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A FINITE DIFFERENCE TECHNIQUE FOR SOLVING VARIABLE-ORDER FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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(Communicated by Mohammad Asadzadeh)

Abstract. In this article, we use a finite difference technique to solve variable - order fractional integro - differential equations (VOFIDEs, for short). In these equations, the variable-order fractional integration(VOFI) and variable-order fractional derivative (VOFD) are described in the Riemann-Liouville's and Caputo's sense,respectively. Numerical experiments, consisting of two examples, are studied. The obtained numerical results reveal that the proposed finite difference technique is very effective and convenient for solving VOFIDEs.
Keywords: Variable-order fractional calculus, fractional integro-differential equation, finite difference method, numerical solution.
MSC(2010): Primary: 26A33; Secondary: 45G05, 45G10, 11Y50.

1. Introduction

Mathematical modeling of real-world problems generally arises in functional equations, for example, integral equations (IEs, for short), integrodifferential equations (IDEs, for short), ordinary and partial differential equations (ODEs and PDEs, for short), and the others. The study of classical IDEs has a long history. Both theoretical and numerical investigations of the subject have seen much development in recent decades [1, 2]. Most of the mathematical formulations of physical phenomena include IDEs, which arise in viscoelasticity [3], risk management models [4], biological [5], and cosmological physics [6]. Many significant

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works can be referred to [7–10] and references cited therein. In [7], an IDE arising as a limit of individual cell-based models is proposed. Under suitable assumptions, the existence, uniqueness, and non-negativity of the solutions and mass conservation are proved. In [8], the numerical solutions of a class of IDE with delay are studied, and the stability of the numerical scheme is discussed. There are several methods that can be used for solving IDEs. In [9], the high-order linear Volterra-Fredholm IDEs is solved by using Taylor polynomials. In [10], a combined form of the Laplace transform method and the Adomian decomposition method is developed for analytic treatment of the nonlinear Volterra IDEs. The combined method is capable of handling both equations of the first and second kind.

As an excellent modeling tool, fractional calculus, in allowing operations of differentiation and integration of arbitrary order(real or complex value, including fractional), has attracted much attention recently [11–19]. The fractional operators are due mainly to that contain the integer order integrals and derivatives as special cases. Moreover, they exhibit the memory property which does not exist in integer order integrals and derivatives. Using this memory property will generate more accurate models in simulating some physical processes. Fractional integro-differential equations have been the focus of many studies by virtue of their dense aspects in diverse fields such as physics [13], biology [20], engineering [11, 21], and other applications [22].

However, the investigation, in recent years, of fractional calculus suggests that many scientific and engineering models can be depicted more accurately via variable-order fractional operators, since the above objectives can exhibit memory property which changes with time and/or spatial locations. The variable-order fractional operator was first suggested by S. Samko and B. Ross [23]. In that study, the authors investigated the properties of variable-order differentiation and integration operators in the sense of Riemann-Liouville. From this point of view, the order of operator is enabled to alternate either as a function of the independent variable of differentiation or integration, or as a function of some other variables. The notion of a variable-order fractional operator is a result of recent advances of fractional calculus. But nevertheless, mainly because of lack of physical interpretation, the research of this new fractional calculus is still in the beginning stage and it is not much known. As in the modeling systems with improving dynamics, the order of the integral or derivative is permitted to alternate over the domain of concern.

This kind of systems contain the mechanics of a variable viscoelastic oscillators [24, 25] and deformation of viscoelastic materials [26]. In parallel to the case of matters in fractional calculus, many definitions of variable-order fractional integral or derivative (VOFI, VOFD) have been proposed [27–29]. In these works, the Caputo variable-order fractional derivative is used [30–34].

Nowadays, some motivated works have been done on the numerical solutions of variable-order fractional differential equations (VOFDEs) [35]. However, many other variable-order fractional differential/integro-differential equations are still not solved. In most cases, to obtain the closed form solution of an equation with variable-order fractional derivative is impossible. The contribution of this paper is two-fold: *First*, we propose a class of variable-order fractional integro-differential equations (VOFIDEs), which is more general than classical IDEs. *Second*, the finite difference scheme is applied to solve this particular VOFIDEs. Our work on numerical solutions of VOFIDEs will be of some importance since little work has been done for the application of variable-order fractional calculus to study the VOFIDEs.

The paper is organized as follows. In Section 2, we present some important preliminaries of variable-order integrals and derivatives. In Section 3, we develop a finite difference method to solve VOFIDEs. The stability of the numerical scheme is discussed. In Section 4, the proposed method is applied to two examples. Finally, a conclusion is given in Section 5.

2. **Variable-order fractional calculus**

In this section, we introduce the mathematical background of variable-order fractional Calculus (VOFC, for short). Formally, the VOFC is similar to the classical fractional calculus. For the comprehensive understanding of fractional calculus, we refer the readers to [12, 14, 15]. Replacing the fractional order with a bounded function, the fractional derivatives and integrals are generalized to the variable-order fractional integral and derivatives (VOFIs and VOFDs, for short) as follows:

Definition 2.1. *The left-sided Riemann-Liouville (R-L) VOFI of order* $q(t) > 0$ *of a function* $f(t)$ *is defined as [23]*

$$\left(I^{q(t)}_{[a,t,RL]}f\right)(t) = \frac{1}{\Gamma(q(t))}\int_a^t \frac{f(s)}{(t-s)^{1-q(s)}}ds, \tag{2.1}$$

provided the integral exists.

Definition 2.2. *The left-sided R-L VOFD of order $q(t) > 0$ of a function $f(t)$ is defined as [23]*

$$\left(D^{q(t)}_{[a,t,RL]}f\right)(t) = \frac{1}{\Gamma(m-q(t))}\left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-q(s)-1} f(s)ds, \tag{2.2}$$

provided the right-side of equation is finite, where $0 \le m-1 < q(t) < m$ and m is a positive integer.

Definition 2.3. The left-sided Caputo VOFD of order $q(t) > 0$ of a function $f(t)$ is defined as [23]

$$\left(D^{q(t)}_{[a,t,C]}f\right)(t) = \frac{1}{\Gamma(m-q(t))}\int_a^t (t-s)^{m-q(s)-1} f^{(m)}(s)ds, \tag{2.3}$$

provided the right hand side of equation is finite, where $0 \le m-1 < q(t) < m$ and m is a positive integer.

There is no need to list more VOFIs and VOFDs here, since in this paper only the R-L VOFI and Caputo VOFD will be employed for defining the VOFIDE. However undoubtedly, we note that many existing fractional integrals and derivatives can have their corresponding VOFIs and VOFDs, by replacing the constant orders with variable orders. Now we would like to make the following remarks:

- Eq. (2.1) is well-defined, since for any $t_j > a$, Eq. (2.1) reduces to the classical Riemann-Liouville integral at $t = t_j$. Moreover, if $t = a$, $\left(I^{q(a)}_{[a,a,RL]}f\right)(a) = 0$.
- The VOFIs and VOFDs is more general than integer and fractional order integrals and derivatives. Taking specific order functions, VOFIs and VOFDs will reduce to the corresponding integer and fractional operators, respectively. For example, let $q(t) = 1$, Eq. (2.1) becomes the integral of $f(s)$. Let $q(t) = 1/2$, Eq. (2.1) reduces to the half-order Riemann-Liouville integral of $f(t)$.
- Using particular discontinuous order functions, the classical ordinary and partial differential equations (ODEs and PDEs) will be unified. For instance, consider differential equation $D^{q(t)}y(t) = f(t,y)$, when $q(t) = 1$, $t \in [0,1)$ and $q(t) = 2$, $t \in [1,2]$, and we specify some conditions as $y(0) = y_0$, $y(1) = y_1$ and $y'(1) = \bar{y}_1$, we obtained a first-order ODE defined on $[0,1)$ and a second-order ODE defined on $[1,2]$, and they are both unified by the same VOFDE.

- This generalized calculus, VOFC, is significantly important to study physical problems with varying memory property, which is changing with time and/or spatial location [29,31]. The variable-order fractional operators can depict those properties clearly.

To simplify the derivations, we drop symbol "C" in the subscript of $D^{q(t)}_{[a,t,C]}$ in equation (2.3) in what follows. In next section, we discuss the model of VOFIDEs and its numerical scheme.

3. Model description and Numerical scheme

In this section, we first define a class of VOFIDEs, and then deduce the numerical scheme of VOFIDEs. Finally, we discuss the stability of the numerical scheme.

3.1. **Model description.** The integro-differential equations (IDEs) play an important role in modeling some physical processes. The integer-order IDE is defined as

$$\left(\frac{d^m}{dt^m}f\right)(t) + \left({}_0I_t^n f\right)(t) = h(t), \tag{3.1}$$

with initial conditions $f^{(k)}(0) = f_0^k$, $k = 0, 1, \cdots, \max\{m, n\} - 1$, where m, n are positive integers. $\frac{d^m}{dt^m}f$ is the m-th order derivative of f with respect to t, and ${}_0I_t^n f$ is the n-th fold integral of f from 0 to t. Generally, f is assumed to be continuously differentiable m times.

Using the R-L VOFI and Caputo VOFD given in Eqs. (2.1) and (2.3), we can generalize a class of VOFIDEs as

$$\left(D^{q_1(t)}_{[0,t]}f\right)(t) + \left(I^{q_2(t)}_{[0,t]}f\right)(t) = h(t), \tag{3.2}$$

where q_1, q_2, f and h are sufficiently good such that Eq. (3.2) is well-defined. q_1 and q_2 can be different. For simplicity, we specify that $0 < q_1(t), q_2(t) < 1$. In this case, we only need one initial condition to specify the solution of Eq. (3.2). However, our discussion can be generalized to other cases of q_1 and q_2.

3.2. **Numerical scheme.** In this part, we develop a finite difference scheme to solve Eq. (3.2). We consider Eq. (3.2) on a domain with a uniform mesh as $0 = t_0 < t_1 < \cdots < t_N = 1$, and $h = t_{j+1} - t_j$ is the step size. For simplicity, $f(t_j)$, $h(t_j)$ and $q(t_j)$ are denoted as f_j, h_j and q_j, respectively.

When $0 < q_1(t) < 1$, the Caputo VOFD is defined as

$$\left(D^{q_1(t)}_{[0,t]} f\right)(t) = \frac{1}{\Gamma(1-q_1(t))} \int_0^t (t-s)^{-q_1(s)} f'(s)ds, \tag{3.3}$$

and can be approximated on the uniform mesh as

$$\begin{aligned}
&\left(D^{q_1(t)}_{[0,t]} f\right)(t_{j+1}) \qquad (3.4)\\
=\ & \frac{1}{\Gamma(1-q_1(t_{j+1}))} \int_0^{t_{j+1}} (t_{j+1}-s)^{-q_1(s)} f'(s)ds\\
\approx\ & \frac{1}{\Gamma(1-q_{1,j+1})} \sum_{k=0}^{j} \frac{f(t_{k+1})-f(t_k)}{h} \int_{t_k}^{t_{k+1}} (t_{j+1}-s)^{-q_1\left(\frac{t_k+t_{k+1}}{2}\right)} ds\\
=\ & \frac{1}{\Gamma(1-q_{1,j+1})} \sum_{k=0}^{j} \frac{f_{k+1}-f_k}{h \times \left[1-q_1\left(\frac{t_k+t_{k+1}}{2}\right)\right]}\\
& \times \left[(t_{j+1}-t_k)^{1-q_1\left(\frac{t_k+t_{k+1}}{2}\right)} - (t_{j+1}-t_{k+1})^{1-q_1\left(\frac{t_k+t_{k+1}}{2}\right)}\right].
\end{aligned}$$

Similarly, the R-L VOFI in Eq. (3.2) can be approximated as

$$\begin{aligned}
&\left(I^{q_2(t)}_{[0,t]} f\right)(t_{j+1}) \qquad (3.5)\\
=\ & \frac{1}{\Gamma(q_2(t_{j+1}))} \int_0^{t_{j+1}} (t_{j+1}-s)^{q_2(s)-1} f(s)ds\\
=\ & \frac{1}{\Gamma(q_{2,j+1})} \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} (t_{j+1}-s)^{q_2(s)-1} f(s)ds\\
\approx\ & \frac{1}{\Gamma(q_{2,j+1})} \sum_{k=0}^{j} \frac{f(t_k)+f(t_{k+1})}{2} \int_{t_k}^{t_{k+1}} (t_{j+1}-s)^{q_2\left(\frac{t_k+t_{k+1}}{2}\right)-1} ds\\
=\ & \frac{1}{2\Gamma(q_{2,j+1})} \sum_{k=0}^{j} \frac{f_k+f_{k+1}}{q_2\left(\frac{t_k+t_{k+1}}{2}\right)}\\
& \times \left[(t_{j+1}-t_k)^{q_2\left(\frac{t_k+t_{k+1}}{2}\right)} - (t_{j+1}-t_{k+1})^{q_2\left(\frac{t_k+t_{k+1}}{2}\right)}\right].
\end{aligned}$$

Hence, substituting Eqs. (3.4) and (3.5) into Eq. (3.2) yields the discretization equation of Eq. (3.2):

$$\sum_{k=0}^{j} B_k^j (f_{k+1} - f_k) + \sum_{k=0}^{j} A_k^j (f_k + f_{k+1}) = h_{j+1}, \tag{3.6}$$

for $j = 0, 1, \cdots, N-1$, where

$$\begin{aligned} B_k^j &= \frac{h^{-1}}{\Gamma(1-q_{1,j+1})\left[1 - q_1\left(\frac{t_k+t_{k+1}}{2}\right)\right]} \\ &\quad \times \left[(t_{j+1}-t_k)^{1-q_1\left(\frac{t_k+t_{k+1}}{2}\right)} - (t_{j+1}-t_{k+1})^{1-q_1\left(\frac{t_k+t_{k+1}}{2}\right)}\right], \\ A_k^j &= \frac{1}{2q_2\left(\frac{t_k+t_{k+1}}{2}\right)\Gamma(q_{2,j+1})} \\ &\quad \times \left[(t_{j+1}-t_k)^{q_2\left(\frac{t_k+t_{k+1}}{2}\right)} - (t_{j+1}-t_{k+1})^{q_2\left(\frac{t_k+t_{k+1}}{2}\right)}\right], \end{aligned}$$

for $k = 0, 1, \cdots, j$.

Remark 3.1. *In Eqs. (3.4) and (3.5), we approximate functions $f'(s)$ linearly. The values of $f(s)$, $q_1(s)$ and $q_2(s)$ in the middle point of subinterval are assumed to be the corresponding values in the whole subinterval. However, the approximate method is not unique. The approximation presented above will lead us to obtain satisfying numerical solutions of VOFIDEs.*

Then we have the linear equation

$$K \cdot F = H, \tag{3.7}$$

where

$$\begin{aligned} F &= [f_0, f_1, \cdots, f_N]^T, \\ H &= [h_0, h_1, \cdots, h_N]^T, \\ K &= \left[\begin{array}{llllll} 1 & 0 & 0 & 0 & 0 & \cdots \\ A_0^0 - B_0^0 & A_0^0 + B_0^0 & 0 & 0 & 0 & \cdots \\ A_0^1 - B_0^1 & A_0^1 + B_0^1 + A_1^1 - B_1^1 & A_1^1 + B_1^1 & 0 & 0 & \cdots \\ A_0^2 - B_0^2 & A_0^2 + B_0^2 + A_1^2 - B_1^2 & A_1^2 + B_1^2 + A_2^2 - B_2^2 & A_2^2 + B_2^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}\right]. \end{aligned}$$

Now, we successfully transform Eq. (3.2) into a linear algebraic equation. Since the coefficient matrix is lower triangular and all the diagonal elements are nonzero, the coefficient matrix K is invertible. Hence, Eq. (3.7) is solvable, which implies that the numerical solutions of Eq. (3.2) are obtained.

Theorem 3.2. *Suppose that the coefficients A_k^j and B_k^j in Eq. (3.6) satisfy*

$$B_{j-1}^j + A_{j-1}^j \leq 2B_j^j, \tag{3.8}$$

for $j = 1, 2, \cdots, N-1$ and $k = 0, 1, 2, \cdots, j$, then finite difference scheme (3.6) is stable and convergent.

Proof. It is obvious that A_k^j and B_k^j are both positive. Therefore we have

$$2A_j^j + B_{j-1}^j + A_{j-1}^j \geq 0. \tag{3.9}$$

Now, we rewrite Eq. (3.6) in an iterative form as

$$\begin{aligned} f_{j+1} = \frac{1}{A_j^j + B_j^j} \Bigg[& h_{j+1} + B_j^j f_j - A_j^j f_j - \sum_{k=0}^{j-1} B_k^j \left(f_{k+1} - f_k \right) \\ & - \sum_{k=0}^{j-1} B_k^j \left(f_{k+1} - f_k \right) \Bigg]. \end{aligned} \tag{3.10}$$

Assume error $\epsilon_{j+1} = f_{j+1} - \bar{f}_{j+1}$, where $\bar{f}_{j+1}$ denotes the exact value of f at $t = t_{j+1}$. According to Eq. (3.10), the error satisfies

$$|\epsilon_{j+1}| \leq \frac{B_j^j - A_j^j - B_{j-1}^j - A_{j-1}^j}{A_j^j + B_j^j} \left| \epsilon_j \right|. \tag{3.11}$$

Using Eq. (3.9), we have

$$\frac{B_j^j - A_j^j - B_{j-1}^j - A_{j-1}^j}{A_j^j + B_j^j} \leq 1. \tag{3.12}$$

According to Eq. (3.8), we have

$$\frac{B_j^j - A_j^j - B_{j-1}^j - A_{j-1}^j}{A_j^j + B_j^j} \geq -1. \tag{3.13}$$

Hence,

$$\left| \frac{B_j^j - A_j^j - B_{j-1}^j - A_{j-1}^j}{A_j^j + B_j^j} \right| \leq 1,$$

which implies that the error in discrete equation (3.6) is bounded and $\epsilon_{j+1} \to 0$ as $j \to +\infty$. Since Eq. (3.6) is linear, by Lax-Richtmyer theorem (see [36]), the numerical scheme is convergent. $\square$

Remark 3.3. *In this Section, we only consider the VOFIDEs with left-sided fractional integral and derivative. However, the numerical scheme discussed above can be generalized to VOFIDEs with right-sided fractional integral and derivative, or both of them. When the VOFIDE is defined with right-sided fractional operators, the initial conditions should specified on the right side boundary point of domain.*

4. **Numerical examples**

To demonstrate the effectiveness of the above numerical scheme, we discuss two numerical examples as below. In these numerical experiments, VOFIDEs are solved with different step sizes, the numerical solutions are graphed and the approximation order of numerical scheme is estimated and tabled.

Example 4.1. As the first example, we consider the following VOFIDE:

$$\left(D^{q_1(t)}_{[0,t]} f\right)(t) + \left(I^{q_2(t)}_{[0,t]} f\right)(t) = h(t),\ f(0) = 0, \tag{4.1}$$

where $q_1(t) = \frac{1}{6}\sin(10\pi t) + \frac{2}{3}$ and $q_2(t) = \exp(t) - t^2$ are the order functions of derivative and integral, respectively. The source term $h(t) = \sin(4t) + \exp(t^2) - 1$.

We solve VOFIDE (4.1) on $[0, 1]$ with step size $\Delta t = 1/8$, $1/16$, $1/32$, $1/64$, $1/128$, and $1/256$. The numerical solutions are displayed in Figure 1. We observe that when the step size reduces, the solution curve becomes stable, which shows that the numerical scheme is stable.

Now, we evaluate the approximation order of numerical scheme in solving Example 4.1. We reduce the step size Δt in halving, then take numerical solution with $\Delta t = 1/512$ as the best approximation since the closed form solution of VOFIDE (4.1) does not exist. The study results are shown in Table 1. We observe that the numerical method has accuracy higher than first order.

In Table 1, it is necessary to point out that the position denoted by star "*", where the approximation order evaluated is negative. This is mainly because of the order functions are not monotonic. Therefore, the computation in Example 4.1 shows that the approximation order of numerical scheme in solving VOFIDEs depends on the order functions.

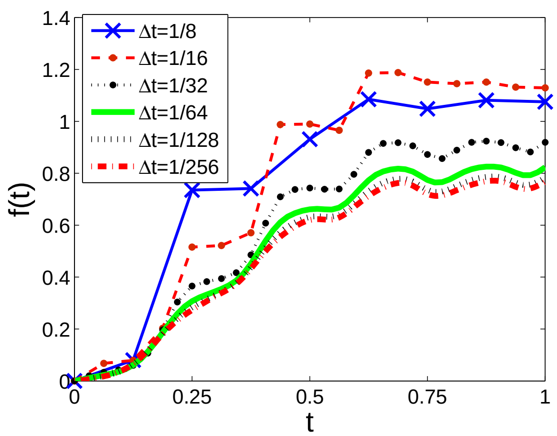

FIGURE 1. Numerical solution of Example 4.1.

TABLE 1. Error of Example 4.1 and approximate order of numerical scheme

step size Δt	absolute error	approximation order
$h = 1/8$	0.46644159	—
$h = 1/16$	0.48198038	"*"
$h = 1/32$	0.17608094	1.4527
$h = 1/64$	0.06877250	1.3563
$h = 1/128$	0.02621251	1.3916
$h = 1/256$	0.00809785	1.6946

Example 4.2. As the second example, we consider the following VOFIDE:

$$\left(D_{[0,t]}^{q_1(t)} f\right)(t) + \left(I_{[0,t]}^{q_2(t)} f\right)(t) = h(t), \ \ f(0) = 1, \tag{4.2}$$

where $q_1(t) = t^2 - t + 0.8$ and $q_2(t) = \exp(\sin(5\pi t))$ are the order functions of derivative and integral, respectively. The source term $h(t) = \cos(t^2)$.

We solve VOFIDE (4.2) on $[0, 1]$ with step size $\Delta t = 1/8$, $1/16$, $1/32$, $1/64$, $1/128$, and $1/256$. The numerical solutions are displayed in Figure 2. We observe that when the step size reduces, the solution curve becomes stable, which shows that the numerical scheme is stable.

Similarly, we evaluate the approximation order of numerical scheme in solving Example 4.2. Again, we reduce the step size Δt in halving,

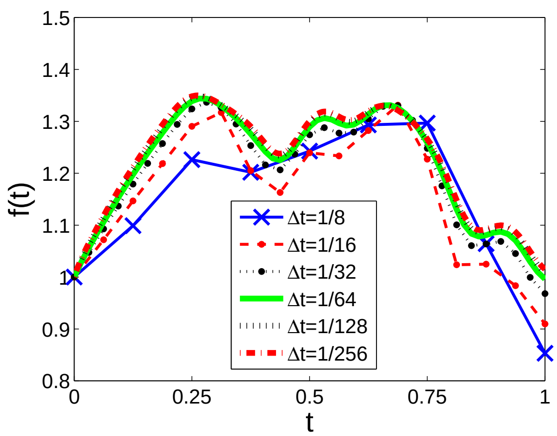

FIGURE 2. Numerical solution of Example 4.2.

then take numerical solution with $\Delta t = 1/512$ as the best approximation since the closed form solution of VOFIDE (4.2) does not exist. The study results are shown in Table 2. We observe that the numerical method has accuracy higher than first order.

TABLE 2. Error of Example 4.2 and approximate order of numerical scheme

step size Δt	absolute error	approximation order
$h = 1/8$	0.16495450	—
$h = 1/16$	0.10839211	0.6058
$h = 1/32$	0.04989323	1.1193
$h = 1/64$	0.02202167	1.1799
$h = 1/128$	0.00905359	1.2824
$h = 1/256$	0.00292889	1.6281

From Tables 1 and 2, we observe that when the step size reduces, the error reduces as well, which demonstrates the effectiveness of our numerical scheme. For simplicity, we just show part of the numerical experiments in those tables. We find that the evaluated approximation order seems always to vary and does not converges to some constant. This is mainly because of the the order functions in VOFIDEs are not constant, and both the numerical experiments show that the approximation order depends on the order functions. Generally, based on the numerical evaluation, our numerical scheme is higher than first order to solve those VOFIDEs.

5. Conclusions

In this paper, we introduced a finite difference technique for solving the variable-order fractional integro-differential equations (VOFIDEs). This work illustrates the validity and potential of proposed method for VOFIDEs. In numerical experiments, we observe that the numerical scheme is higher than first order, and approximation order evaluated is always varying around some constant. This is mainly caused by the order functions employed in VOFIDEs are not monotonic. Generally, the better numerical solutions can be obtained quickly by reducing the step size. The main ideas of this technique can be further used to solve other problems in fractional calculus, as well as other VOFIEs and VOFDEs.

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