.** *Let $X$ be a vector space, and $p$ be a seminorm on $X$. For each $x \in X$, define the linear functional $\widehat{x}$ on $X_p^*$ by $\widehat{x}(\varphi) = \varphi(x)$, for $\varphi \in X_p^*$. Then, for all $x \in X$,*

$$p''(\widehat{x}) = p(x).$$

**Proof.** We have,

$$\begin{aligned} p''(\widehat{x}) &= \sup\{|\widehat{x}(\varphi)| : \ p'(\varphi) \leq 1, \ \varphi \in X_p^*\} \\ &= \sup\{|\varphi(x)| : \ p'(\varphi) \leq 1, \ \varphi \in X_p^*\} \\ &= p(x). \end{aligned}$$

□

**Definition 3.5.** Let $X$ be a vector space, $p$ be a seminorm on $X$, $\Lambda \in X_p^*$ and $G$ be a subspace of $X$. $\Lambda$ is called a $p$-extension of the linear functional $f : G \longrightarrow \mathbf{R}$, if $\Lambda|_G = f$ and $p'(\Lambda) = p'(f)$. $p'(\Lambda)$ and $p'(f)$ are computed relative to the domains of $\Lambda$ and $f$, explicitly as:

$$p'(\Lambda) = \sup\{|\Lambda(x)| : \ p(x) \leq 1, \ x \in X\}$$

and

$$p'(f) = \sup\{|f(x)| : \ p(x) \leq 1, \ x \in G\}.$$

**Theorem 3.6.** *Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$ and $f \in X_p^* \backslash G^{\perp}$. Then, there is an one-to-one correspondence between the set $P_{G^{\perp}}^{p'}(f)$ and the set of all $g \in X_p^*$ such that $\widehat{g}$ is a $p$-extension of $\widehat{f}|_{G^{\perp\perp}}$, given by $g \mapsto f - g$.*

**Proof.** If $g \in P_{G^{\perp}}^{p'}(f)$, then it is clear that $\widehat{f}|_{G^{\perp\perp}} = \widehat{f-g}|_{G^{\perp\perp}}$, and from Theorem 3.2 and Corollary 3.4 we have,

$$\begin{aligned} p'''(\widehat{f-g}) &= p'(f-g) \\ &= p'((f-g)|_{G^{\perp\perp}}) \\ &= p'''(\widehat{(f-g)}|_{G^{\perp\perp}}) \\ &= p'''(\widehat{f}|_{G^{\perp\perp}}). \end{aligned}$$

Therefore, $\widehat{f-g}$ is a $p$-extension of $\widehat{f}|_{G^{\perp\perp}}$. We show that this map is onto. Suppose $\widehat{h}$ is a $p$-extension of $\widehat{f}|_{G^{\perp\perp}}$. Let $g = f - h$. We show that $g \in P_{G^{\perp}}^{p'}(f)$. For this, since $\widehat{f-g}$ is a $p$-extension of $\widehat{f}|_{G^{\perp\perp}}$, then $\widehat{f}|_{G^{\perp\perp}} = \widehat{f-g}|_{G^{\perp\perp}}$ and $p'''(\widehat{f}|_{G^{\perp\perp}}) = p'''(\widehat{f-g})$. Now, we have,

$$\begin{aligned} p'((f-g)|_{G^{\perp\perp}}) &= p'''(\widehat{(f-g)}|_{G^{\perp\perp}}) \\ &= p'''(\widehat{f}|_{G^{\perp\perp}}) \\ &= p'''(\widehat{f-g}) \\ &= p'(f-g). \end{aligned}$$

Therefore, by Theorem 3.2, $g \in P_{G^{\perp}}^{p'}(f)$. □

**Theorem 3.7.** *Let $X$ be a vector space, $G$ be a subspace of $X$, $p$ be a seminorm on $X$, $f \in G_p^*$ and let $\widetilde{f} \in X_p^*$ be a $p$-extension of $f$. Then, there is a one-to-one correspondence between the set of all $p$-extensions of $f$ to $X$ and the set $P_{G^{\perp}}^{p'}(\widetilde{f})$, given by $g \longrightarrow \widetilde{f} - g$.*

**Proof.** Suppose $g \in X_p^*$ is a $p$-extension of $f$. Then, $g|_G = f$ and $p'(f) = p'(g)$. For all $\varphi \in G^{\perp\perp}$,

$$\widehat{g}(\varphi) = \varphi(g) = \varphi(\widetilde{f}) = \widehat{\widetilde{f}}(\varphi),$$

since $g|_G = \widetilde{f}|_G$. Also $\widetilde{f}|_G = f$, and thus $\widehat{\widetilde{f}}|_{G^{\perp\perp}} = \widehat{f}$ and

$$p'''(\widehat{g}) = p'(g) = p'(f) = p'''(\widehat{f}) = p'''(\widehat{\widetilde{f}}|_{G^{\perp\perp}}).$$

So by Theorem 3.6, $\widetilde{f} - g \in P^{p'}_{G^{\perp}}(\widetilde{f})$.

Now, suppose that $g \in P^{p'}_{G^{\perp}}(\widetilde{f})$. Then, $\widehat{\widetilde{f} - g}$ is a $p$-extension of $\widehat{\widetilde{f}}|_{G^{\perp\perp}}$. We know,

$$(\widetilde{f} - g)(x) = f(x),$$

for all $x \in G$, and

$$p'(\widetilde{f} - g) = p'''(\widehat{\widetilde{f} - g}) = p'''(\widehat{\widetilde{f}}|_{G^{\perp\perp}}) = p'''(\widehat{f}) = p'(f).$$

That is, $\widetilde{f} - g$ is a $p$-extension of $f$. □

**Corollary 3.8.** *Let $X$ be a vector space, $G$ be a subspace of $X$, and $p$ be a seminorm on $X$. Then, the following statements are equivalent:*
*a) Every non-zero $f \in G^*_p$ has a unique $p$-extension of $X$.*
*b) For each $f \in G^*_p \backslash G^{\perp}$, there is a unique $g \in X^*_p$ such that $\widehat{g}$ is a $p$-extension of $\widehat{f}|_{G^{\perp\perp}}$.*
*c) $G^{\perp}$ is a $p'$-Chebyshev subspace of $X^*_p$.*

**Example 3.9.** Let $f$ be a linear functional on a real vector space $X$. Then, $p(x) = |f(x)|$ gives a seminorm on $X$. If $f$ is nonzero and $dim(X) > 1$, then $p$ is seminorm that is not a norm. Now, let $G_1 = kerf$ and $G_2 = X\backslash kerf$. Then, $G_1$ is a subspace of $X$. For $y \in G_1$, we have $f(y) = 0$, and therefore for all $x \in X\backslash G_1$, $x\perp^p G_1$. Also, since for every $y \in G_2$, $f(y) \neq 0$, then,

$$\inf{}_{\alpha} p(x + \alpha y) \leq |f(x) + (-\frac{f(x)}{f(y)})f(y)| = 0.$$

Therefore, for all $x \in X\backslash G_2$, $x\perp^p G_2$.

**Example 3.10.** The elements of $X = L^2$ are the real Lebesgue measurable functions $f$ on $[0,1]$. We can define a seminorm $p$ on $X$ by:

$$p(f) = \{\int_0^1 |f(t)|^2 dt\}^{\frac{1}{2}},\ f \in X.$$

It is clear that $p$ is a seminorm on $X$ and is not a norm. Now, we can apply Theorem 3.7 and Theorem 3.6 to this example.

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