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SINGULAR VALUE INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES

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Abstract. In this note, we obtain some singular values inequalities for positive semidefinite matrices by using block matrix technique. Our results are similar to some inequalities shown by Bhatia and Kittaneh in [Linear Algebra Appl. 308 (2000) 203-211] and [Linear Algebra Appl. 428 (2008) 2177-2191].
Keywords: Singular values, Positive semidefinite matrices, Block matrix technique.
MSC(2010): Primary: 15A60; Secondary: 15A18, 15A42.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. We shall always denote the singular values of A by $s_1(A) \geq \cdots \geq s_n(A) \geq 0$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. Let $A, B \in M_n$ be Hermitian, the order relation $A \geq B$ means, as usual, that $A - B$ is positive semidefinite. We use the direct sum notation $A \oplus B$ for the block-diagonal operator $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ defined on $M_n \oplus M_n$.

Bhatia and Kittaneh ([5], p.206) proved that if $A, B \in M_n$ are positive semidefinite, then

$$2s_j\left(A^{1/2}(A+B)B^{1/2}\right) \leq s_j\left((A+B)^2\right) \tag{1.1}$$

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for all $j = 1, \cdots, n$. Bhatia and Kittaneh ([6], p.2186) generalized the inequality (1.1) to the following form:

$$2s_j\left(A^{1/2}(A+B)^r B^{1/2}\right) \leq s_j\left((A+B)^{r+1}\right), \quad r \geq 0 \tag{1.2}$$

for all $j = 1, \cdots, n$.

In Section 2, we first present an inequality for singular values, which is similar to the inequalities (1.2). After that, we generalize the inequality (1.1), which is a special case of the inequality (1.2). Section 3 contains some remarks.

2. **Main results**

In this section, we first present an inequality, which is similar to the inequality (1.2). To do this, we need the following lemmas.

Lemma 2.1. *([2], Theorem 1) Let $f(t)$ be an operator monotone function and $A, B \in M_n$ be positive semidefinite. Then*

$$\left(\frac{A+B}{2}\right)^{1/2}(f(A)+f(B))\left(\frac{A+B}{2}\right)^{1/2} \leq Af(A)+Bf(B).$$

Lemma 2.2. *([8], Theorem 1) Let $A, B, X \in M_n$ such that $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0$. Then*

$$s_j(X) \leq \frac{1}{2}s_j\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

for all $j = 1, \cdots, n$.

Theorem 2.3. *Let $A, B \in M_n$ be positive semidefinite and suppose that*

$$K = \left(A^{1/(q+1)} + B^{1/(q+1)}\right)^{1/2}, \quad 0 \leq q \leq 1.$$

Then, we have

$$s_j\left(A^{q/2(q+1)}K(A+B)^r KB^{q/2(q+1)}\right) \leq s_j\left((A+B)^{1+r}\right), \quad r \geq 0$$

for all $j = 1, \cdots, n$.

Proof. It is known that the function $f(t) = t^q, \ \ 0 \leq q \leq 1$ is operator monotone on $(0, \infty)$. So, by Lemma 2.1, we have

$$\frac{1}{2}(A+B)^{1/2}(A^q+B^q)(A+B)^{1/2} \leq A^{1+q}+B^{1+q}.$$

It is known that if $X \le Y$, then $ZXZ^* \le ZYZ^*$. It follows that

$$\begin{aligned}\frac{1}{2}\left(A^{1+q}+B^{1+q}\right)^{r/2} L\left(A^q+B^q\right) L\left(A^{1+q}+B^{1+q}\right)^{r/2} \\ \le \left(A^{1+q}+B^{1+q}\right)^{1+r},\end{aligned}$$

where $L=(A+B)^{1/2}$. Note that the matrices XY and YX have the same eigenvalues. Thus, by this last inequality, we obtain

$$\frac{1}{2}\lambda_j\left(\left(A^q+B^q\right) L\left(A^{1+q}+B^{1+q}\right)^r L\right) \le \lambda_j\left(\left(A^{1+q}+B^{1+q}\right)^{1+r}\right). \tag{2.1}$$

Let

$$X=(A+B)^{1/2}\left(A^{1+q}+B^{1+q}\right)^{r/2}.$$

The inequality (2.1) is equivalent to

$$\frac{1}{2}\lambda_j\left(\left(A^q+B^q\right) XX^*\right) \le \lambda_j\left(\left(A^{1+q}+B^{1+q}\right)^{1+r}\right). \tag{2.2}$$

Except for trivial zeros, the eigenvalues of $\left(A^q+B^q\right) XX^*$ are the same as the following matrix

$$\begin{bmatrix} A^{q/2} & B^{q/2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A^{q/2} & 0 \\ B^{q/2} & 0 \end{bmatrix}\begin{bmatrix} XX^* & 0 \\ 0 & 0 \end{bmatrix},$$

and in turn, these are the same as the eigenvalues of

$$\begin{aligned}\begin{bmatrix} A^{q/2} & 0 \\ B^{q/2} & 0 \end{bmatrix}\begin{bmatrix} XX^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A^{q/2} & B^{q/2} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A^{q/2}XX^* & 0 \\ B^{q/2}XX^* & 0 \end{bmatrix}\begin{bmatrix} A^{q/2} & B^{q/2} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A^{q/2}XX^*A^{q/2} & A^{q/2}XX^*B^{q/2} \\ B^{q/2}XX^*A^{q/2} & B^{q/2}XX^*B^{q/2} \end{bmatrix} \\ &\ge 0.\end{aligned}$$

So, by Lemma 2.2 and the inequality (2.2) , we have

$$s_j\left(A^{q/2}XX^*B^{q/2}\right) \le \lambda_j\left(\left(A^{1+q}+B^{1+q}\right)^{1+r}\right).$$

That is,

$$s_j\left(A^{q/2(q+1)}K\left(A+B\right)^r KB^{q/2(q+1)}\right) \le s_j\left(\left(A+B\right)^{1+r}\right)$$

on replacing A by $A^{1/(q+1)}$ and B by $B^{1/(q+1)}$. This completes the proof. □

Next, we generalize the inequality (1.1).

Theorem 2.4. *Let $A, B \in M_n$ be positive semidefinite and suppose that*

$$K = \left(A^{1/(q+1)} + B^{1/(q+1)}\right)^{1/2}, \quad 0 \le q \le 1.$$

Then, we have

$$s_j\left(A^{1/2}K\left(A^{q/(1+q)} + B^{q/(1+q)}\right)KB^{1/2}\right) \le s_j\left((A+B)^2\right)$$

for all $j = 1, \cdots, n$.

Proof. By the inequality (2.1) with $r = 1$, we have

$$\frac{1}{2}\lambda_j\left((A^q + B^q)(A+B)^{1/2}\left(A^{1+q} + B^{1+q}\right)(A+B)^{1/2}\right) \le \lambda_j\left(\left(A^{1+q} + B^{1+q}\right)^2\right).$$

Let

$$X = (A+B)^{1/2} A^{(1+q)/2}, \quad Y = (A+B)^{1/2} B^{(1+q)/2}.$$

Then

$$\frac{1}{2}\lambda_j\left((A^q + B^q)(XX^* + YY^*)\right) \le \lambda_j\left(\left(A^{1+q} + B^{1+q}\right)^2\right). \tag{2.3}$$

Except for trivial zeros, the eigenvalues of $(A^q + B^q)(XX^* + YY^*)$ are the same as those of

$$M = \begin{bmatrix} A^q + B^q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X^* & 0 \\ Y^* & 0 \end{bmatrix}.$$

Meanwhile, we know that the eigenvalues of M are the same as those of

$$\begin{aligned} \begin{bmatrix} X^* & 0 \\ Y^* & 0 \end{bmatrix} \begin{bmatrix} A^q + B^q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} X^*(A^q + B^q) & 0 \\ Y^*(A^q + B^q) & 0 \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} X^*(A^q + B^q)X & X^*(A^q + B^q)Y \\ Y^*(A^q + B^q)X & Y^*(A^q + B^q)Y \end{bmatrix} \\ &\ge 0. \end{aligned}$$

So, by Lemma 2.2, we have

$$s_j\left(X^*(A^q + B^q)Y\right) \le \frac{1}{2}\lambda_j\left((A^q + B^q)(XX^* + YY^*)\right). \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$s_j\left(A^{(1+q)/2}(A+B)^{1/2}(A^q + B^q)(A+B)^{1/2}B^{(1+q)/2}\right) \le \lambda_j\left(\left(A^{1+q} + B^{1+q}\right)^2\right),$$

which is equivalent to

$$s_j\left(A^{1/2}K\left(A^{q/(1+q)}+B^{q/(1+q)}\right)KB^{1/2}\right)\le s_j\left((A+B)^2\right)$$

for all $j=1,\cdots,n$. This completes the proof. □

3. Remarks

Remark 3.1. Putting $q=0$ in Theorem 2.3, we obtain the inequality (1.1). Putting $q=1$ in Theorem 2.4, we have

$$s_j\left(A^{1/2}\left(A^{1/2}+B^{1/2}\right)^2B^{1/2}\right)\le s_j\left((A+B)^2\right),\ \ j=1,\cdots,n.$$

For more information on singular value inequalities for positive semidefinite matrices the reader is referred to [3, 4, 10].

Remark 3.2. Let $A,X,B\in M_n$ such that A and B are positive semidefinite. The inequality (1.1) is equivalent to

$$2s_j\left(A^{3/2}B^{1/2}+A^{1/2}B^{3/2}\right)\le s_j\left((A+B)^2\right),\ \ j=1,\cdots,n. \tag{3.1}$$

Recently, Drury ([7], Theorem 1) proved that

$$4s_j\left(AB\right)\le s_j\left((A+B)^2\right),\ \ j=1,\cdots,n, \tag{3.2}$$

which is a question posed by Bhatia and Kittaneh ([5], p.204). Having the inequalities (3.1) and (3.2), it is natural to raise the following question: For $\frac{1}{2}\le v\le\frac{3}{2}$, is it true that

$$2s_j\left(A^vB^{2-v}+A^{2-v}B^v\right)\le s_j\left((A+B)^2\right),\ \ j=1,\cdots,n? \tag{3.3}$$

An inequality weaker than (3.3) is

$$\left\|A^vB^{2-v}+A^{2-v}B^v\right\|\le\frac{1}{2}\left\|(A+B)^2\right\|. \tag{3.4}$$

This is true. In fact, it is known ([3], p.265) that the function

$$g\left(r\right)=\left\|A^rB^{1-r}+A^{1-r}B^r\right\|$$

is convex on $[0,1]$. Replacing A by A^2, B by B^2, and $2r$ by v, we know that the function

$$f\left(v\right)=\left\|A^vB^{2-v}+A^{2-v}B^v\right\|$$

is convex on $[0,2]$. It follows that this function is also convex on $\left[\frac{1}{2},\ \frac{3}{2}\right]$. So, by the convexity of the function $f\left(v\right)$, we obtain the inequality (3.4).

Zhan ([9], Theorem 6) has proved that if $\frac{1}{2} \le v \le \frac{3}{2}$ and $-2 < t \le 2$, then

$$\left\| A^v X B^{2-v} + A^{2-v} X B^v \right\| \le \frac{2}{t+2} \left\| A^2 X + tAXB + XB^2 \right\|$$

The special case $X = I, t = 2$

$$\left\| A^v B^{2-v} + A^{2-v} B^v \right\| \le \frac{1}{2} \left\| A^2 + 2AB + B^2 \right\|$$

is an inequality weaker than (3.4).

Remark 3.3. Let $A, B \in M_n$ be positive semidefinite and suppose that $0 \le v \le 2$. An inequality weaker than (3.3) is

$$s_j \left(A^v B^{2-v} + A^{2-v} B^v \right) \le s_j \left(A^2 + B^2 \right), \quad j = 1, \cdots, n. \tag{3.5}$$

This is a conjecture by Zhan ([10], Conjecture 3.4), settled in the affirmative by Audenaert ([2], Theorem 2). Bhatia and Kittaneh ([6], p.2182) stated that the following inequalities

$$s_j \left(A^v B^{2-v} + B^{2-v} A^v \right) \le s_j \left(A^2 + B^2 \right), \quad j = 1, \cdots, n, \tag{3.6}$$

$$s_j \left(A^v B^{2-v} + B^v A^{2-v} \right) \le s_j \left(A^2 + B^2 \right), \quad j = 1, \cdots, n \tag{3.7}$$

are not always true. The inequalities (3.6) and (3.7) are similar to the inequality (3.5). The inequality (3.6) is not always true even for scalars a and b unless $v = 1$. By Theorem 2.7 of [1], we have

$$\begin{aligned} s_j \left(A^v B^{2-v} + B^{2-v} A^v \right) &\le \tfrac{1}{2} s_j \left(\left(A^v + B^{2-v} \right)^2 \oplus \left(A^v - B^{2-v} \right)^2 \right) \\ &\le s_j \left(\left(A^{2v} + B^{4-2v} \right) \oplus \left(A^{2v} + B^{4-2v} \right) \right) \end{aligned}$$

for all $j = 1, \cdots, n$. This inequality is similar to the inequality (3.6) and has been obtained by Bhatia and Kittaneh ([6], Proposition 6.2). For $v = 1$, the inequality (3.7) is

$$s_j \left(AB + BA \right) \le s_j \left(A^2 + B^2 \right), \quad j = 1, \cdots, n.$$

This is not possible by the example

$$A = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

In fact, we have

$$s_2 \left(AB + BA \right) = 4.1421 \ge 1.9600 = s_2 \left(A^2 + B^2 \right).$$

Meanwhile, we also know that the inequality (3.7) does not hold for $v = 0.5$ and $v = 1.5$ with the same matrices above. On the other hand, if $AB = BA$, then by the the inequality (3.5), we know that the inequality (3.7) holds for all $0 \le v \le 2$.

Remark 3.4. Let A, B be positive semidefinite and $0 \le v \le 2$. Here, we give an inequality for singular values, which is similar to the inequality (3.7). It is known that

$$\begin{bmatrix} A^{2v} & A^vB^{2-v} \\ B^{2-v}A^v & B^{4-2v} \end{bmatrix} = \begin{bmatrix} A^v & 0 \\ B^{2-v} & 0 \end{bmatrix} \begin{bmatrix} A^v & B^{2-v} \\ 0 & 0 \end{bmatrix} \ge 0 \tag{3.8}$$

and

$$\begin{bmatrix} B^{2v} & B^vA^{2-v} \\ A^{2-v}B^v & A^{4-2v} \end{bmatrix} = \begin{bmatrix} B^v & 0 \\ A^{2-v} & 0 \end{bmatrix} \begin{bmatrix} B^v & A^{2-v} \\ 0 & 0 \end{bmatrix} \ge 0. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{bmatrix} A^{2v}+B^{2v} & A^vB^{2-v}+B^vA^{2-v} \\ B^{2-v}A^v+A^{2-v}B^v & A^{4-2v}+B^{4-2v} \end{bmatrix} \ge 0.$$

So, by Theorem 2.1 of [1], we have

$$s_j\left(A^vB^{2-v}+B^vA^{2-v}\right) \le s_j\left(\left(A^{2v}+B^{2v}\right) \oplus \left(A^{4-2v}+B^{4-2v}\right)\right)$$

for all $j = 1, \cdots, n$.

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