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ON z-IDEALS OF POINTFREE FUNCTION RINGS

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Abstract. Let L be a completely regular frame and $\mathcal{R}L$ be the ring of continuous real-valued functions on L. We show that the lattice $\mathrm{Zid}(\mathcal{R}L)$ of z-ideals of $\mathcal{R}L$ is a normal coherent Yosida frame, which extends the corresponding $C(X)$ result of Martínez and Zenk. This we do by exhibiting $\mathrm{Zid}(\mathcal{R}L)$ as a quotient of $\mathrm{Rad}(\mathcal{R}L)$, the frame of radical ideals of $\mathcal{R}L$. The saturation quotient of $\mathrm{Zid}(\mathcal{R}L)$ is shown to be isomorphic to the Stone-Čech compactification of L. Given a morphism $h\colon L \to M$ in **CRegFrm**, Zid creates a coherent frame homomorphism $\mathrm{Zid}(h)\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$ whose right adjoint maps as $(\mathcal{R}h)^{-1}$, for the induced ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$. Thus, $\mathrm{Zid}(h)$ is an s-map, in the sense of Martìnez [18], precisely when $\mathcal{R}(h)$ contracts maximal ideals to maximal ideals.
Keywords: Frame, ideal, z-ideal.
MSC(2010): Primary: 13A15; Secondary: 06D22, 54C30.

1. Introduction

An ideal I of a ring A (all our rings are commutative with identity) is a z-ideal if whenever two elements of A are in the same set of maximal ideals and I contains one of the elements, then it also contains the other. A study of z-ideals in rings generally has been carried out by Mason [21]. In [20] Martínez and Zenk show that, for any compact Hausdorff space X, the lattice of z-ideals of $C(X)$ is a coherent normal Yosida frame when ordered by inclusion.

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Now, although every $C(X)$ is isomorphic to an $\mathcal{R}L$, there are frames L for which $\mathcal{R}L$ is not isomorphic to any $C(X)$. Thus, our Proposition 3.5, which states that $\mathrm{Zid}(\mathcal{R}L)$ is a coherent normal Yosida frame, is a proper extension of the result of Martínez and Zenk. Concerning normality, $\mathrm{Zid}(\mathcal{R}L)$ is actually coherently normal (Proposition 3.7) in the sense of [3].

For any $L \in \mathbf{CRegFrm}$, the map σ_L which sends an ideal in $\mathrm{Zid}(\mathcal{R}L)$ to the join of cozero elements of its members is a dense onto frame homomorphism $\mathrm{Zid}(\mathcal{R}L) \to L$. Given any morphism $h\colon L \to M$ in $\mathbf{CRegFrm}$, Zid creates a coherent homomorphism $\mathrm{Zid}(h)\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$, which leads to a commutative square in $\mathbf{Frm}$ for which $h{\cdot}\sigma_L = \sigma_M \cdot \mathrm{Zid}(h)$ (Proposition 4.3). The square in question is round (in the terminology of [13]) if and only if h is λ-map, as defined in [13]. That is, $(\sigma_M)_* \cdot h = \mathrm{Zid}(h) \cdot (\sigma_L)_*$ if and only if $(\lambda_M)_* \cdot h = h^\lambda \cdot (\lambda_L)_*$, where h^λ denotes the λ-lift of h to Lindelöf coreflections (Proposition 4.4). On the other hand, replacing the morphisms h and $\mathrm{Zid}(h)$ in the square with their right adjoints yields a commutative square if and only if h is a *perfect map*, in the sense that its right adjoint preserves directed joins. That is, $\sigma_L \cdot \mathrm{Zid}(h)_* = h_* \cdot \sigma_M$ if and only if h is a perfect map (Proposition 4.5). Weakening the notion of flatness of a frame homomorphism $h\colon L \to M$ (meaning that h_* is a lattice homomorphism), we call h *coz-flat* if $h_*|_{\mathrm{Coz}\, M}$ is a lattice homomorphism. It turns out that if h_* takes cozero elements to cozero elements, then $\mathrm{Zid}(h)$ is flat if and only if $\mathrm{Zid}(h)_*$ is a frame homomorphism if and only if h is coz-flat. (Proposition 5.2).

2. **Preliminaries**

Our references for the general theory of frames are [15] and [22]. An element c of a frame L is said to be *compact* if for any $S \subseteq L$, $c \leq \bigvee S$ implies $c \leq \bigvee T$, for some finite $T \subseteq S$. If the top of L is compact we say the frame itself is compact. We denote the set of all compact elements of L by $\mathfrak{k}(L)$. An *algebraic frame* is one which is $\bigvee$-generated by its compact elements. If L is a compact algebraic frame such that $a \wedge b \in \mathfrak{k}(L)$ for all $a, b \in \mathfrak{k}(L)$, then L is called *coherent*. A frame homomorphism $h\colon L \to M$ between coherent frames is called *coherent* in case it takes compact elements to compact elements.

By a *point* of L we mean an element $p < 1$ such that $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. Points of a frame are also called *prime elements*.

The points of any regular frame L are precisely those elements which are maximal in the poset $L \setminus \{1\}$. We denote the set of all points of L by $\mathrm{Pt}(L)$. A frame *has enough points* if every element is a meet of points above it. Every compact regular frame has enough points if one assumes (as we do throughout) the Prime Ideal Theorem. Frames that have enough points are also said to be *spatial*. In [20] a frame is called a *Yosida frame* if each of its compact elements is a meet of maximal elements.

We regard the Stone-Čech compactification of L, denoted βL, as the frame of completely regular ideals of L. We denote the right adjoint of the join map $j_L\colon \beta L \to L$ by r_L, and recall that $r_L(a) = \{x \in L \mid x \prec\prec a\}$.

Regarding the f-ring $\mathcal{R}L$ of *continuous real-valued functions* on L, we use the notation of [4]. When we say an element $\alpha \in \mathcal{R}L$ is *positive*, we shall mean that $\alpha \geq \mathbf{0}$. A crucial link between $\mathcal{R}L$ and L is given by the *cozero map*, $\mathrm{coz}\colon \mathcal{R}L \to L$, the properties of which we shall use freely without comment.

A ring is said to be *reduced* if it has no nonzero nilpotent element. An f-ring has *bounded inversion* if every $a \geq 1$ is invertible. It is shown in [4] that $\mathcal{R}L$ is a reduced f-ring with bounded inversion. Every frame homomorphism $h\colon L \to M$ induces a ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ which sends an element α of $\mathcal{R}L$ to the composite $h \cdot \alpha$. Furthermore, $\mathrm{coz}(h \cdot \alpha) = h(\mathrm{coz}\,\alpha)$.

A *cozero element* of L is an element of the form $\mathrm{coz}\,\alpha$ for some $\alpha \in \mathcal{R}L$. The *cozero part* of L, denoted $\mathrm{Coz}\,L$, is the regular sub-σ-frame consisting of all the cozero elements of L. General properties of $\mathrm{Coz}\,L$ can be found in [8].

We recall from [10] the following types of ideals. For any $I \in \beta L$, put

$$\boldsymbol{M}^{I} = \{\alpha \in \mathcal{R}L \mid r_L(\mathrm{coz}\,\alpha) \subseteq I\},$$

and observe that $\boldsymbol{M}^{I}$ is an ideal of $\mathcal{R}L$ which is proper if and only if $I \neq 1_{\beta L}$. For any $a \in L$ we abbreviate $\boldsymbol{M}^{r_L(a)}$ as $\boldsymbol{M}_a$, and observe that

$$\boldsymbol{M}_a = \{\alpha \in \mathcal{R}L \mid \mathrm{coz}\,\alpha \leq a\}.$$

Since, for any $I, J \in \beta L$, $\boldsymbol{M}^{I} = \boldsymbol{M}^{J}$ implies $I = J$ (see proof of [10, Lemma 4.15]), it follows that, for any $a, b \in L$, $\boldsymbol{M}_a = \boldsymbol{M}_b$ if and only if $a = b$. Maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}^{I}$, for $I \in \mathrm{Pt}(\beta L)$ (see [10]).

3. The frame of z-ideals of $\mathcal{R}L$

In the introduction we have recalled the algebraic definition of z-ideal. For the ring $\mathcal{R}L$ we have the following characterisation in terms of the cozero map which we shall need throughout. The second one corresponds to the definition of z-ideals in $C(X)$ [14].

Lemma 3.1. *The following are equivalent for an ideal Q of $\mathcal{R}L$.*

(1) *Q is a z-ideal.*
(2) *For any $\alpha, \beta \in \mathcal{R}L$, $\alpha \in Q$ and $\operatorname{coz}\alpha = \operatorname{coz}\beta$ imply $\beta \in Q$.*
(3) *For any $\alpha, \beta \in \mathcal{R}L$, $\alpha \in Q$ and $\operatorname{coz}\beta \leq \operatorname{coz}\alpha$ imply $\beta \in Q$.*
(4) *$Q = \bigcup\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in Q\}$.*

Proof. The equivalence of (1) and (2) is shown in [11, Corollary 3.8].

(2) $\Rightarrow$ (3): Assume $\alpha \in Q$ and $\operatorname{coz}\beta \leq \operatorname{coz}\alpha$. Then $\operatorname{coz}\beta = \operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \operatorname{coz}(\alpha\beta)$. Since Q is an ideal and $\alpha \in Q$, we have that $\alpha\beta \in Q$. Therefore, by (2), $\beta \in Q$.

(3) $\Rightarrow$ (4): Clearly, $Q \subseteq \bigcup\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in Q\}$ because, for any $\tau \in \mathcal{R}L, \tau \in \boldsymbol{M}_{\operatorname{coz}\tau}$. To see the reverse inclusion, let $\alpha \in Q$, and consider any $\beta \in \boldsymbol{M}_{\operatorname{coz}\alpha}$. This means $\operatorname{coz}\beta \leq \operatorname{coz}\alpha$, so that, by (3), $\beta \in Q$, showing that $\boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq Q$, and hence the desired inclusion.

(4) $\Rightarrow$ (2): Let $\alpha \in Q$ and β be an element of $\mathcal{R}L$ with $\operatorname{coz}\alpha = \operatorname{coz}\beta$. Then

$$\beta \in \boldsymbol{M}_{\operatorname{coz}\beta} = \boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq Q,$$

and hence (2) follows. □

We denote by $\operatorname{Zid}(\mathcal{R}L)$ the lattice of z-ideals of $\mathcal{R}L$ partially ordered by inclusion. Recall that an ideal I of a ring A is called a *radical* ideal if, for any $a \in A$, $a^2 \in I$ implies $a \in I$. It follows from the lemma above that every z-ideal is radical. The lattice $\operatorname{Rad}(A)$ of radical ideals of A is a coherent frame whose compact elements are the finitely generated radical ideals [2]. In particular, since $\mathcal{R}L$ is a Gelfand ring, in the sense of [5], $\operatorname{Rad}(\mathcal{R}L)$ is a normal coherent frame. We prove that $\operatorname{Zid}(\mathcal{R}L)$ is a normal coherent Yosida frame by showing that $\operatorname{Zid}(\mathcal{R}L) = \operatorname{Fix}(z)$, where $z\colon \operatorname{Rad}(\mathcal{R}L) \to \operatorname{Rad}(\mathcal{R}L)$ denotes the z-nucleus on $\operatorname{Rad}(\mathcal{R}L)$. Let us recall the pertinent definitions from [19].

We write $\operatorname{Max}(L)$ for the set of maximal elements of a frame L, and remind the reader that "maximal" is understood to mean maximal different from the top. Martínez and Zenk [19] define the *archimedean* *nucleus* $\mathrm{ar}\colon L \to L$ by

$$\mathrm{ar}(x) = \bigwedge\{m \in \mathrm{Max}(L) \mid x \le m\},$$

and the *z-nucleus* $z\colon L \to L$ by

$$z(x) = \bigvee\{\mathrm{ar}(c) \mid c \le x,\ c \text{ compact}\}.$$

Elements of $\mathrm{Fix}(z)$ are then called *z-elements* of L, and are characterised as follows:

$$x \in \mathrm{Fix}(z) \quad \Leftrightarrow \quad \text{for every } c \in \mathfrak{k}(L),\ \mathrm{ar}(c) \le x \text{ whenever } c \le x.$$

We shall use this characterization to show that z-ideals of $\mathcal{R}L$ are precisely the z-elements of $\mathrm{Rad}(\mathcal{R}L)$. It therefore behoves us to determine which elements of $\mathrm{Rad}(\mathcal{R}L)$ are meets of the maximal ones. Since maximal radical ideals of $\mathcal{R}L$ are precisely the maximal ideals, and intersections of z-ideals are z-ideals, we need to determine which ideals of $\mathcal{R}L$ are intersections of maximal ideals. The following lemma tells us which.

Lemma 3.2. *An ideal of $\mathcal{R}L$ is an intersection of maximal ideals iff it is of the form $\boldsymbol{M}^I$, for some $I \in \beta L$.*

Proof. Let $\{I_\lambda \mid \lambda \in \Lambda\}$ be a collection of elements of βL, and put $I = \bigwedge_{\lambda\in\Lambda} I_\lambda$. We claim that

$$\bigcap_{\lambda\in\Lambda} \boldsymbol{M}^{I_\lambda} = \boldsymbol{M}^I.$$

For any $\alpha \in \mathcal{R}L$ we have

$$\begin{aligned}
\alpha \in \bigcap_{\lambda\in\Lambda} \boldsymbol{M}^{I_\lambda} \quad &\Leftrightarrow \quad \alpha \in \boldsymbol{M}^{I_\lambda} \quad \text{for every } \lambda \\
&\Leftrightarrow \quad r_L(\operatorname{coz}\alpha) \le I_\lambda \quad \text{for every } \lambda \\
&\Leftrightarrow \quad r_L(\operatorname{coz}\alpha) \le \bigwedge_{\lambda\in\Lambda} I_\lambda \\
&\Leftrightarrow \quad \alpha \in \boldsymbol{M}^I,
\end{aligned}$$

which proves the claim. Now it follows from this that an intersection of maximal ideals is of the form $\boldsymbol{M}^I$, for some $I \in \beta L$. On the other hand, let $I \in \beta L$. If $I = 1_{\beta L}$, then $\boldsymbol{M}^I = \mathcal{R}L$, so that it is the empty meet of maximal ideals. So suppose $I < 1_{\beta L}$. Since βL has enough points, $I = \bigwedge\{J \in \mathrm{Pt}(\beta L) \mid I \le J\}$. Thus,

$$\boldsymbol{M}^I = \bigcap\{\boldsymbol{M}^J \mid J \in \mathrm{Pt}(\beta L) \text{ and } J \ge I\};$$

an intersection of maximal ideals. □

Remark 3.3. *As an aside, we deduce from this lemma that the Jacobson radical of* $\mathcal{R}L$ *is the zero ideal. Indeed, in light of* βL *being spatial,* $\bigwedge \mathrm{Pt}(\beta L) = 0_{\beta L}$*, and hence*

$$\mathrm{Jac}(\mathcal{R}L) = \bigcap\{\boldsymbol{M}^I \mid I \in \mathrm{Pt}(\beta L)\} = \boldsymbol{M}^{0_{\beta L}} = \{\mathbf{0}\}.$$

Lemma 3.4. $\mathrm{Zid}(\mathcal{R}L) = \mathrm{Fix}(z)$*, for the z-nucleus on* $\mathrm{Rad}(\mathcal{R}L)$.

Proof. Observe that, by Lemma 3.2, $\mathrm{ar}(\boldsymbol{M}_{\mathrm{coz}\,\alpha}) = \boldsymbol{M}_{\mathrm{coz}\,\alpha}$, for every $\alpha \in \mathcal{R}L$. Recall that the compact elements of $\mathrm{Rad}(\mathcal{R}L)$ are precisely the finitely generated radical ideals. Let K be a compact element of $\mathrm{Rad}(\mathcal{R}L)$, generated by $\alpha_1, \dots, \alpha_m$, say. For brevity, write $\alpha = \alpha_1^2 + \cdots + \alpha_m^2$, and note that $K \subseteq \boldsymbol{M}_{\mathrm{coz}\,\alpha}$, so that

$$\mathrm{ar}(K) \subseteq \mathrm{ar}(\boldsymbol{M}_{\mathrm{coz}\,\alpha}) = \boldsymbol{M}_{\mathrm{coz}\,\alpha}.$$

Next, let I be a point of βL with $K \subseteq \boldsymbol{M}^I$. Since $\alpha \in K$, we have $r_L(\mathrm{coz}\,\alpha) \subseteq I$, and hence $\boldsymbol{M}_{\mathrm{coz}\,\alpha} = \boldsymbol{M}^{r_L(\mathrm{coz}\,\alpha)} \subseteq \boldsymbol{M}^I$. Therefore

$$\boldsymbol{M}_{\mathrm{coz}\,\alpha} \subseteq \bigcap\{M \in \mathrm{Max}(\mathcal{R}L) \mid K \subseteq M\} = \mathrm{ar}(K),$$

and hence $\mathrm{ar}(K) = \boldsymbol{M}_{\mathrm{coz}\,\alpha}$.

Now if Q is a z-ideal and K a compact element of $\mathrm{Rad}(\mathcal{R}L)$ with $K \subseteq Q$, then, an argument as above shows that $\mathrm{ar}(K) \subseteq Q$. Therefore Q is a z-element of $\mathrm{Rad}(\mathcal{R}L)$. Conversely, suppose $Q \in \mathrm{Rad}(\mathcal{R}L)$ is a z-element. Let $\alpha \in Q$, and consider the radical ideal $[\alpha]$ of $\mathcal{R}L$ generated by α. It is a compact element of $\mathrm{Rad}(\mathcal{R}L)$ with $[\alpha] \subseteq Q$. Since Q is a z-element, we have $Q \supseteq \mathrm{ar}([\alpha]) = \boldsymbol{M}_{\mathrm{coz}(\alpha^2)} = \boldsymbol{M}_{\mathrm{coz}\,\alpha}$. But this implies Q is a z-ideal, by Lemma 3.1. □

Proposition 3.5. $\mathrm{Zid}(\mathcal{R}L)$ *is a normal coherent Yosida frame with*

$$\mathfrak{k}(\mathrm{Zid}(\mathcal{R}L)) = \{\boldsymbol{M}_{\mathrm{coz}\,\alpha} \mid \alpha \in \mathcal{R}L\}.$$

Proof. That $\mathrm{Zid}(\mathcal{R}L)$ is a normal coherent Yosida frame follows from the properties of the z-nucleus which are summarised in [18, Definition & Remarks 3.3.1]. Furthermore,

$$\mathfrak{k}(\mathrm{Zid}(\mathcal{R}L)) = z[\mathfrak{k}(\mathrm{Rad}(\mathcal{R}L))].$$

Now, for any $K \in \mathfrak{k}(\mathrm{Rad}(\mathcal{R}L))$, we have

$$z(K) = \bigvee\{\mathrm{ar}(T) \mid T \in \mathfrak{k}(\mathrm{Rad}(\mathcal{R}L)), T \leq K\} = \mathrm{ar}(K) = \boldsymbol{M}_{\mathrm{coz}\,\alpha},$$

for some $\alpha \in \mathcal{R}L$, as we observed in the foregoing proof. Also, for any $\beta \in \mathcal{R}L$,

$$\boldsymbol{M}_{\operatorname{coz}\beta} = \operatorname{ar}([\beta]) = z([\beta]);$$

and so $\mathfrak{k}(\operatorname{Zid}(\mathcal{R}L)) = \{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in \mathcal{R}L\}$. □

We show next that $\operatorname{Zid}(\mathcal{R}L)$ has a property which is a stronger version of normality. In [3], Banaschewski calls a frame L *coherently normal* if it is coherent and, for each compact $c \in L$, the frame $\downarrow c$ is normal. We show below that $\operatorname{Zid}(\mathcal{R}L)$ is coherently normal. We will use [7, Lemma 1] which (paraphrased) states:

> *For any elements a and b of a σ-frame A, there exist u and v in A such that $u \wedge v = 0$ and $a \vee u = b \vee v = a \vee b$.*

In the proof of this result it is clear that $u \leq b$ and $v \leq a$. Martínez [17] says an algebraic frame L *has disjointification* if for each pair of compact elements $a, b \in L$, there exist disjoint compact elements $c \leq a$ and $d \leq b$ in L with $a \vee b = a \vee d = b \vee c$. He then remarks that if L has FIP (meaning that $a \wedge b \in \mathfrak{k}(L)$ for all $a, b \in \mathfrak{k}(L)$), then it is coherently normal if and only if it has disjointification.

Let us observe an easy lemma for use in the upcoming result and later.

Lemma 3.6. *For any $c, d \in \operatorname{Coz} L$, $\boldsymbol{M}_c \vee \boldsymbol{M}_d = \boldsymbol{M}_{c\vee d}$.*

Proof. Clearly, $\boldsymbol{M}_{c\vee d}$ is an upper bound for the set $\{\boldsymbol{M}_c, \boldsymbol{M}_d\}$. Now let H be a z-ideal containing $\boldsymbol{M}_c$ and $\boldsymbol{M}_d$. Take positive $\alpha, \beta \in \mathcal{R}L$ with $\operatorname{coz}\alpha = c$ and $\operatorname{coz}\beta = d$. If $\gamma \in \boldsymbol{M}_{c\vee d}$, then $\operatorname{coz}\gamma \leq \operatorname{coz}(\alpha+\beta)$, implying $\gamma \in H$ as H is a z-ideal. □

Proposition 3.7. $\operatorname{Zid}(\mathcal{R}L)$ *is coherently normal.*

Proof. Let $\alpha \in \mathcal{R}L$ and suppose $Q \vee R = \boldsymbol{M}_{\operatorname{coz}\alpha}$ for some $Q, R \in \operatorname{Zid}(\mathcal{R}L)$. Thus

$$\boldsymbol{M}_{\operatorname{coz}\alpha} = \bigvee_{\tau \in Q} \boldsymbol{M}_{\operatorname{coz}\tau} \vee \bigvee_{\rho \in R} \boldsymbol{M}_{\operatorname{coz}\rho},$$

so that, by compactness, there exists a positive $\gamma \in Q$ and a positive $\delta \in R$ such that $\boldsymbol{M}_{\operatorname{coz}\alpha} = \boldsymbol{M}_{\operatorname{coz}\gamma} \vee \boldsymbol{M}_{\operatorname{coz}\delta}$. Consequently, $\operatorname{coz}\alpha = \operatorname{coz}\gamma \vee \operatorname{coz}\delta$. By the result quoted from [7], there exist $\mu, \nu \in \mathcal{R}L$ such that $\operatorname{coz}\mu \leq \operatorname{coz}\gamma$, $\operatorname{coz}\nu \leq \operatorname{coz}\delta$ and

$$\operatorname{coz}\mu \wedge \operatorname{coz}\nu = 0 \quad \text{and} \quad \operatorname{coz}\gamma \vee \operatorname{coz}\nu = \operatorname{coz}\delta \vee \operatorname{coz}\mu = \operatorname{coz}\gamma \vee \operatorname{coz}\delta.$$

Since Q and R are z-ideals, $\boldsymbol{M}_{\operatorname{coz}\mu} \subseteq Q$ and $\boldsymbol{M}_{\operatorname{coz}\nu} \subseteq R$. Thus, $\boldsymbol{M}_{\operatorname{coz}\mu}$ and $\boldsymbol{M}_{\operatorname{coz}\nu}$ are elements of $\operatorname{Zid}(\mathcal{R}L)$ which witness the normality of ${\downarrow}\boldsymbol{M}_{\operatorname{coz}\alpha}$. □

For what follows, given a compact frame L, we let $s_L\colon L \to L$ denote the *saturation* nucleus on L (see, for instance, [3] or [18]). Traditionally, we write SL for the frame $\operatorname{Fix}(s_L)$. It is known that for any coherent frame L and any $x \in L$,

$$x = s(x) \quad \Leftrightarrow \quad x = \bigwedge\{m \in \operatorname{Max}(L) \mid x \leq m\}.$$

It follows therefore from Lemma 3.2 that the saturation of $\operatorname{Rad}(\mathcal{R}L)$ is

$$S\big(\operatorname{Rad}(\mathcal{R}L)\big) = \{\boldsymbol{M}^{\boldsymbol{I}} \mid I \in \beta L\}.$$

Since $\operatorname{Rad}(\mathcal{R}L)$ and $\operatorname{Zid}(\mathcal{R}L)$ have exactly the same maximal elements, we deduce immediately that

$$S\big(\operatorname{Rad}(\mathcal{R}L)\big) = S\big(\operatorname{Zid}(\mathcal{R}L)\big).$$

Now observe that the map $I \mapsto \boldsymbol{M}^{\boldsymbol{I}}$ is a frame homomorphism from βL into $S(\operatorname{Rad}(\mathcal{R}L))$. As remarked in the Preliminaries, this map is one-one. It is also clearly onto. We therefore have the following result.

Proposition 3.8. $S(\operatorname{Zid}(\mathcal{R}L)) = S(\operatorname{Rad}(\mathcal{R}L)) \cong \beta L$.

Remark 3.9. *For any Gelfand ring A, let* $\operatorname{JRad}(A)$ *be the frame of its Jacobson radical ideals. Banaschewski* [5] *observes that* $\operatorname{JRad}(A) = S(\operatorname{Rad}(A))$*, for any Gelfand ring A. The foregoing proposition can therefore also be deduced from the work of Banaschewski and Sioen* [9] *in which they show that the frame* $\operatorname{JRad}(\mathcal{R}L)$ *of Jacobson radical ideals of $\mathcal{R}L$ is the compact completely regular coreflection of L.*

Regularity of algebraic frames is studied in detail in [19]. We close this section by investigating when $\operatorname{Zid}(\mathcal{R}L)$ is regular. We will use Banaschewski's characterization [3, Lemma 1.5] of regularity in normal coherent frames. Recall that a *P-frame* is one in which every cozero element is complemented.

Proposition 3.10. $\operatorname{Zid}(\mathcal{R}L)$ *is regular iff L is a P-frame.*

Proof. Suppose L is P-frame, and consider any positive $\alpha \in \mathcal{R}L$. Since L is a P-frame, there exists a positive $\beta \in \mathcal{R}L$ such that $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = 0$ and $\operatorname{coz}\alpha \vee \operatorname{coz}\beta = 1$. Then $\boldsymbol{M}_{\operatorname{coz}\alpha} \wedge \boldsymbol{M}_{\operatorname{coz}\beta} = \{\mathbf{0}\}$ and $\boldsymbol{M}_{\operatorname{coz}\alpha} \vee \boldsymbol{M}_{\operatorname{coz}\beta} = \boldsymbol{M}_{\operatorname{coz}(\alpha+\beta)} = \top$. Therefore every compact element of $\operatorname{Zid}(\mathcal{R}L)$ is complemented, hence $\operatorname{Zid}(\mathcal{R}L)$ is regular in view of [3, Lemma 1.5].

Conversely, suppose $\mathrm{Zid}(\mathcal{R}L)$ is regular. Then, by [3, Lemma 1.5] again, $\boldsymbol{M}_{\operatorname{coz}\alpha}$ is complemented in $\mathrm{Zid}(\mathcal{R}L)$, for any positive $\alpha \in \mathcal{R}L$. Pick $Q \in \mathrm{Zid}(\mathcal{R}L)$ such that

$$\boldsymbol{M}_{\operatorname{coz}\alpha} \wedge Q = \{\mathbf{0}\} \quad \text{and} \quad \boldsymbol{M}_{\operatorname{coz}\alpha} \vee Q = \top.$$

The latter implies

$$\boldsymbol{M}_{\operatorname{coz}\alpha} \vee \bigvee\{\boldsymbol{M}_{\operatorname{coz}\gamma} \mid \gamma \in Q\} = \top,$$

so that, by compactness of the frame $\mathrm{Zid}(\mathcal{R}L)$, there is a positive $\beta \in Q$ such that

$$\top = \boldsymbol{M}_{\alpha} \vee \boldsymbol{M}_{\beta} = \boldsymbol{M}_{\operatorname{coz}(\alpha+\beta)},$$

which implies $\operatorname{coz}\alpha \vee \operatorname{coz}\beta = 1$. But now $\boldsymbol{M}_{\operatorname{coz}\alpha} \wedge \boldsymbol{M}_{\operatorname{coz}\beta} = \{\mathbf{0}\}$, since $\boldsymbol{M}_{\operatorname{coz}\beta} \subseteq Q$, and so $\operatorname{coz}\beta$ is a complement of $\operatorname{coz}\alpha$. Therefore L is a P-frame. □

Since a frame L is a P-frame if and only if every ideal of $\mathcal{R}L$ is a z-ideal [11, Proposition 3.9], we have the following corollary.

Corollary 3.11. *If* $\mathrm{Zid}(\mathcal{R}L)$ *is regular, then* $\mathrm{Zid}(\mathcal{R}L) = \mathrm{Rad}(\mathcal{R}L)$.

4. Some commutative squares associated with z-ideals

Given a completely regular frame L, we wish to establish a frame homomorphism $\sigma_L\colon \mathrm{Zid}(\mathcal{R}L) \to L$ in such a way that, for any frame homomorphism $h\colon L \to M$, the wedge

$$\begin{array}{ccc} \mathrm{Zid}(\mathcal{R}L) & & \mathrm{Zid}(\mathcal{R}M) \\ {\scriptstyle\sigma_L}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_M} \\ L & \xrightarrow{\;h\;} & M \end{array}$$

is completable to a commutative square

$$\begin{array}{ccc} \mathrm{Zid}(\mathcal{R}L) & \xrightarrow{\mathrm{Zid}(h)} & \mathrm{Zid}(\mathcal{R}M) \\ {\scriptstyle\sigma_L}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_M} \\ L & \xrightarrow{\;h\;} & M \end{array} \qquad (\ddagger)$$

with a coherent homomorphism $\mathrm{Zid}(h)\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$.

Lemma 4.1. *For any* $L \in \mathbf{CRegFrm}$*, the map* $\sigma_L\colon \mathrm{Zid}(\mathcal{R}L) \to L$ *given by*

$$\sigma_L(Q) = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\}$$

is a dense onto frame homomorphism.

Proof. Clearly, σ_L takes the bottom to the bottom, and the top to the top. Let $Q, R \in \mathrm{Zid}(\mathcal{R}L)$. Then, by the properties of the cozero map,

$$\begin{aligned}\sigma_L(Q) \wedge \sigma_L(R) &= \bigvee_{\alpha\in Q} \operatorname{coz}\alpha \wedge \bigvee_{\beta\in R} \operatorname{coz}\beta \\ &= \bigvee\{\operatorname{coz}(\alpha\beta) \mid \alpha \in Q, \beta \in R\} \\ &\leq \bigvee\{\operatorname{coz}\gamma \mid \gamma \in Q \cap R\} \\ &= \sigma_L(Q \cap R).\end{aligned}$$

Since σ_L clearly preserves order, it follows that σ_L preserves binary meets. Next, let $\{Q_i \mid i \in I\} \subseteq \mathrm{Zid}(\mathcal{R}L)$, and put $a = \bigvee\limits_{i\in I} \sigma_L(Q_i)$. For any $i \in I$, if $\alpha \in Q_i$, then $\operatorname{coz}\alpha \leq a$, and hence $Q_i \subseteq \boldsymbol{M}_a$. Since $\boldsymbol{M}_a \in \mathrm{Zid}(\mathcal{R}L)$, it follows that $\bigvee\limits_{i\in I} Q_i \leq \boldsymbol{M}_a$. Thus,

$$\sigma_L\Big(\bigvee_{i\in I} Q_i\Big) \leq \sigma_L(\boldsymbol{M}_a) = a,$$

the latter in view of complete regularity. Consequently, $\sigma_L(\bigvee\limits_{i\in I} Q_i) = \bigvee\limits_{i\in I} \sigma_L(Q_i)$, and hence σ_L is a frame homomorphism, which is clearly dense. It is onto since, for any $b \in L$, $\sigma_L(\boldsymbol{M}_b) = b$. □

Remark 4.2. *The homomorphism* σ_L *maps precisely as that employed by Banaschewski* [4, Proposition 12] *in showing that the frame of closed* ℓ*-ideals of* $\mathcal{R}^*L$ *realises the Stone-Čech compactification of* L*. There should therefore be no wonder that our proof (in certain places) is modelled on that of Banaschewski.*

For use in the upcoming proposition, we recall from [15, page 64] that if A and B are coherent frames, then any lattice homomorphism $\mathfrak{k}(A) \to \mathfrak{k}(B)$ extends uniquely to a coherent frame homomorphism $A \to B$.

Proposition 4.3. *Let* $h\colon L \to M$ *be a frame homomorphism between completely regular frames. The map* $\mathrm{Zid}(h)\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$ *defined by*

$$\mathrm{Zid}(h)(Q) = \bigvee\{\boldsymbol{M}_{\operatorname{coz}(h\cdot\alpha)} \mid \alpha \in Q\}$$

is the unique frame homomorphism making the square (‡) *above commute.*

Proof. Define $\bar{h}\colon \mathfrak{k}(\mathrm{Zid}(\mathcal{R}L)) \to \mathfrak{k}(\mathrm{Zid}(\mathcal{R}M))$ by $\bar{h}(\boldsymbol{M}_{\operatorname{coz}\alpha}) = \boldsymbol{M}_{\operatorname{coz}(h\cdot\alpha)}$. A routine calculation shows that this is a lattice homomorphism. Its extension to a frame homomorphism $\mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$ is precisely the map $\mathrm{Zid}(h)$. So we are left with verifying commutativity of the diagram, and uniqueness of the map with this property. By coherence, it suffices to show that $\sigma_M \cdot \mathrm{Zid}(h)$ agrees with $h\cdot\sigma_L$ on $\mathfrak{k}(\mathrm{Zid}(\mathcal{R}L))$. For any $\alpha \in \mathcal{R}L$,

$$\begin{aligned}\big(\sigma_M \cdot \mathrm{Zid}(h)\big)(\boldsymbol{M}_{\operatorname{coz}\alpha}) &= \bigvee\{\operatorname{coz}\gamma \mid \gamma \in \boldsymbol{M}_{\operatorname{coz}(h\cdot\alpha)}\}\\ &= \operatorname{coz}(h\cdot\alpha)\\ &= h(\operatorname{coz}\alpha)\\ &= h\Big(\bigvee\{\operatorname{coz}\tau \mid \tau \in \boldsymbol{M}_{\operatorname{coz}\alpha}\}\Big)\\ &= \big(h\cdot\sigma_L\big)(\boldsymbol{M}_{\operatorname{coz}\alpha}).\end{aligned}$$

Finally, to show uniqueness, suppose $g\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$ is a coherent map with $\sigma_M \cdot g = h\cdot\sigma_L$. We shall be done if we can show that g agrees with $\mathrm{Zid}(h)$ on compact elements. Let $\alpha \in \mathcal{R}L$, and, by coherence, pick $\gamma \in \mathcal{R}M$ such that $g(\boldsymbol{M}_{\operatorname{coz}\alpha}) = \boldsymbol{M}_{\operatorname{coz}\gamma}$. Then $\big(\sigma_M\cdot g\big)(\boldsymbol{M}_{\operatorname{coz}\alpha}) = \big(h\cdot\sigma_L\big)(\boldsymbol{M}_{\operatorname{coz}\alpha})$ implies $\operatorname{coz}\gamma = \operatorname{coz}(h\cdot\alpha)$, so that $g(\boldsymbol{M}_{\operatorname{coz}\alpha}) = \mathrm{Zid}(h)(\boldsymbol{M}_{\operatorname{coz}\alpha})$. This completes the proof. □

In [18], Martínez calls a homomorphism $h\colon L \to M$ between compact frames an *s-map* in case there is a frame homomorphism $S(h)\colon SL \to SM$ making the square below commute:

$$\begin{array}{ccc} L & \xrightarrow{\;h\;} & M\\ {\scriptstyle s_L}\big\downarrow & & \big\downarrow{\scriptstyle s_M}\\ SL & \xrightarrow{\;S(h)\;} & SM\end{array}$$

Letting ϱL stand for the subframe of L generated by the regular subframes of L, and denoting the inclusion map $\varrho L \to L$ by ϱ_L, he then shows that there is a frame homomorphism $\varrho(h)\colon \varrho L \to \varrho M$ which makes the square

$$\begin{array}{ccc} \varrho L & \xrightarrow{\varrho(h)} & \varrho M \\ {\scriptstyle \varrho_L}\downarrow & & \downarrow{\scriptstyle \varrho_M} \\ L & \xrightarrow{h} & M \end{array}$$

commute. Furthermore, in Proposition 3.1.1 he shows that a homomorphism $h\colon L \to M$ is an s-map if and only if the foregoing square has the property that when the downward morphisms are replaced with their right adjoints, the resulting square is also commutative.

Here we will find a necessary and sufficient condition on a frame homomorphism $h\colon L \to M$ between completely regular frames for the square (‡) to have similar properties. Actually we will do more. Recall that a frame homomorphism $h\colon L \to M$ is called *perfect* if its right adjoint preserves directed joins. This is equivalent to saying its right adjoint preserves joins of ideals of M. We will show that the diagram obtained from (‡) by replacing the horizontal morphisms with their right adjoints commutes if and only if h is a perfect map.

A word of caution is in order. Whereas in Martínez's case the maps $(\varrho_L)_*$ are also frame homomorphisms, no such claim is made here. Our diagrams sporting arrows which are right adjoints are not necessarily in **Frm**.

To start, observe that the right adjoint of $\sigma_L\colon \mathrm{Zid}(\mathcal{R}L) \to L$ is given by

$$(\sigma_L)_*(a) = \boldsymbol{M}_a.$$

Indeed, $\boldsymbol{M}_a$ is an element of $\mathrm{Zid}(\mathcal{R}L)$ mapped under a (actually mapped to a) by σ_L, and if Q is a member of $\mathrm{Zid}(\mathcal{R}L)$ with $\sigma_L(Q) \le a$, then $\bigvee\limits_{\alpha\in Q} \operatorname{coz}\alpha \le a$, which implies $Q \subseteq \boldsymbol{M}_a$.

Recall from [16] that Lindelöf frames are coreflective in **CRegFrm**. The coreflection of L is the frame λL of σ-ideals of $\operatorname{Coz} L$ with the coreflection map $\lambda_L\colon \lambda L \to L$ given by join. For any $a \in L$ let $[a]$ be the σ-ideal of $\operatorname{Coz} L$ given by

$$[a] = \{c \in \operatorname{Coz} L \mid c \le a\}.$$

The right adjoint of λ_L is the map $(\lambda_L)_*(a) = [a]$. Every homomorphism $h\colon L \to M$ has a λ-*lift*, $h^\lambda\colon \lambda L \to \lambda M$, which is the unique frame homomorphism making the diagram

$$\begin{array}{ccc} \lambda L & \xrightarrow{h^\lambda} & \lambda M \\ {\scriptstyle\lambda_L}\downarrow & & \downarrow{\scriptstyle\lambda_M} \\ L & \xrightarrow{h} & M \end{array}$$

commute. The homomorphism h^λ maps as follows: For any $s \in \operatorname{Coz} M$ and $I \in \lambda L$,

$$s \in h^\lambda(I) \quad \Leftrightarrow \quad s \leq h(c) \ \text{ for some } c \in I.$$

In [13], a homomorphism $h \colon L \to M$ is called a λ-*map* if the diagram

$$\begin{array}{ccc} \lambda L & \xrightarrow{h^\lambda} & \lambda M \\ {\scriptstyle(\lambda_L)_*}\uparrow & & \uparrow{\scriptstyle(\lambda_M)_*} \\ L & \xrightarrow{h} & M \end{array}$$

commutes; that is, if $(\lambda_M)_* \cdot h = h^\lambda \cdot (\lambda_L)_*$. Since the comparison

$$h^\lambda \cdot (\lambda_L)_* \leq (\lambda_M)_* \cdot h$$

always holds, it follows that h is a λ-map if and only if $[h(a)] \subseteq h^\lambda([a])$ for every $a \in L$.

Proposition 4.4. *The square*

$$\begin{array}{ccc} \operatorname{Zid}(\mathcal{R}L) & \xrightarrow{\operatorname{Zid}(h)} & \operatorname{Zid}(\mathcal{R}M) \\ {\scriptstyle(\sigma_L)_*}\uparrow & & \uparrow{\scriptstyle(\sigma_M)_*} \\ L & \xrightarrow{h} & M \end{array}$$

commutes iff h is a λ-map.

Proof. ($\Leftarrow$) Suppose h is a λ-map. Since $\sigma_M \cdot \operatorname{Zid}(h) = h \cdot \sigma_L$, the comparison

$$\operatorname{Zid}(h) \cdot (\sigma_L)_* \leq (\sigma_M)_* \cdot h$$

does hold. So we need only to show that, for any $a \in L$,

$$(\sigma_M)_* h(a) \leq \operatorname{Zid}(h)(\sigma_L)_*(a).$$

The left side of this inequality is $\boldsymbol{M}_{h(a)}$, and the right side is

$$\operatorname{Zid}(h)(\boldsymbol{M}_a) = \bigvee \{\boldsymbol{M}_{\operatorname{coz}(h\cdot\alpha)} \mid \alpha \in \boldsymbol{M}_a\} = \bigcup \{\boldsymbol{M}_{\operatorname{coz}(h\cdot\alpha)} \mid \operatorname{coz}\alpha \leq a\},$$

since the join is directed, and $\alpha \in \boldsymbol{M}_a$ if and only if $\operatorname{coz}\alpha \leq a$. Let $\gamma \in \boldsymbol{M}_{h(a)}$. Then $\operatorname{coz}\gamma \leq h(a)$, and hence $\operatorname{coz}\gamma \in [h(a)]$. Since h is a λ-map, by hypothesis, $[h(a)] \subseteq h^\lambda([a])$, and hence there is a $\delta \in \mathcal{R}L$ such that

$$\operatorname{coz}\delta \leq a \quad \text{and} \quad \operatorname{coz}\gamma \leq h(\operatorname{coz}\delta) = \operatorname{coz}(h \cdot \delta).$$

This shows that γ is in the ideal on the right side of the desired inequality.

($\Rightarrow$) Let $a \in L$, and consider any $c \in \operatorname{Coz} M$ with $c \leq h(a)$. Pick $\gamma \in \mathcal{R}M$ with $c = \operatorname{coz}\gamma$. Then $\gamma \in \boldsymbol{M}_{h(a)}$, so that, by the current hypothesis, $\gamma \in \boldsymbol{M}_{h(\operatorname{coz}\alpha)}$ for some $\alpha \in \mathcal{R}L$ with $\operatorname{coz}\alpha \leq a$. This shows that $c \in h^\lambda([a])$, whence h is a λ-map. □

In the proposition that follows it is the horizontal morphisms in the diagram (‡) that we replace with their right adjoints. We shall need to know how the right adjoint of $\operatorname{Zid}(h)$ maps. To calculate it, recall that coherent maps are perfect. Now, the equality $\sigma_M \cdot \operatorname{Zid}(h) = h \cdot \sigma_L$ implies $\operatorname{Zid}(h)_* \cdot (\sigma_M)_* = (\sigma_L)_* \cdot h_*$, so that, for any $a \in M$,

$$\operatorname{Zid}(h)_*(\boldsymbol{M}_a) = \boldsymbol{M}_{h_*(a)}.$$

Thus, for any $Q \in \operatorname{Zid}(\mathcal{R}M)$,

$$\begin{aligned}
\operatorname{Zid}(h)_*(Q) &= \operatorname{Zid}(h)_*\Big(\bigvee_{\alpha\in Q} \boldsymbol{M}_{\operatorname{coz}\alpha}\Big)\\
&= \bigvee_{\alpha\in Q} \operatorname{Zid}(h)_*(\boldsymbol{M}_{\operatorname{coz}\alpha})\\
&= \bigvee_{\alpha\in Q} \boldsymbol{M}_{h_*(\operatorname{coz}\alpha)}.
\end{aligned}$$

Since the last join is directed, we can also express this as

$$\operatorname{Zid}(h)_*(Q) = \bigcup\{\boldsymbol{M}_{h_*(\operatorname{coz}\alpha)} \mid \alpha \in Q\} = (\mathcal{R}h)^{-1}[Q].$$

The last equality is verified by a routine calculation.

Proposition 4.5. *For any morphism $h\colon L \to M$ in* **CRegFrm**, *the square*

$$\begin{array}{ccc}
\operatorname{Zid}(\mathcal{R}L) & \xleftarrow{\operatorname{Zid}(h)_*} & \operatorname{Zid}(\mathcal{R}M)\\
{\scriptstyle\sigma_L}\downarrow & & \downarrow{\scriptstyle\sigma_M}\\
L & \xleftarrow{h_*} & M
\end{array}$$

commutes iff h is a perfect map.

Proof. ($\Leftarrow$) Assume h is a perfect map. We must show that $\sigma_L \cdot \mathrm{Zid}(h)_* = h_* \cdot \sigma_M$. For any $Q \in \mathrm{Zid}(\mathcal{R}M)$, we have

$$h_*\sigma_M(Q) = h_*\Big(\bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\}\Big) = \bigvee\{h_*(\operatorname{coz}\alpha) \mid \alpha \in Q\},$$

since the join is directed. On the other hand,

$$\sigma_L\big(\mathrm{Zid}(h)_*(Q)\big) = \sigma_L\Big(\bigvee\{\boldsymbol{M}_{h_*(\operatorname{coz}\alpha)} \mid \alpha \in Q\}\Big) = \bigvee h_*(\operatorname{coz}\alpha) \mid \alpha \in Q\}.$$

($\Rightarrow$) Assume $h_* \cdot \sigma_M = \sigma_L \cdot \mathrm{Zid}(h)_*$. Let I be an ideal of M, and define $Q \in \mathrm{Zid}(\mathcal{R}M)$ by

$$Q = \bigvee\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \operatorname{coz}\alpha \in I\} = \bigcup\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \operatorname{coz}\alpha \in I\}.$$

Observe that

$$\sigma_M(Q) = \bigvee\{\operatorname{coz}\alpha \mid \operatorname{coz}\alpha \in I\} = \bigvee I,$$

by complete regularity. Thus

$$\begin{aligned} h_*\Big(\bigvee I\Big) &= (h_* \cdot \sigma_M)(Q) \\ &= (\sigma_L \cdot \mathrm{Zid}(h)_*)(Q) \\ &= \bigvee\{\operatorname{coz}\rho \mid \rho \in (\mathcal{R}h)^{-1}[Q]\} \\ &= \bigvee\{\operatorname{coz}\rho \mid h \cdot \rho \in Q\} \\ &\leq \bigvee\{h_*h(\operatorname{coz}\rho) \mid h \cdot \rho \in Q\} \\ &= \bigvee\{h_*\big(\operatorname{coz}(h \cdot \rho)\big) \mid h \cdot \rho \in Q\} \\ &\leq \bigvee\{h_*(x) \mid x \in I\} \qquad \text{since } \tau \in Q \Rightarrow \operatorname{coz}\tau \in I. \end{aligned}$$

It follows therefore that $h_*(\bigvee I) = \bigvee h_*[I]$, whence h is a perfect map. $\square$

We conclude by remarking that, in view of the fact that $\mathrm{Zid}(h)_*$ maps as $(\mathcal{R}h)^{-1}$, it follows from [18, Proposition 3.2.2] that

> $\mathrm{Zid}(h)$ *is an s-map iff the ring homomorphism* $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ *contracts maximal ideals to maximal ideals.*

In [12] we show that the assignments $L \mapsto \mathrm{Zid}(\mathcal{R}L)$ and $h \mapsto \mathrm{Zid}(h)$ define a functor **CRegFrm** $\to$ **CohFrm**; and we study some properties of this functor and another induced by d-ideals of $\mathcal{R}L$.

5. A note on flatness

We remind the reader that a frame homomorphism $h\colon L \to M$ is said to be *flat* if $h_*\colon M \to L$ is a lattice homomorphism. Usually, h is assumed to be onto, but we shall relax that requirement. Weakening this, we say h is *coz-flat* if $h_*(0) = 0$ and $h_*(a \vee b) = h_*(a) \vee h_*(b)$ for all $a, b \in \operatorname{Coz} L$. Observe that coz-flatness is a genuine weakening of flatness. Indeed, for any non-normal non-compact completely regular frame L, the join map $\beta L \to L$ is coz-flat, but not flat. We aim to show that for a homomorphism h whose right adjoint sends cozero elements to cozero elements, $\mathrm{Zid}(h)$ is flat precisely when h is coz-flat. We need a lemma.

Lemma 5.1. *Let $h\colon L \to M$ be a morphism in* **CRegFrm**. *For all $S, T \in \mathrm{Zid}(\mathcal{R}L)$ and $Q, R \in \mathrm{Zid}(\mathcal{R}M)$ we have:*

(1) $S \vee T = \bigvee\{\boldsymbol{M}_{\operatorname{coz}\gamma} \mid \gamma \in S + T\} = \bigcup\{\boldsymbol{M}_{\operatorname{coz}\gamma} \mid \gamma \in S + T\}$
(2) $\mathrm{Zid}(h)_*(Q \vee R) = \bigvee\{\boldsymbol{M}_{h_*(\operatorname{coz}\tau)} \mid \tau \in Q + R\}$.

Proof. (1) Observe that the join is directed, and hence equals the union. The rest is easy to check.

(2) Again, observe that the join is directed, and hence, by the first part,

$$\begin{aligned} \mathrm{Zid}(h)_*(Q \vee R) &= (\mathcal{R}h)^{-1}\Big(\bigcup\{\boldsymbol{M}_{\operatorname{coz}\tau} \mid \tau \in Q + R\}\Big) \\ &= \bigcup\{(\mathcal{R}h)^{-1}(\boldsymbol{M}_{\operatorname{coz}\tau}) \mid \tau \in Q + R\} \\ &= \bigvee\{\boldsymbol{M}_{h_*(\operatorname{coz}\tau)} \mid \tau \in Q + R\}. \end{aligned}$$

□

Proposition 5.2. *Let $h\colon L \to M$ be a morphism in* **CRegFrm**. *Consider the following statements.*

(1) $\mathrm{Zid}(h)$ *is flat.*
(2) $\mathrm{Zid}(h)_*$ *is a frame homomorphism.*
(3) h *is coz-flat.*

We have that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *Furthermore, if h_* takes cozero elements to cozero elements, then all three statements are equivalent.*

Proof. Since $\mathrm{Zid}(h)$ is a coherent map, and hence its right adjoint preserves directed joins, it follows that (1) and (2) are equivalent.

(1) ⇒ (3): Banaschewski [6, Lemma 2] has shown that h is dense if and only if $\mathcal{R}h$ is one-one. Thus, h is dense if and only if $(\mathcal{R}h)^{-1}\{\mathbf{0}\} = \{\mathbf{0}\}$, that is, if and only if $\mathrm{Zid}(h)_*(\bot) = \bot$. So we need only show preservation of binary joins of cozero elements. Let $a, b \in \mathrm{Coz}\, M$. By (1),

$$\begin{aligned}\boldsymbol{M}_{h_*(a\vee b)} &= \mathrm{Zid}(h)_*(\boldsymbol{M}_{a\vee b})\\ &= \mathrm{Zid}(h)_*(\boldsymbol{M}_a \vee \boldsymbol{M}_b)\\ &= \mathrm{Zid}(h)_*(\boldsymbol{M}_a) \vee \mathrm{Zid}(h)_*(\boldsymbol{M}_b) \quad \text{since } \mathrm{Zid}(h) \text{ is flat}\\ &= \boldsymbol{M}_{h_*(a)} \vee \boldsymbol{M}_{h_*(b)}.\end{aligned}$$

Applying the map σ_L yields $h_*(a \vee b) = h_*(a) \vee h_*(b)$, as required.

(3) ⇒ (1): Let $Q, R \in \mathrm{Zid}(\mathcal{R}M)$. Then

$$\begin{aligned}\mathrm{Zid}(h)_*(Q \vee R) &= \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}\,\gamma)} \mid \gamma \in Q + R\} \quad \text{by the lemma above}\\ &= \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}(\alpha+\beta))} \mid \alpha \in Q, \beta \in R\}\\ &\le \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}\,\alpha \vee \mathrm{coz}\,\beta)} \mid \alpha \in Q, \beta \in R\}\\ &= \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}\,\alpha) \vee h_*(\mathrm{coz}\,\beta)} \mid \alpha \in Q, \beta \in R\},\end{aligned}$$

the last step because h is coz-flat. Now since $h_*[\mathrm{Coz}\, M] \subseteq \mathrm{Coz}\, L$, this implies

$$\begin{aligned}\mathrm{Zid}(h)_*(Q \vee R) &\le \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}\,\alpha)} \vee \boldsymbol{M}_{h_*(\mathrm{coz}\,\beta)} \mid \alpha \in Q, \beta \in R\}\\ &= \bigvee_{\alpha\in Q} \boldsymbol{M}_{h_*(\mathrm{coz}\,\alpha)} \vee \bigvee_{\beta \in R} \boldsymbol{M}_{h_*(\mathrm{coz}\,\beta)}\\ &= \mathrm{Zid}(h)_*(Q) \vee \mathrm{Zid}(h)_*(R),\end{aligned}$$

which proves the nontrivial inequality of the desired equality. □

We end with the following observation. We remarked above that coz-flatness is strictly weaker than flatness. However,

a coz-flat perfect homomorphism into a completely regular frame is flat.

To see this, let $\phi\colon A \to B$ be such a homomorphism, and let $b_1, b_2 \in B$. By complete regularity,

$$\phi_*(b_1 \vee b_2) = \phi_*\Big(\bigvee\{c \vee d \mid c, d \in \mathrm{Coz}\, B, c \le b_1 \text{ and } d \le b_2\}\Big).$$

The set whose join is displayed is directed, so

$$\begin{aligned}\phi_*(b_1 \vee b_2) &= \bigvee\{\phi_*(c \vee d) \mid c, d \in \operatorname{Coz} B, c \leq b_1 \text{ and } d \leq b_2\}\\ &= \bigvee\{\phi_*(c) \vee \phi_*(d) \mid c, d \in \operatorname{Coz} B, c \leq b_1 \text{ and } d \leq b_2\}\\ &= \bigvee\{\phi_*(c) \mid c \in \operatorname{Coz} B, c \leq b_1\}\\ &\qquad\qquad \vee \bigvee\{\phi_*(d) \mid d \in \operatorname{Coz} B, d \leq b_2\}\\ &= \phi_*(b_1) \vee \phi_*(b_2) \quad \text{since } \phi \text{ is perfect.}\end{aligned}$$

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Professor F. Azarpanah has shown us a draft [1] of a work he has done on the lattice of z-ideals of $C(X)$. Our work does not overlap with his; neither in style nor in content.

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