ISSN: 1017-060X (Print)



ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 40 (2014), No. 3, pp. 677-688

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Published by Iranian Mathematical Society http://bims.ims.ir

Bull. Iranian Math. Soc. Vol. 40 (2014), No. 3, pp. 677–688 Online ISSN: 1735-8515

WHEN EVERY P-FLAT IDEAL IS FLAT

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(Communicated by Teo Mora)

ABSTRACT. In this paper, we study the class of rings in which every P-flat ideal is flat and which will be called PFF-rings. In particular, Von Neumann regular rings, hereditary rings, semi-hereditary ring, PID and arithmetical rings are examples of PFF-rings. In the context domain, this notion coincide with Prüfer domain. We provide necessary and sufficient conditions for $R = A \propto E$ to be a PFF-ring where A is a domain and E is a K-vector space, where K := qf(A) or A is a local ring such that ME := 0. We give examples of non-fqp PFF-ring, of non-arithmetical PFF-ring, of non-semihereditary PFF-ring. Also, we investigate the stability of this property under localization and homomorphic image, and its transfer to finite direct products. Our results generate examples which enrich the current literature with new and original families of rings that satisfy this property.

Keywords: *PFF*-rings, *P*-flat module, trivial extension. MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.

1. Introduction

All rings considered in this paper are assumed to be commutative, with identity elements and all modules are unitary.

We start by recalling a few definitions. An *R*-module *M* is called *P*-flat if, for any $(s, x) \in R \times M$ such that $sx := 0, x \in (0:s)M$. If *M*

O2014 Iranian Mathematical Society

Article electronically published on June 17, 2014.

Received: 13 January 2012, Accepted: 15 May 2013.

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is flat, then M is naturally P-flat. When R is a domain, M is P-flat if and only if its torsion-free. When R is arithmetic ring, then any P-flat module is flat (by [8, p. 236]). Also, every P-flat cyclic module is flat (by [8, Proposition 1(2)]). P-flatness coincides with torsion-freeness in the sense of [17]. P-flat modules are also called (1, 1)-flatness in [21].

Let R be a ring and let M be an R-module. As usual, we use fdR(M) to denote the usual flat dimensions of M. If R is an integral domain, we denote its quotient field by qf(R). In this paper, we are interested to those rings in which every P-flat ideal is flat and which will be called PFF-rings. In particular, semi-hereditary ring, PID and arithmetical rings are examples of PFF-rings. In [2, Theorem 4.11, Corollary 4.12, Corollary 5.5 and Theorem 5.7], the authors showed that P-flat ideals, in a perfect ring which is self $(\aleph_0, 1)$ -injective and in a quasi-Frobenius ring are projective (and so flat). So, perfect ring which is self $(\aleph_0, 1)$ -injective is PFF-rings. In particular, locally perfect (1, 1)-coherent and locally self $(\aleph_0, 1)$ -injective is PFF-ring.

Let A be a ring and E be an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by (a, e)(a', e') := (aa', ae' + a'e). For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $p \propto E$, where p is a prime (resp., maximal) ideal of A by [4, Theorem 3.2]. Suitable background on commutative trivial ring extensions is [4, 9, 12, 15].

The purpose of this paper is to give some simple methods in order to construct PFF-rings. For this, we investigate the stability of the PFF-property under localization and homomorphic image, and its transfer to various contexts of constructions such as trivial ring extensions and finite direct products. Our results generate original examples which enrich the current literature with new families of rings satisfying the PFF-pr

2. Main results

Let R be a commutative ring and M be an R-module. We will use the following notations and basic notions: $Z(R) := \{a \in R | ax := 0 \text{ for some } 0 \neq x \in R\}$ denotes the set of zero divisors of R.

 $Z(M) := \{a \in R/ax := 0 \text{ for some } 0 \neq x \in M\}$ denotes the set of zero divisors of M.

 $Ann(M) := \{a \in R/ax := 0 \text{ for all } 0 \neq x \in M\}$ denotes the annihilator of M.

T(R) denotes the total ring of quotients of R, that is, the localization of R by the set of all its non zero divisors.

qf(R) denotes the quotient field of R.

The homological interpretation of the notion of P-flat modules is given as follows :

Proposition 2.1. Let R be a ring. An R-module M is P-flat if and only if $Tor_1^R(M; R/aR) = 0$ for every $a \in R$.

Proof. Assume that M is a P-flat R-module. For $a \in R$, consider the map $1 \otimes \lambda_a : M \otimes aR \to M \otimes R$ where $\lambda_a : aR \to R$ is the inclusion. If $m \otimes a \in Ker(1 \otimes \lambda_a)$, where $m \in M$, then $m \otimes a = 0$ in $M \otimes R$; hence am = 0 in M (because $M \otimes R \cong M$ via $m \times r \to rm$). By hypothesis, $m = \sum_j s_j m_j$, where $s_j \in (0 : a)$ and $m_j \in M$. Thus, $m \otimes a = \sum_j s_j m_j \otimes a = \sum_j m_j \otimes s_j a = 0$. Thus, $Ker(1 \otimes \lambda_a) = \{0\}$, and so $Tor_1^R(M, R/aR) = 0$.

Assume that $Tor_1^R(M; R/aR) = 0$ for every $a \in R$. Thus, for every $a \in R$, the map $M \otimes aR \to M$; defined by $m \times a \mapsto am$ is injective. For any $a \in R$, we have the exact sequence of R-modules

$$0 \longrightarrow (0:a) \hookrightarrow^{\iota} R \longrightarrow^{f} aR \longrightarrow 0$$

with f(1) = a. It is clear that $1 \otimes m \in Ker(f \otimes 1_M) = Im(\iota \otimes 1_M)$. Hence, $1 \otimes m = \Sigma_j s_j \otimes m_j$, where $s_j \in (0:a)$ and $m_j \in M$. Thus, $1 \otimes m = 1 \otimes (\Sigma_j s_j m_j)$. Therefore, $m = \Sigma_j s_j m_j$. Consequently, M is P-flat module.

In a domain context, the *PFF*-domain become a Prüfer domain:

Remark 2.2. Let R be a domain. Then R is a PFF-domain if and only if it is a Prüfer domain.

Proof. R is a *PFF*-domain if and only if $wdim(R) \leq 1$ if and only if R is a Prüfer domain (by [10]).

Hence, any non Prüfer domain is a non *PFF*-ring.

Remark 2.3. Any principal ideal ring is a PFF-ring.

Proof. Assume that R is a principal ideal ring and let I be a P-flat ideal of R. I is flat by [8, Proposition 1], as desired.

Proposition 2.4. Every local ring (R, M) with $M^2 = 0$ is a PFF-ring.

We need the following Lemma before proving Proposition 2.4.

Lemma 2.5. Let (A, M) be a local ring and I be a P-flat ideal of A. Then IM = I or Ann(I) = 0.

Proof. Let I be a P-flat ideal of A. Assume that $Ann(I) \neq 0$ and we have to show that IM = I. Let x be a nonzero element of I. Then there exists a nonzero element d of Ann(I) such that dx = 0. Hence, there exist $(y_i)_{i=1,..,n}$ a family of elements of I and $(c_i)_{i=1,..,n}$ a family of elements of I and $(c_i)_{i=1,..,n}$ a family of elements of (0:d) such that $x = \sum_{i=1}^{n} y_i c_i$ since I is P-flat. Therefore, $x \in IM$ and so I = IM, as desired. \Box

Proof of Proposition 2.4. We claim that there exists no proper *P*-flat ideal of *R*. On the contrary let *I* be a nonzero proper *P*-flat ideal of *R*. Then, Ann(I) = 0 by Lemma 2.5, a contradiction since Ann(I) = M, as desired.

We know that an arithmetical ring is a PFF-ring. Now, we construct an example of a non-arithmetical PFF-ring.

Example 2.6. Let K be a field and let tand u be indeterminates over K. Set $R = K[t, u]/(t, u)^2$. From [11, Example 4.4], R is local ring with maxiaml ideal \mathfrak{m} with $\mathfrak{m}^2 = 0$, and so every P-flat ideal of R is flat but R is non-arithmetical ring.

The next result establish the transfer of the PFF-property to localization.

Proposition 2.7. Let R be a commutative ring. Then R is a PFF-ring provided R_p is a PFF-ring, for each prime (resp., maximal) ideal p of R.

Before proving Proposition 2.7, we establish the following lemmas.

Lemma 2.8. Let R be a commutative ring and let M be an R-module. Then the following conditions are equivalent: Cheniour and Mahdou

- (1) M is P-flat.
- (2) The canonical map: $M \otimes_R Ra \to M \otimes_R R$ is injective for any $a \in R$.
- (3) Tor(M, R/Ra) := 0 for all $a \neq 0 \in R$.

Proof. 1) \Rightarrow 2) Assume that M is a P-flat R-module and let a be a nonzero element of R. Our aim is to show that $u: M \otimes_R Ra \to M \otimes_R R$, where $u(m \otimes a) = ma$ is injective. Let $m \otimes a \in Keru$ then, ma = 0. Since M is a P-flat R-module, there exists $(\beta_i)_{i=1..n} \in (0:a)^n$ and $(m_i)_{i=1..n} \in M^n$ such that $m = \sum_{i=1}^n \beta_i m_i$. Hence, $m \otimes a = \sum_{i=1}^n \beta_i m_i \otimes a = \sum_{i=1}^n m_i \otimes \beta_i a = 0$, as desired.

2) \Rightarrow 1) Assume that the canonical map: $M \otimes_R Ra \to M \otimes_R R$ is injective for any $a \in R$. We have to show that if ma = 0 where $m \in M$ and $a \in R$, then $m \in (0:a)M$. Since $u(m \otimes a) = ma = 0$, it follows from 2) that $m \otimes a = 0$. Consider the map $f: R \to Ra$ such that f(1) = a. Then the exact sequence $0 \to kerf \to R \to Ra \to 0$ yields the exact sequence $kerf \otimes M \to R \otimes M \to Ra \otimes M \to 0$, where $(f \otimes 1_M)(1 \otimes m) =$ $a \otimes m = 0$. Hence $(1 \otimes m) \in ker(f \otimes 1_M) = Im(i \otimes 1_M)$, and so there exists $(y_i, m_i)_{1 \leq i \leq n} \in (kerf \times M)$ such that $1 \otimes m = (i \otimes 1_M)(\sum_{1 \leq i \leq n} i(y_i)m_i) =$ $\sum_{1 \leq i \leq n} (i(y_i) \otimes m_i) = 1 \otimes \sum_{1 \leq i \leq n} i(y_i)m_i$. Therefore, $\sum_{1 \leq i \leq n} i(y_i)m_i =$ m and $i(y_i)a = i(y_ia) = i(f(y_i)) = i(0) = 0$ and thus M is P-flat. 2 \Leftrightarrow 3 The proof is straightforward and may be left to the reader.

Lemma 2.9. Let R be a commutative ring and let M be an R-module. The following statements are equivalent.

- (1) M is P-flat.
- (2) M_p is *P*-flat for every prime ideal *p* of *R*.
- (3) M_m is P-flat for every maximal ideal m of R.

Proof. $1 \Rightarrow 2$ by [17, Lemma 3.1].

 $2 \Rightarrow 3$ is straightforward.

 $3 \Rightarrow 1$. Assume that M_m is *P*-flat for every maximal ideal of *R*. Using Lemma 2.8 we need to prove that the morphism $f: M \otimes Ra \to M \otimes R$ is injective for every nonzero element *a* of *R*. For each maximal ideal *m* of *R*, the morphism $f_m: M_m \otimes Ra_m \to M_m \otimes R_m$ is injective (since M_m is *P*-flat) and so *f* is injective by [19, Proposition 4.90]. \Box

Proof of Proposition 2.7. Let I be a P-flat ideal of R, then I_p is P-flat for every prime (resp maximal) ideal p of R by Lemma 2.9 and so it is flat since R_p is PFF-ring. Hence, I is flat and so R is a PFF-ring, as desired.

Now, we study the transfer of a PFF-property between a ring A and $A \propto E$, the trivial ring extension of A by E, where E be an A-module. The main result (Theorem 2.10) enriches the literature with original examples of PFF-ring. Recall that if E is an A-module, then $Z(E) := \{a \in A \text{ such that } ae = 0 \text{ for some } e(\neq 0) \in E\}.$

Theorem 2.10. Let A be a ring, E be an A-module, and let $R := A \propto E$ be a trivial ring extension of A by E. Then:

- (1) Assume that A is a domain and E is a K-vector space, where K := qf(A). Then R is a PFF-ring if and only if A is a PFF-ring.
- (2) Assume that (A, M) be a local ring such that ME := 0. Then R is a PFF-ring if and only if A is a PFF-ring.

The proof of the theorem relies on the following lemmas which are of independent interest.

Lemma 2.11. Let (A, M) be a local ring, E be an A-module such that ME := 0, $R := A \propto E$ be a trivial ring extension of A by E, and let I be a nonzero ideal of A. Then $J := I \propto 0$ is a P-flat ideal of R if and only if I is a P-flat ideal of A.

Proof. Assume that J is a P-flat ideal of R. Let x be a nonzero element of I and let a be an element of A such that xa = 0. Then $(x, 0) \in J$, $(a, 0) \in R$ and (x, 0)(a, 0) = (0, 0). Since J is a P-flat R-module, then there exists $(y_i, 0)_{i=1,..,n}$ a family of elements of J and $(c_i, f_i)_{i=1,..,n}$ a family of elements of (0 : (a, 0)) such that $(x, 0) = \sum_{i=1}^{n} (y_i, 0)(c_i, f_i)$. Therefore, $x = \sum_{i=1}^{n} y_i c_i$, where $y_i \in I$ and $ac_i = 0$ for all $1 \leq i \leq n$). Hence, I is a P-flat A-module.

Conversely, assume that I is a P-flat ideal of A and let (x, 0) be a nonzero element of $J := I \propto 0$ and (d, e) be an element of $A \propto E$ such that (x, 0)(d, e) = (0, 0). Then, xd = 0. Since I is a P-flat ideal of A, then there exists $(y_i)_{i=1,..,n}$ a family of elements of I and $(c_i)_{i=1,..,n}$ a family of elements of (0:d) such that $x = \sum_{i=1}^{n} y_i c_i$. Therefore, $(x, 0) = (\sum_{i=1}^{n} y_i c_i, 0) = \sum_{i=1}^{n} (y_i, 0)(c_i, 0)$, where $(y_i, 0) \in J (:= I \propto 0)$ and $(d, e)(c_i, 0) = (0, 0)$ for all $1 \leq i \leq n$. Hence, $J := I \propto 0$ is a P-flat ideal of R.

Lemma 2.12. Let A be a domain, E be an A-module, F be a nonzero sub-module of E, and let $R := A \propto E$ be a trivial ring extension of A by E. Then $0 \propto F$ is not a P-flat R-module.

Proof. Let F be a nonzero sub-module of E. Two cases are possible: **Case 1:** Z(F) = 0. Let $(0, f) \neq (0, 0)$ and $(0, e) \neq (0, 0)$ be two elements of $0 \propto F$ such that (0, f)(0, e) = (0, 0). Hence, $(0, f) \notin (0 : (0, e))(0 \propto F) = 0$ since $(0 : (0, e)) = 0 \propto E$ (since Z(F) = 0). Therefore, $0 \propto F$ is not a P-flat R-module.

Case 2: $Z(F) \neq 0$. Let $d(\neq 0) \in Z(F)$ and $f(\neq 0) \in F$ such that df = 0. Hence, (d, 0)(0, f) = (0, 0) and $(0 : (d, 0) \subseteq 0 \propto E$ (since A is a domain), and so $(0, f) \notin (0 : (d, 0))(0 \propto F) = 0$. Therefore, $0 \propto F$ is not a P-flat R-module.

Lemma 2.13. Let A be a domain, E be a K-vector space where K := qf(A), $R := A \propto E$ be a trivial ring extension of A by E, and let I be a nonzero ideal of A. Then $J := I \propto E$ is a P-flat ideal of R.

Proof. Let (x, f) be a nonzero element of $J := I \propto E$ and (d, e) be an element of $A \propto E$ such that (x, f)(d, e) = (0, 0). Then, xd = 0 and xe + df = 0. Since A is a domain, then two cases are possible: x = 0 and df = 0, or $x \neq 0$, d = 0 and xe = 0.

Case 1: x = 0 and df = 0. Then x = 0 and d = 0, since $(x, f) \neq (0, 0)$. Hence $(0, f) = (0, b^{-1}f)(b, 0)$ for some nonzero element *b* of *I* and so $(0, f) \in (0 : (0, e))(I \propto E)$ (since $(0 : (0, e)) = 0 \propto E$), as desired.

Case 2: $x \neq 0$, d = 0 and xe = 0. Then e = 0, and so $(x, f) \in (0 : (0,0))(I \propto E)$, as desired.

Hence, $J := I \propto 0$ is a *P*-flat ideal of *R*.

Proof. Theorem 2.10. 1) Assume that A is a PFF-domain (that is a Prüfer domain). Let J be a nonzero P-flat ideal of R, we need to prove that J is a flat ideal of R. By [4, Corollary 3.4], $J := I \propto E$ or $J := 0 \propto E'$ for some ideal I of A or some submodule E' of E. We omit the case $0 \propto E'$ by Lemma 2.12, then $J := I \propto E$. But $J := I \propto E$ is flat ideal of R by [1, Theorem 8] since I is flat ideal of A (since A is a Prüfer domain) and E is flat A-module. So, we conclude that R is a PFF-ring, as desired.

Conversely, assume that R is a PFF-ring and let I be a nonzero ideal of A. We need to prove that I is a flat ideal of A. But $J := I \propto E$ is a P-flat ideal of R by Lemma 2.13. Hence, $J := I \propto E$ is a flat ideal of R (since R is a PFF-ring) and so I is a flat ideal of A by [1, Theorem 8]. Therefore, R is a PFF-ring, as desired.

2) Assume that A is a *PFF*-ring and let J be a P-flat ideal of R. By Lemma 2.5, we may assume that $J(M \propto E) := J$. Then $J := J(M \propto E)$

 $E) \subseteq (M \propto E)(M \propto E) = M^2 \propto 0$ and so $J := I \propto 0$ for some ideal I of A. Hence, I is a P-flat ideal of A (by Lemma 2.11 since J is a P-flat ideal of R) and so I is a flat ideal of A (since A is a PFF-ring). Therefore, J is a flat ideal of R by Lemma 2.11, as desired.

Conversely, assume that R is a PFF-ring and let I be a P-flat ideal of A. Then $J := I \propto 0$ is a P-flat ideal of R by Lemma 2.11 and so J is a flat ideal of R (since R is a PFF-ring). Therefore I is a flat ideal of A, as desired. And this completes the proof of Theorem 2.10.

Corollary 2.14. Let (A, M) be a local ring with $M^2 = 0$ and let E be a nonzero A/M -vector space. Then $A \propto E$ is a PFF-ring.

Next, we give an example of non- $fqp \ PFF$ -ring, an example of non-arithmetical PFF-ring, and an example of non-semihereditary PFF-ring.

A domain is Prüfer if all its non-zero finitely generated ideals are invertible. There are well-known extensions of this notion to arbitrary rings (with zero divisors). Namely, for a ring R:

1) R is semihereditary, i.e., every finitely generated ideal of R is projective;

2) R is arithmetical, i.e., every finitely generated ideal of R is locally principal;

3) R is an fqp-ring, i.e., every finitely generated ideal of R is quasiprojective (see [3]).

All these forms coincide in the context of domains to a Prüfer domain. See for instance [3, 5, 6, 10, 11].

As an application of Theorem 2.10, one can construct new examples of non- $fqp \ PFF$ -rings, non-arithmetical PFF-rings, and non-semihereditary PFF-rings as shown below.

Example 2.15. Let (A, m) be a local PFF-ring with $m^2 \neq 0$, E be a nonzero (A/m)-vector space, and let $R := A \propto E$ be the trivial ring extension of A by E. Then:

1) R is a PFF-ring by Theorem 2.10(2).

2) R is non-fqp-ring by [3, Theorem 4.4].

Example 2.16. Let \mathbb{Z} be the ring of integers, $\mathbb{Q} := qf(\mathbb{Z})$, \mathbb{R} be the field of real numbers and let $S := \mathbb{Z} \propto \mathbb{R}$. Then: 1) S is a PFF-ring by Theorem 2.10(1). 2) S is non-semihereditary since S non-coherent by [15, Theorem 3.1].

Example 2.17. Let A be a PFF-domain (that is a Prüfer domain) which is not a field, K := qf(A), and let $R := A \propto K$ be the trivial ring

extension of A by K. Then: 1) R is a PFF-ring by Theorem 2.10(1).

2) R is non-arithmetical by [6, Theorem 3.1].

Our next result establish the transfer of PFF property to a particular homomorphic image.

Proposition 2.18. Let R be a ring and let I be an ideal of R. 1) Assume that $I_m \in \{R_m, 0\}$ and R_m is a PFF-ring for every maximal ideal m of R containing I. Then, R/I is a PFF-ring. 2) Assume that I be a pure ideal of R. Then, R/I is a PFF-ring if so is R.

Before proving Proposition 2.18, we establish the following Lemma.

Lemma 2.19. Let $0 \to A \to B \to C \to 0$ be an exact sequence of *R*-modules. If *C* and *A* are *P*-flat, then so is *B*.

Proof. Let a be a nonzero element of R. For each $a \in R$, we have the exact sequence:

$$0 = Tor_1^R(A, R/aR) \longrightarrow Tor_1^R(B, R/aR) \longrightarrow Tor_1^R(C, R/aR) = 0$$

Hence, $Tor_1^R(B, R/aR) = 0$ and so B is P-flat.

Lemma 2.20. Let $f : R \to S$ be a ring homomorphism making S a P-flat R-module. If an S-module E is P-flat as an S-module, then E is P-flat as an R-module.

Proof. Assume that E is P-flat as an S-module and we must to show that E is P-flat as an R-module. Let x be a nonzero element of E and r be an element of R such that xr = 0. Since xr = xf(r) and E is S P-flat, then, there exists $(x_i)_{i=1,..,n}$ a family of elements of E and $(s_i)_{i=1,..,n}$ a family of elements of (0: f(r)) such that $x = \sum_{i=1}^n x_i s_i$. On the other hand $s_i r = s_i f(r) = 0$ for every $i \in \{1,..,n\}$. Since S is R P-flat, then, there exists $(s_{ij})_{j=1,..,p}$ a family of elements of S and

 $(r_j)_{j=1,\dots,p}$ a family of elements of (0:r) such that $s_i = \sum_{j=1}^p s_{ij}r_j$ for every $i \in \{1,\dots,n\}$. Hence $x = \sum_{i=1}^n x_i s_i = \sum_{i=1}^n x_i \sum_{j=1}^p s_{ij}r_j = \sum_{j=1}^p \sum_{i=1}^n x_i s_{ij}r_j = \sum_{j=1}^p y_j r_j$, where $y_j = \sum_{i=1}^n x_i s_{ij} \in E$. Therefore, E is P-flat as R-module.

Proof of Proposition 2.18. 1) Using Proposition 2.7, we need to prove that $(R/I)_M$ is a *PFF*-ring whenever M is a maximal ideal of R/I. Let M be a maximal ideal of R, then there exist a maximal ideal m of R containing I, such that M = m/I. From the hypothesis conditions $I_m \in \{R_m, 0\}$ and by [16, Theorem 3.17], we obtain $(R/I)_M \cong R_m/I_m \cong R_m$ or 0. Hence $(R/I)_M$ is an *PFF*-ring. So, we have the desired result. 2) Assume that R is a *PFF*-ring and let J/I be a *P*-flat ideal of R/I.

2) Assume that *R* is a *PFF*-ring and let *J*/*I* be a *F*-hat ideal of *R*/*I*. Then, by Lemma 2.18 *J*/*I* is a *P*-flat *R*-module and by Lemma 2.19, *J* is a *P*-flat ideal of *R* (using the exact sequence: $0 \rightarrow I \rightarrow J \rightarrow J/I \rightarrow 0$, where *I* and *J*/*I* are *P*-flat *R*-modules). Since *R* is a *PFF*-ring, then *J* is flat. Hence, *J*/*I* is a flat ideal of *R* and so, *R*/*I* is a *PFF*-ring, as desired.

Example 2.21. Let R be a von Neumann regular ring and let I be an ideal of R. Then R/I is a PFF-ring.

Proof. Since R is a von Neumann regular ring, R_m should be a field for every maximal ideal of R. So, I is a pure ideal of R.

Our last result is to transfer the PFF property to finite direct products.

Proposition 2.22. Let $(R_i)_{i=1,..,n}$ be a family of rings. Then, $\prod_{i=1}^{n} R_i$ is a PFF-ring if and only if so is R_i for each i = 1,..,n.

Proof. The proof is done by induction on n and it suffices to check it for n = 2.

Assume that $R_1 \times R_2$ is a *FPP*-ring. Let I_1 be a *P*-flat ideal of R_1 . It is easy to check that $I_1 \times 0$ is a *P*-flat ideal of $R_1 \times R_2$, and hence flat. By [7, Lemma 3.7], I_1 is a flat ideal of R_1 and R_1 is a *PFF*-ring. Similarly, we prove that R_2 is a *PFF*-ring.

Conversely, assume that R_i is a PFF-ring for each i = 1, 2. Let I be a P-flat ideal of $R_1 \times R_2$. Then, $I = I_1 \times I_2$ where I_1 and I_2 are respectively ideals of R_1 and R_2 . We easily check that I_1 is a P-flat ideal of R_1 and I_2 is a P-flat ideal of R_2 . Hence, I_1 is a flat ideal of R_1

and I_2 is a flat ideal of R_2 and by [7, Lemma 3.7], we conclude that I is a flat ideal of $R_1 \times R_2$, as desired.

Acknowledgments

The authors would like to express their sincere thanks to the referee for his/her helpful suggestions and comments, which have greatly improved the paper.

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