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ON THE EXISTENCE OF FIXED POINTS FOR CONTRACTION MAPPINGS DEPENDING ON TWO FUNCTIONS

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ABSTRACT. In this paper we study the existence of fixed points for mappings defined on complete metric spaces, satisfying a general contractive inequality depending on two additional mappings.

Keywords: Metric space, fixed point, contractive mapping, sequentially convergent.

MSC(2010): Primary: 47H09; Secondary: 47H10.

1. Introduction

In 2009, Beiranvand, Moradi, Omid and Pazandeh [4] introduced new classes of contractive mappings: the so-called T -contraction and T -contractive mappings, which extend the Banach's contraction principle and the Edelstein's fixed point theorem [8]. Subsequently, the authors of this paper considered various extensions of classic contraction type of mappings, (more specifically: Kannan, Zamfirescu, weak-contractions and also the so-called $D(a, b)$ class) by considering their contractive inequalities depending on another mapping T . For these classes of contractions, conditions for the existence and uniqueness of fixed points, as well as for its asymptotic behavior are given [11–13]. Since then, extensions of other well known classes of contractive type of mappings have been given similarly, see e.g., [1–3, 5–7, 14–21]. We would like to

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point out that most of these extensions were considered on cone metric spaces, whose usefulness is a bit controversial (see, e.g., [9, 10]), however the results are valid even in the particular case of metric spaces.

The purpose of this paper is to analyse the existence and uniqueness of fixed points for a self-mapping S defined on a complete metric space satisfying a contraction inequality depending on two extra mappings. As a consequence of our study, we extend some fixed points theorems given in [4] as well as the classic Banach's Contraction Principle. Also, we are going to prove a localization fixed point result for mappings on this class. Examples showing the applicability of our results are provided.

2. Main results

In this section, first we introduce the notion of TR -contraction mapping, then we prove an existence and uniqueness fixed point theorem for mappings in this class.

Definition 2.1. Let (M, d) be a metric space and $T, R, S : M \rightarrow M$ three functions. A mapping S is said to be a TR -contraction if there is $a \in [0, 1)$ constant such that

$$d(TSx, RSy) \leq a \cdot d(Tx, Ry)$$

for all $x, y \in M$.

Example 2.2. Let \mathbb{R} with the usual metric $d(x, y) = |x - y|$. We are going to consider the functions $T, R, S : \mathbb{R} \rightarrow \mathbb{R}$ defined by $Tx = e^{-x}$, $Rx = 2e^{-x}$ and $Sx = x + 1$. Then:

- (1) Clearly S is not a contraction;
- (2) S is a TR -contraction. In fact,

$$\begin{aligned} d(TSx, RSy) &= |TSx - RSy| \\ &= \frac{1}{e} |e^{-x} - 2e^{-y}| \\ &= \frac{1}{e} d(Tx, Ry) \leq ad(Tx, Ry) \end{aligned}$$

where, $\frac{1}{e} \leq a < 1$.

Recall that a mapping T is said to be *sequentially convergent* if we have, for every sequence (y_n) , if $T(y_n)$ is convergent, then (y_n) also is convergent. The mapping T is said to be *subsequentially convergent* if

we have, for every sequence (y_n) , if $T(y_n)$ is convergent, then (y_n) has a convergent subsequence.

We would like to point out that in [4] was proved that if (M, d) is a sequentially compact (cone) metric space, then every function $T : M \rightarrow M$ is subsequentially convergent and every continuous (non-constant) function $T : M \rightarrow M$ is sequentially convergent.

Our first result is the following.

Theorem 2.1. *Let (M, d) be a complete metric space and let $T, R : M \rightarrow M$ be one to one and continuous functions. Let $S : M \rightarrow M$ be a TR -contraction continuous function. Then*

(i) *For every $x_0 \in M$,*

$$\lim_{n, m \rightarrow \infty} d(TS^n x_0, RS^m x_0) = 0;$$

(ii) *There exist $y_0, z_0 \in M$ such that*

$$(2.1) \quad \lim_{n \rightarrow \infty} TS^n x_0 = y_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} RS^n x_0 = z_0;$$

(iii) *If T (or R) is subsequentially convergent, then $(S^n x_0)$ has a convergent subsequence, and there exists a unique $v_0 \in M$ such that*

$$(2.2) \quad Sv_0 = v_0;$$

(iv) *If T (or R) is a sequentially convergent, then for each $x_0 \in M$ the iterate sequence $(S^n x_0)$ converges to v_0 .*

Proof. Let $x_0 \in M$, and (x_n) the Picard iteration associate to S given by $x_{n+1} = Sx_n = S^n x_0$, $n = 0, 1, \dots$. Notice that

$$d(Tx_n, Rx_{n+1}) = d(TS^{n-1}x_0, RS^n x_0) \leq ad(TS^{n-2}x_0, RS^{n-1}x_0)$$

hence, recursively we obtain

$$(2.3) \quad d(TS^{n-1}x_0, RS^n x_0) \leq a^{n-1}d(Tx_0, Rx_0).$$

which, taking limits, implies that

$$(2.4) \quad \lim_{n \rightarrow \infty} d(TS^{n-1}x_0, RS^n x_0) = 0.$$

Now, let $m, n \in \mathbb{N}$ with $m > n + 1$. Then

$$\begin{aligned} d(TS^n x_0, RS^m x_0) &\leq d(TS^n x_0, RS^{n+1} x_0) + d(RS^{n+1} x_0, TS^{n+2} x_0) \\ &\quad + \dots + \Delta \end{aligned}$$

where

$$\Delta := \begin{cases} d(TS^{m-1}x_0, RS^m x_0), & \text{if } m - n \text{ is odd} \\ d(TS^m x_0, RS^m x_0), & \text{otherwise.} \end{cases}$$

Since for any $k \in \mathbb{N}$ can be proved similar to (2.3) that

$$(2.5) \quad d(RS^k x_0, TS^{k+1} x_0) \rightarrow 0, \quad \text{as } k \rightarrow \infty$$

then, by (2.3) and (2.5), we have that:

$$\lim_{n,m \rightarrow \infty} d(TS^n x_0, RS^m x_0) = 0,$$

which proves (i). To prove (ii) notice that

$$d(TS^n x_0, TS^m x_0) \leq d(TS^n x_0, RS^{n+1} x_0) + d(RS^{n+1} x_0, TS^{n+2} x_0) + \dots + \nabla$$

here

$$\nabla := \begin{cases} d(RS^m x_0, TS^m x_0), & \text{if } m - n \text{ is odd} \\ d(RS^{m-1} x_0, TS^m x_0), & \text{otherwise.} \end{cases}$$

As it was proved above, from (2.3) and (2.5), taking limits in the above inequality, we conclude that

$$\lim_{n,m \rightarrow \infty} d(TS^n x_0, TS^m x_0) = 0$$

thus, from the fact that (M, d) is a complete metric space, the sequence $(TS^n x_0)$ converges. Similarly it can be proved that

$$\lim_{n,m \rightarrow \infty} d(RS^n x_0, RS^m x_0) = 0$$

i.e., the sequence $(RS^n x_0)$ is also convergent. Therefore the limit in (2.1) exists, so (ii) is proved. To prove (iii), we are going to consider that both T and R are subsequentially convergent. This assumption implies that $(S^n x_0)$ has a convergent subsequence. Hence, there exists $v_0 \in M$ and $(n_i)_{i=1}^\infty$ such that

$$(2.6) \quad \lim_{i \rightarrow \infty} S^{n_i} x_0 = v_0.$$

From the fact that T and R are two continuous functions, we have

$$\lim_{i \rightarrow \infty} TS^{n_i} x_0 = Tv_0 \quad \text{and} \quad \lim_{i \rightarrow \infty} RS^{n_i} x_0 = Rv_0$$

from equality (2.1) we conclude that

$$Tv_0 = y_0 \quad \text{and} \quad Rv_0 = z_0.$$

Since S is continuous, then from (2.6) we get that

$$\lim_{i \rightarrow \infty} S^{n_i+1} x_0 = Sv_0,$$

also that

$$\lim_{i \rightarrow \infty} TS^{n_i+1}x_0 = TSv_0 \quad \text{and} \quad \lim_{i \rightarrow \infty} RS^{n_i+1}x_0 = RSv_0.$$

Again by (2.1), the following equalities hold

$$\lim_{i \rightarrow \infty} TS^{n_i+1}x_0 = y_0 \quad \text{and} \quad \lim_{i \rightarrow \infty} RS^{n_i+1}x_0 = z_0,$$

hence

$$TSv_0 = y_0 = Tv_0 \quad \text{and} \quad RSv_0 = z_0 = Rv_0,$$

from the injectivity of T and R it follows that

$$Sv_0 = v_0.$$

Now, we are going to prove that the fixed point is unique. Let us suppose that another $u_0 \in M$ is such that $Su_0 = u_0$. Since S is a TR -contraction, then

$$(2.7) \quad d(TSv_0, RSu_0) \leq ad(Tv_0, Ru_0)$$

but, on the other hand, $d(TSv_0, RSu_0) = d(Tv_0, Ru_0)$. Therefore from (2.7) we have that $a \geq 1$ which is false. Thus the fixed point of S is unique. Finally, if T and R are sequentially convergent, $(S^n x_0)$ is convergent and replacing (n_i) by (n) in (2.6), the corresponding values of the limit is v_0 , which proves (iv). \square

From Theorem 2.1 we obtain the following:

Corollary 2.2 ([4], Theorem 2.6). *Let (M, d) be a complete metric space and $T : M \rightarrow M$ be a one to one, continuous and subsequentially convergent mapping. Then every T -contraction continuous function $S : M \rightarrow M$ has a unique fixed point. Moreover, if T is sequentially convergent, then for each $x_0 \in M$, the sequence $(S^n x_0)$ converges to the fixed point of S .*

Notice that if we take $Tx = Rx = x$ in Theorem 2.1, then we obtain the Banach's Contraction Principle.

The following result is a fixed point localization of Theorem 2.1.

Theorem 2.3. *Let (M, d) be a complete metric space and $T, R : M \rightarrow M$ be injective, continuous and subsequentially convergent mappings. For $c > 0$ and $x_0 \in M$, set*

$$B(Tx_0, c) = \{y \in M : d(Tx_0, y) \leq c\}.$$

Suppose $S : M \rightarrow M$ is a TR -contraction continuous mapping for all $x, y \in B(Tx_0, c)$ and $d(Tx_0, STx_0) \leq \varepsilon_1 < c$. Then S has a unique fixed point in $B(Tx_0, c)$.

Proof. In order to show that the hypotheses of Theorem 2.1 (iii) holds we need only to prove that $B(Tx_0, c)$ is complete. Also we have to show that $Sx \in B(Tx_0, c)$ for all $x \in B(Tx_0, c)$. First we are going to prove that $B(Tx_0, c)$ is complete: Suppose that $(y_n) \subset B(Tx_0, c)$ is a Cauchy sequence. By the completeness of M , there exist $y \in M$ such that $y_n \rightarrow y$, as $n \rightarrow \infty$.

Thus, we have

$$d(Tx_0, y) \leq d(y_n, Tx_0) + d(y_n, y) \leq c + d(y_n, y)$$

since $y_n \rightarrow y$, as $n \rightarrow \infty$, $d(y_n, y) \rightarrow 0$. Hence $d(Tx_0, y) \leq c$ and $y \in B(Tx_0, c)$. Therefore, $B(Tx_0, c)$ is complete.

On the other hand, for every $x \in B(Tx_0, c)$,

$$d(Tx_0, Sx) \leq d(Tx_0, STx_0) + d(STx_0, Sx),$$

since $d(Tx_0, STx_0) \leq \varepsilon_1$ and S is continuous, then we can conclude that

$$d(Tx_0, Sx) \leq \varepsilon_1 + \varepsilon_2 \leq c.$$

I.e., $Sx \in B(Tx_0, c)$. So the proof is complete. \square

Remark 1. Notice that the conclusion of the above theorem remains valid when we replace $B(Tx_0, c)$ by the set $B(Rx_0, c) := \{y \in M : d(Rx_0, y) \leq c\}$ and assuming in this case that $d(Rx_0, SRx_0) \leq \varepsilon_1 < c$.

Corollary 2.4. Let (M, d) be a complete metric space and $T, R : M \rightarrow M$ be one to one, continuous and subsequentially convergent mappings. Suppose that $S : M \rightarrow M$ is a mapping such that, S^n is a TR -contraction for some $n \in \mathbb{N}$ and furthermore a continuous function. Then S has a unique fixed point in M .

Proof. From Theorem 2.1, we have that S^n has a unique fixed point $z \in M$, that is, $S^n z = z$. But $S^n(Sz) = S(S^n z) = Sz$, so Sz is also a fixed point of S^n . Hence $Sz = z$, i.e., z is a fixed point of S . Since the fixed point of S is also fixed point of S^n , then the fixed point of S is unique. \square

The following examples show that we cannot omit the subsequentially convergence hypothesis of the function T (or R) in Theorem 2.1 (iii).

Example 2.3. Let us consider the Example 2.2. I.e., consider \mathbb{R} with the usual metric defined by $d(x, y) = |x - y|$. Let $T, R, S : \mathbb{R} \rightarrow \mathbb{R}$ be three functions defined by $Tx = e^{-x}$, $Rx = 2e^{-x}$ and $Sx = x + 1$. As we see, S is a TR -contraction but T is not subsequentially convergent, because $Tn \rightarrow 0$ as $n \rightarrow \infty$ but the sequence (n) has not any convergent subsequence and S has not a fixed point.

Example 2.4. Let $M := [1, 10^4]$ with the usual metric $d(x, y) = |x - y|$. Consider the mappings $T, R, S : M \rightarrow M$ defined by $Sx = e^{\frac{1}{10^4}(x-1)}$, $Tx = x^{1/2}$ and $R(x) = x^{1/10}$. One can show that

- (1) $d(Sx, Sy) \leq \frac{18}{10}d(x, y)$. I.e., S is not a Banach contraction on M .
- (2) $d(TSx, RSy) \leq \frac{8}{10}d(Tx, Ry)$. I.e., S is a TR -contraction for $a \in [0, 8/10]$.

Since the mapping T (and R) is non decreasing, continuous and injective, then it is sequentially convergent on M . Thus, from Theorem 2.1, we have that $x_0 = 1$ is the unique fixed point of S on M .

In what follows, by \mathcal{F} we mean the family of mappings whose members are either contractive, nonexpansive or α -contraction ($0 < \alpha < 1$) mappings.

Theorem 2.5. Let (M, d) be a complete metric space, let $T, R : M \rightarrow M$ be one to one and continuous mappings in \mathcal{F} and $S : M \rightarrow M$ a TR -contraction continuous function. Then:

- (i) For every $x_0 \in M$, the iterate sequence $(S^n x_0)$ converges;
- (ii) There exists a unique $v_0 \in M$ such that

$$Sv_0 = v_0;$$

- (iii) The iterate sequence $(S^n x_0)$ converges to the fixed point of S .

Proof. (i) Let $x_0 \in M$ and $(S^n x_0)$ the Picard iterate sequence

$$x_{n+1} = Sx_n = S^n x_0, \quad n = 0, 1, \dots$$

Since T and R are in \mathcal{F} , then

$$d(TS^n x_0, TS^m x_0) \leq ad(S^n x_0, S^m x_0)$$

and

$$d(RS^n x_0, RS^m x_0) \leq bd(S^n x_0, S^m x_0)$$

with $0 < a, b \leq 1$. On the other hand, notice that

$$d(TS^n x_0, RS^m x_0) \leq d(TS^n x_0, TS^m x_0) + d(TS^m x_0, RS^n x_0) \\ + d(RS^n x_0, RS^m x_0)$$

therefore, we get

$$d(TS^n x_0, RS^m x_0) \leq (a + b)d(S^n x_0, S^m x_0) + d(TS^m x_0, RS^n x_0).$$

Due to the fact that $d(TS^m x_0, RS^n x_0) \geq 0$, then we can choose a number $c > 0$ such that

$$0 \leq d(S^n x_0, S^m x_0) \leq \frac{c}{(a + b)} d(TS^n x_0, RS^m x_0).$$

Taking limit ($m, n \rightarrow \infty$) in the above inequality and by the proof of Theorem 2.1, we conclude that

$$\lim_{n, m \rightarrow \infty} d(S^n x_0, S^m x_0) = 0$$

or, equivalently, that $(S^n x_0)$ is convergent. The rest of the proof is similar to the proof of Theorem 2.1 with obvious changes. \square

Final remarks

In this paper we consider contraction mappings whose contractive inequality depends on two extra functions. To our knowledge, the consideration of other classic classes of contractive type of mappings such as Kannan, Chatterjea, Hardy-Rogers, Riech etc., for the generalization of their contractive inequality as it was presented here, has not yet been done.

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