Title:
Cat\(^1\)-polygroups and pullback cat\(^1\)-polygroups

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CAT$^1$-POLYGROUPS AND PULLBACK CAT$^1$-POLYGROUPS

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Abstract. In this paper, we give the notions of crossed polymodule and cat$^1$-polygroup as a generalization of Loday's definition. Then, we define the pullback cat$^1$-polygroup and we obtain some results in this respect. Specially, we prove that by a pullback cat$^1$-polygroup we can obtain a cat$^1$-group.

Keywords: polygroup, crossed polymodule, cat$^1$-group, cat$^1$-polygroup, pullback cat$^1$-polygroup.


1. Introduction

Crossed module was presented by Whitehead in [24]. So many applications of crossed module have been made by mathematicians. A very important application of crossed module is cat$^1$-group structure. Loday showed that the category of crossed module is equivalent to the category of cat$^1$-group in [21]. This application gave the new direction to crossed module. So many applications of cat$^1$-groups have been found by several mathematicians. After defining cat$^1$-group structure mathematicians have tried to study these categories. Important calculation examples of these categories were given by Brown and Wensley in [6] and [7]. The other important application of crossed module is defining pullback crossed module. Pullback crossed module was defined by Brown and Wensley in [6] and [7]. They gave many examples and applications of pullback crossed module in their work. Other cat$^1$-groups application is
Pullback cat\(^1\)-group which was defined by Alp using the equivalence between the category of crossed module and the category of cat\(^1\)-groups in [2]. GAP [17] program calculations of these categories were presented by Alp and Wensley in [3]. Crossed polymodule and its application derivation and actor crossed module were presented by Alp and Davvaz. In this paper, we use the same idea to define cat\(^1\)-polygroups and pullback cat\(^1\)-polygroups in Loday and Alp’s way. We study the connections between crossed polymodules and cat\(^1\)-polygroups. We present some basic definitions and results of polygroups and crossed polymodules in Section 2. In Section 3, we give the definition of cat\(^1\)-polygroup and some properties of cat\(^1\)-polygroups. In the last section, we define the concept of pullback cat\(^1\)-polygroup and we obtain some results in this respect. Specially, we prove that by a pullback cat\(^1\)-polygroup we can obtain a cat\(^1\)-group.

2. Polygroups and crossed polymodules

The polygroup theory is a natural generalization of the group theory. In a group the composition of two elements is an element, while in a polygroup the composition of two elements is a set. Polygroups have been applied in many areas, such as geometry, lattice theory, combinatorics and color schemes. There exists a rich bibliography: publications appeared within 2012 can be found in “Polygroup Theory and Related Systems” by Davvaz [12]. This book contains the principal definitions endowed with examples and the basic results of the theory. Applications of hypergroups appear in special subclasses like polygroups that were studied by Comer [8], also see [12, 13, 14]. Specially, Comer and Davvaz developed the algebraic theory for polygroups. A polygroup is a completely regular, reversible in itself multigroup. We recall the following definition from [8]. A polygroup is a multi-valued system \(\mathcal{M} = \langle P, \circ, e, (\ )^{-1} \rangle\), with \(e \in P\), \((\ )^{-1} : P \rightarrow P\), \(\circ : P \times P \rightarrow \mathcal{P}(P)\), where the following axioms hold for all \(x, y, z \in P\): (1) \((x \circ y) \circ z = x \circ (y \circ z)\), (2) \(e \circ x = x \circ e = x\), (3) \(x \in y \circ z\) implies \(y \in x \circ z^{-1}\) and \(z \in y^{-1} \circ x\). In this definition, \(\mathcal{P}(P)\) is the set of all non-empty subsets of \(P\), and if \(x \in P\) and \(A, B\) are non-empty subsets of \(P\), then \(A \circ B = \bigcup_{a \in A, b \in B} a \circ b\), \(x \circ B = \{x\} \circ B\) and \(A \circ x = A \circ \{x\}\). The following elementary facts about polygroups follow easily from the axioms: \(e \in x \circ x^{-1} \cap x^{-1} \circ x\), \(e^{-1} = e\) and \((x^{-1})^{-1} = x\). In the rest of this section we present the facts about
polygroups that underlie the subsequent material. For further discussion of polygroups, we refer the readers to Davvaz’s book [12]. Many important examples of polygroups are collected in [12] such as Double coset algebra, Prenowitz algebras, Conjugacy class polygroups, Character polygroups, Extension of polygroups, and Chromatic polygroups. Clearly, every group is a polygroup. If \( K \) is a non-empty subset of \( P \), then \( K \) is called a subpolygroup of \( P \) if \( e \in K \) and \( < K, \circ, e, (\ )^{-1} > \) is a polygroup. The subpolygroup \( N \) of \( P \) is said to be normal in \( P \) if \( a^{-1} \circ N \circ a \subseteq N \), for every \( a \in P \). There are several kinds of homomorphisms between polygroups [12]. In this paper, we apply only the notion of strong homomorphism. Let \( < P, \circ, e, (\ )^{-1} > \) and \( < P', \ast, e, (\ )^{-1} > \) be two polygroups. A mapping \( \phi \) from \( P \) into \( P' \) is said to be a strong homomorphism if \( \phi(e) = e \) and for all \( a, b \in P \), \( \phi(a \circ b) = \phi(a) \ast \phi(b) \), for all \( a, b \in P \). A strong homomorphism \( \phi \) is said to be an isomorphism if there is an isomorphism from \( P \) to \( P' \). The defining condition for a strong homomorphism is also valid for sets, i.e., if \( A, B \) are non-empty subsets of \( P \), then it follows that \( f(A \circ B) = f(A) \ast f(B) \). By using the concept of generalized permutation, in [10], Davvaz defined permutation polygroups and action of a polygroup on a set. For the definition of crossed polymodule, we need the notion of polygroup action.

**Definition 2.1.** [10] Let \( P = < P, \circ, e, (\ )^{-1} > \) be a polygroup and \( \Omega \) be a non-empty set. A map \( \alpha : P \times \Omega \to P^*(\Omega) \), where \( \alpha(g, \omega) := g^\omega \) is called a (left) polygroup action on \( \Omega \) if the following axioms hold:

1. \( e^\omega = \omega \),
2. \( h^\omega \circ (g \circ \omega) = h \circ g \circ \omega \), where \( gA = \bigcup_{a \in A} ga \) and \( B \omega = \bigcup_{b \in B} b \omega \), for all \( A \subseteq \Omega \) and \( B \subseteq P \),
3. \( \bigcup_{\omega \in \Omega} g^\omega = \Omega \),
4. for all \( g \in P \), \( a \in g^b \equiv b \in g^{-1}a \).

**Example 2.2.** Suppose that \( < P, \circ, e, (\ )^{-1} > \) is a polygroup. Then, \( P \) acts on itself by conjugation. Indeed, if we consider the map \( \alpha : P \times P \to P^*(P) \) by \( \alpha(g, x) = g^x := g \circ x \circ g^{-1} \), then

1. \( e^x = x \),
2. \( h^x = h(g \circ x \circ g^{-1}) = h \circ g \circ x \circ (h \circ g)^{-1} = (h \circ g) \circ x \circ (h \circ g)^{-1} = \bigcup_{b \in h \circ g} (b \circ x \circ b^{-1}) = \bigcup_{b \in h \circ g} b^x = h \circ g \circ x \).
(3) $\bigcup_{x \in P} g_x = \bigcup_{x \in P} g \circ x \circ g^{-1} = P$,

(4) if $a \in g b = g \circ b \circ g^{-1}$, then $g \in a \circ g \circ b^{-1}$ and hence $b^{-1} \in g^{-1} \circ a^{-1} \circ g$. This implies that $b \in g^{-1} \circ a \circ g$.

Note that the above definition is a generalization of the group action. Let $G$ be a group and $\Omega$ be a non-empty set. A (left) group action is a binary operator from $G \times \Omega$ to $\Omega$ that satisfies the following two axioms: $gh\omega = g(h\omega)$ and $e\omega = \omega$, for all $g, h \in G$ and $\omega \in \Omega$. Now, we present the notion of crossed polymodule and main results about fundamental relation on polygroups and fundamental crossed polymodule.

Definition 2.3. A crossed polymodule $\mathcal{X} = (C, P, \partial, \alpha)$ consists of polygroups $\langle C, \ast, e, (\ )^{-1} \rangle$ and $\langle P, \circ, e, (\ )^{-1} \rangle$ together with a strong homomorphism $\partial : C \rightarrow P$ and a (left) action $\alpha : P \times C \rightarrow P^\ast(C)$ on $C$, satisfying the conditions:

(1) $\partial(p c) = p \circ \partial(c) \circ p^{-1}$, for all $c \in C$ and $p \in P$,

(2) $\partial(c)c' = c \ast c' \ast c^{-1}$, for all $c, c' \in C$.

When we wish to emphasize the codomain $P$, we call $\mathcal{X}$ a crossed $P$-polymodule. The strong homomorphism $\partial : C \rightarrow P$ is called the boundary homomorphism.

Example 2.4. A conjugation crossed polymodule is an inclusion of a normal subpolygroup $N$ of $P$, with action given by conjugation. In particular, for any polygroup $P$ the identity map $Id_P : P \rightarrow P$ is a crossed polymodule with the action of $P$ on itself by conjugation. Indeed, there are two canonical ways in which a polygroup $P$ may be regarded as a crossed polymodule: via the identity map or via the inclusion of the trivial subpolygroup.

Example 2.5. If $C$ is a $P$-polymodule, then there is a well defined action $\alpha$ of $P$ on $C$. This together with the zero homomorphism yields a crossed polymodule $(C, P, 0, \alpha)$.

Example 2.6. The direct product of $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed polymodules has source $C_1 \times C_2$, range $P_1 \times P_2$ and boundary homomorphism $\partial_1 \times \partial_2$ with $P_1 \times P_2$ acting obviously on $C_1 \times C_2$.

Note that the above definition is a generalization of the notion of crossed module. We recall that a crossed module $X = (M, G, \partial, \tau)$ consists of groups $M$ and $G$ together with a homomorphism $\partial : M \rightarrow G$ and a (left) action $\tau : G \times M \rightarrow M$ on $M$, satisfying the conditions:
\[ \partial(gm) = g\partial(m)g^{-1}, \text{ for all } m \in M, \ g \in G, \ \text{and} \ \partial(m)m' = mm'm^{-1}, \text{ for all } m, m' \in M. \]

**Theorem 2.7.** Every crossed module is a crossed polymodule.

**Proof.** Since every group is a polygroup, the proof is straightforward. \qed

**Definition 2.8.** Let \( \mathcal{X} = (C, \partial, \alpha) \) be a crossed polymodule and \( \iota : Q \to P \) be a morphism of polygroups. Then \( \iota^*\mathcal{X} = (\iota^*C, Q, \partial^*, \alpha^*) \) is the pullback of \( \mathcal{X} \) by \( \iota \), where \( \iota^*C = \{(q, c) \in Q \times C \mid \iota(q) = \partial(c) \} \) and \( \partial^*(q, c) = q \). The polygroup action of \( Q \) on \( \iota^*C \) is given by

\[ q(q_1, c) = \{(x, y) \mid (x, y) \in (q \circ q_1 \circ q^{-1}, c) \}. \]

**Theorem 2.9.** \( \iota^*\mathcal{X} = (\iota^*C, Q, \partial^*, \alpha^*) \) is a crossed polymodule.

**Proof.** The verification of crossed polymodule axioms is similar to the crossed module axioms in [6]. \qed

Let \( < P, \circ, e, (\cdot)^{-1} > \) be a polygroup. We define the relation \( \beta^*_P \) as the smallest equivalence relation on \( P \) such that the quotient \( P/\beta^*_P \), the set of all equivalence classes, is a group. In this case \( \beta^*_P \) is called the fundamental equivalence relation on \( P \) and \( P/\beta^*_P \) is called the fundamental group. The product \( \odot \) in \( P/\beta^*_P \) is defined as follows: \( \beta^*_P(x) \odot \beta^*_P(y) = \beta^*_P(z) \), for all \( z \in \beta^*_P(x) \odot \beta^*_P(y) \). This relation is introduced by Koskas [18] and studied mainly by Corsini [9], Leoreanu-Fotea et al. [19, 20] and Freni [15, 16] concerning hypergroups, Vougiouklis [23] for \( H_v \)-groups, Davvaz for polygroups [11, 22], and many others. We consider the relation \( \beta_P \) as follows:

\[ x \beta_P y \iff \text{there exist } z_1, \ldots, z_n \text{ such that } \{x, y\} \subseteq \prod_{i=1}^n z_i. \]

Freni in [15] proved that for hypergroups \( \beta = \beta^* \). Since polygroups are certain subclass of hypergroups, we have \( \beta^*_P = \beta_P \). The kernel of the canonical map \( \varphi_P : P \to P/\beta^*_P \) is called the core of \( P \) and is denoted by \( \omega_P \). Here we denote by \( \omega_P \) the unit of \( P/\beta^*_P \). It is easy to prove
the following statements: \( \omega_P = \beta_P^e(x) \) and \( \beta_P^e(x)^{-1} = \beta_P^e(x^{-1}) \), for all \( x \in P \).

**Lemma 2.10.** [9] \( \omega_P \) is a subpolygroup of \( P \).

**Lemma 2.11.** [1] Let \( \omega_P, \omega_Q \) and \( \omega_{P \times Q} \) be the cores of \( P, Q \) and \( P \times Q \), respectively. Then, \( \omega_{P \times Q} = \omega_P \times \omega_Q \).

Throughout the paper, for the polygroupos \( < P, \circ, e, (\cdot)^{-1} >, < C, \ast, e, (\cdot)^{-1} > \) and \( < Q, \cdot, e, (\cdot)^{-1} > \), we denote the binary operations of the fundamental groups \( P/\beta_P^e, C/\beta_C^e \) and \( Q/\beta_Q^e \) by \( \circ, \ast \) and \( \cdot \), respectively.

**Proposition 2.12.** [4] Let \( < C, \ast, e, (\cdot)^{-1} > \) and \( < P, \circ, e, (\cdot)^{-1} > \) be two polygroups and let \( \partial : C \rightarrow P \) be a strong homomorphism. Then, \( \partial \) induces a group homomorphism \( \mathcal{D} : C/\beta_C^e \rightarrow P/\beta_P^e \) by setting

\[
\mathcal{D}(\beta_C^e(c)) = \beta_P^e(\partial(c)), \text{ for all } c \in C.
\]

We say the action of \( P \) on \( C \) is productive, if for all \( c \in C \) and \( p \in P \) there exist \( c_1, \ldots, c_n \) in \( C \) such that \( p^c = c_1 \ast \ldots \ast c_n \).

**Example 2.13.** The action defined in Example 2.2 is productive.

Let \( < C, \ast, e, (\cdot)^{-1} > \) and \( < P, \circ, e, (\cdot)^{-1} > \) be two polygroups and let \( \alpha : P \times C \rightarrow \mathcal{P}^e(C) \) be a productive action on \( C \). We define the map \( \psi : P/\beta_P^e \times P/\beta_C^e \rightarrow \mathcal{P}^e(P/\beta_C^e) \) as follows:

\[
\psi(\beta_P^e(p), \beta_C^e(c)) = \{ \beta_C^e(x) \mid x \in \bigcup_{y \in \beta_C^e(c)} z y \}.
\]

By definition of \( \beta_C^e \), since the action of \( P \) on \( C \) is productive, we conclude that \( \psi(\beta_P^e(p), \beta_C^e(c)) \) is singleton, i.e., we have

\[
\psi : P/\beta_P^e \times P/\beta_C^e \rightarrow P/\beta_C^e,
\]

\[
\psi(\beta_P^e(p), \beta_C^e(c)) = \beta_C^e(x), \text{ for all } x \in \bigcup_{y \in \beta_C^e(c)} z y.
\]

We denote \( \psi(\beta_P^e(p), \beta_C^e(c)) = [\beta_P^e(p)] \beta_C^e(c) \).

**Proposition 2.14.** [4] Let \( < C, \ast, e, (\cdot)^{-1} > \) and \( < P, \circ, e, (\cdot)^{-1} > \) be two polygroups and let \( \alpha : P \times C \rightarrow \mathcal{P}^e(C) \) be a productive action on \( C \). Then, \( \psi \) is an action of the group \( P/\beta_P^e \) on the group \( P/\beta_C^e \).
Theorem 2.15. [4] Let $X = (C, P, \partial, \alpha)$ be a crossed polymodule such that the action of $P$ on $C$ is productive. Then, $X_{\beta*} = (C/\beta_C^*, P/\beta_P^*, \mathcal{D}, \psi)$ is a crossed module.

3. Cat$^1$-polygroups

Cat$^1$-groups are the first in a series of models for homotopy $n$-types introduced by Loday. According to [21], Loday’s definition of a cat$^1$-group consists of groups $G$ and $S$, an embedding $k : S \to G$ and epimorphisms $t, h : G \to S$ satisfying (1) $tk = hk = Id_S$, (2) $[\ker t, \ker h] = \{1_G\}$. Now, we give a generalization of Loday’s definition. First, we need the following definition of the kernel homomorphism of polygroups. Let $\langle P; \circ; e; ( )^1 \rangle$ and $\langle C; *; e; ( )^1 \rangle$ be two polygroups and $\phi : P \to C$ be a strong homomorphism. The core-kernel of $\phi$ is defined by

$$\ker^* \phi = \{ x \in P \mid \phi(x) \in \omega_C \}.$$

Definition 3.1. A cat$^1$-polygroup $C = (k; t, h : P \to C)$ consists of polygroups $P$ and $C$, two strong epimorphisms $t, h : P \to C$ and an embedding $k : C \to P$ satisfying

\begin{align*}
\text{CAT-P-1} : & \quad tk = hk = Id_C, \\
\text{CAT-P-2} : & \quad [x, y] \subseteq w_P, \forall x \in \ker^* t, \forall y \in \ker^* h,
\end{align*}

where $[x, y] = \{ z \mid z \in x \circ y \circ x^{-1} \circ y^{-1} \}$.

The maps $t, h$ are called the source and target.

Lemma 3.2. Condition CAT-P-2 is equivalent to, for all $x, y \in P$,

$$[\beta^*_P(x), \beta^*_P(y)] = w_P = 1_{P/\beta^*_P}.$$

Proof. $[x, y] \subseteq w_P$ if $x \circ y \circ x^{-1} \circ y^{-1} \subseteq w_P$ if $\beta^*_P(x \circ y \circ x^{-1} \circ y^{-1}) = w_P$ if $\beta^*_P(x) \otimes \beta^*_P(y) \otimes \beta^*_P(x^{-1}) \otimes \beta^*_P(y^{-1}) = w_P$ if $\beta^*_P(x) \otimes \beta^*_P(y) \otimes \beta^*_P(x^{-1}) \otimes \beta^*_P(y^{-1}) = w_P$. \hfill \Box

Theorem 3.3. A cat$^1$-group is a cat$^1$-polygroup.

Proof. If $P$ and $C$ are groups, then $\omega_P = \{ e \}$, $\ker^* t = \ker t$ and $\ker^* h = \ker h$. \hfill \Box

Theorem 3.4. If $X = (C, P, \partial, \alpha)$ is a crossed polymodule, then $(k; t, h : P/\beta^*_P \ltimes C/\beta^*_C \to P/\beta^*_P)$ is a cat$^1$-group.
Proof. According to Theorem 2.15, we know \((C/\beta_C, P/\beta_P, D, \psi)\) is a crossed module. Now, we can consider

\[
P/\beta_P \times C/\beta_C \xrightarrow{h,t} P/\beta_P \]

where

\[
h(\beta_P^*(p), \beta_C^*(c)) = D(\beta_C^*(c)) \circ \beta_P^*(p),
\]

\[
t(\beta_P^*(p), \beta_C^*(c)) = \beta_P^*(p),
\]

\[
k(\beta_P^*(p)) = (\beta_P^*(p), w_C).
\]

Then

\[
h|_{P/\beta_P^*} = t|_{P/\beta_P^*} = Id_P
\]

and \([\ker h, \ker t] = 1_{P/\beta_P^* \ltimes C/\beta_C^*}\). Therefore we obtain a cat\(^1\)-group. □

Lemma 3.5. For a cat\(^1\)-polygroup \(C = (k, t, h : P \to C)\),

\[
P/\beta_P^* \cong \ker t^* \ltimes C/\beta_C^*,
\]

where \(t^* : P/\beta_P^* \to C/\beta_C^*\), \(t^*(\beta_P^*(p)) = \beta_C^*(t(p))\) and \(k^* : C/\beta_C^* \to P/\beta_P^*, k^*(\beta_C^*(c)) = \beta_P^*(k(c))\).

Proof. We define \(f : P/\beta_P^* \to \ker t^* \ltimes C/\beta_C^*\) by

\[
f(\beta_P^*(p)) = (k^*t^*(\beta_P^*(p)) \otimes \beta_P^*(p), t^*(\beta_P^*(p)))
\]

and \(g : \ker t^* \ltimes C/\beta_C^* \to P/\beta_P^*\) by

\[
g(\beta_P^*(p), \beta_C^*(c)) = k^*(\beta_P^*(p)) \otimes \beta_C^*(c)).
\]

It is not difficult to see that \(f, g\) are homomorphisms and \(f\) is the inverse of \(g\). □

Note that in the previous lemma, since \(\ker t^* \trianglelefteq P/\beta_P^*\) and \(k^*(C/\beta_C^*) \trianglelefteq P/\beta_P^*\) there is an action of \(k^*(C/\beta_C^*)\) on \(\ker t^*\) by conjugation. Hence, the semi-direct product \(\ker t^* \ltimes C/\beta_C^*\) is defined.

Theorem 3.6. If \(C = (k, t, h : P \to C)\) a cat\(^1\)-polygroup, then by putting \(S = \ker t^*\) and \(D = h^*|_{\ker t^*}\), we obtain a crossed module.
Proof. The action of $C/\beta_C$ on $S$ is conjugation in $P/\beta_P^*$. Now, if $\beta_P^*(x) \in \ker^*$ and $\beta_P^*(y) \in \ker h^*$, then
\[
\beta_P^*(x) = (w_C, \beta_P^*(a)), \quad \beta_P^*(y) = (D(\beta_P^*(b)), \beta_P^*(b^{-1})),
\]
for all $\beta_P^*(a), \beta_P^*(b) \in S$. Thus,
\[
\beta_P^*(x) \circ \beta_P^*(y) = (w_C, \beta_P^*(a)) \circ (D(\beta_P^*(b)), \beta_P^*(b^{-1})) = (D(\beta_P^*(b)), D(\beta_P^*(b)) \beta_P^*(a) \circ \beta_P^*(b^{-1}))
\]
\[
\beta_P^*(y) \circ \beta_P^*(x) = (D(\beta_P^*(b)), \beta_P^*(b^{-1})) \circ (w_C, \beta_P^*(a)) = (D(\beta_P^*(b)), w_C \beta_P^*(b^{-1}) \circ \beta_P^*(a)) = (D(\beta_P^*(b)), \beta_P^*(b^{-1}) \circ \beta_P^*(a))
\]
Thus, the equality $\beta_P^*(x) \circ \beta_P^*(y) = \beta_P^*(y) \circ \beta_P^*(x)$ is equivalent to $D(\beta_P^*(b)) \beta_P^*(a) = \beta_P^*(b^{-1}) \circ \beta_P^*(a) \circ \beta_P^*(b)$. □

Corollary 3.7. The following diagram shows all the results obtained and thus gives their relations.

\[
\begin{array}{ccc}
\text{Cat}^1 - \text{groups} & \xrightarrow{\text{Inc}} & \text{Cat}^1 - \text{polygroups} \\
\downarrow \cong & & \downarrow \phi_{\beta^*} \\
\text{Crossed modules} & & \text{Crossed polymodules} \\
\downarrow \phi_{\beta^*} & & \downarrow \text{Inc} \\
\end{array}
\]

4. Pullback cat$^1$-polygroups

In this section, we define the pullback cat$^1$-polygroup and we obtain some results in this respect. Specially, we prove that by a pullback cat$^1$-polygroup we can obtain a cat$^1$-group.
Definition 4.1. A pullback cat¹-polygroup is defined as follows.

Let \( C = (k; t, h : P \to C) \) be a cat¹-polygroup and let \( \iota : Q \to C \) be a strong homomorphism. Define \( \iota^{**}C = (k^{**}; t^{**}, h^{**} : \iota^{**}P \to Q) \) to be the pullback of \( P \), where

\[
\iota^{**}P = \{(q_1, p, q_2) \in Q \times P \times Q \mid \iota(q_1) = t(p), \iota(q_2) = h(p)\},
\]

\( t^{**}(q_1, p, q_2) = q_1, \ h^{**}(q_1, p, q_2) = q_2 \) and \( k^{**}(q) = (q, k\iota(q), q) \). Multiplication in \( \iota^{**}P \) is componentwise. The pair \((\pi, \iota)\) is a morphism of cat¹-polygroups, where \( \pi : \iota^{**}P \to P, (q_1, p, q_2) \mapsto p \).

Theorem 4.2. By a pullback cat¹-polygroup, we have a cat¹-polygroup.

Proof. We verify the cat¹-polygroup axioms. For the first axiom, we have

\[
\iota^{**}k^{**} = \iota^{**}(q, kt(q), q) = q,
\]

\[
h^{**}k^{**}(q) = h^{**}(q, kt(q), q) = q.
\]

Thus, \( \iota^{**}k^{**} = h^{**}k^{**} = Id_Q \) and CAT-P-1 is satisfied.

In order to prove the second condition, suppose that \( x = (q_1', p_1, q_1) \in \ker^*t^{**}, y = (q_2, p_2, q_2') \in \ker^*h^{**} \). Then, \( t^{**}(q_1', p_1, q_1) = q_1' \in \omega_Q \) and \( h^{**}(q_2, p_2, q_2') = q_2' \in \omega_Q \). By Lemma 2.10, \( \omega_Q \) is a subpolygroup of \( Q \). We show that it is also normal. Suppose that \( b \in Q \) and \( a \in \omega_Q \) are
arbitrary. For each $z \in b \cdot a \cdot b^{-1}$, we have
\[
\beta^*_Q(z) = \beta^*_Q(b) \otimes \beta^*_Q(a) \otimes \beta^*_Q(b^{-1}) \\
= \beta^*_Q(b) \otimes \omega_Q \otimes \beta^*_Q(b^{-1}) \\
= \beta^*_Q(y) \otimes \beta^*_Q(b^{-1}) \\
= \beta^*_Q(b \cdot b^{-1}) \\
= \beta^*_Q(e) = \omega_Q.
\]
So, $z \in \omega_Q$. Therefore, we conclude that $q_1' \cdot q_2 \cdot q_1'^{-1} \cdot q_2' \in \omega_Q$ and $q_1 \cdot q_2' \cdot q_1'^{-1} \cdot q_2'' \in \omega_Q$.

On the other hand, by the definition of $t^*$, we obtain
\[
t(q_1') = t(p_1) \in t(\omega_Q) \quad \text{and} \quad t(q_2') = h(p_2) \in t(\omega_Q).
\]
Now, we show that $t(\omega_Q) \subseteq \omega_C$. Since $e \in \omega_Q$, $t(e) \in \omega_C$. Now, suppose that there exists $a \in \omega_Q$ such that $t(a) \in \omega_C$. Since $a, e \in \omega_Q$, $\beta^*_C(t(a)) \neq \omega_C$. On the other hand, $t(e) = e \in \omega_C$ and so $\beta^*_C(t(e)) = \omega_C$. Thus, $\beta^*_C(t(e)) \neq \beta^*_C(t(a))$. This implies that $t^*(\beta^*_Q(e)) \neq t^*(\beta^*_Q(a))$, which is a contradiction. Hence, $t(p_1) \in \omega_C$ and $h(p_2) \in \omega_C$. Thus,
\[
p_1 \in \ker t \quad \text{and} \quad p_2 \in \ker h.
\]
Now, we have
\[
[x, y] = x \boxtimes y \boxtimes x^{-1} \boxtimes y^{-1} \\
= \{(q, p, q') \mid q \in q_1' \cdot q_2 \cdot q_1'^{-1} \cdot q_2', \ p \in [p_1, p_2], \ q' \in q_1 \cdot q_2' \cdot q_1'^{-1} \cdot q_2'' \} \\
\subseteq \omega_Q \times \omega_Q \times \omega_Q.
\]
Therefore, CAT-P-2 is also satisfied. □

**Theorem 4.3.** If $\iota^* \mathcal{X}$ is the pullback of the crossed polymodule $\mathcal{X}$ over $\iota: Q \to P$ and if $\mathcal{A}, \mathcal{B}$ are the cat^1-groups obtained from $\mathcal{X}, \iota^* \mathcal{X}$ respectively, then $\mathcal{B} \cong \iota^* \mathcal{A}$.

**Proof.**

\[
\begin{array}{ccc}
\iota^* C & \longrightarrow & C \\
\downarrow \iota^* & & \downarrow \iota \\
Q & \longrightarrow & P
\end{array}
\quad
\begin{array}{ccc}
\iota^* C/\beta^*_C & \longrightarrow & C/\beta^*_C \\
\downarrow \iota^* & & \downarrow \iota^* \\
Q/\beta^*_Q & \longrightarrow & P/\beta^*_P
\end{array}
\]
Starting with the pullback crossed polymodule $\iota^* X = (\iota^*, Q, \partial^*, \alpha^*)$, where $\partial^* : \iota^* C \to Q$, the source polygroup of $B$ is defined as the semi-direct product $Q/\beta_Q^* \ltimes \iota^* C/\beta_C^*).

\[
Q/\beta_Q^* \ltimes \iota^* C/\beta_C^* \xrightarrow{\iota^* h^*} P/\beta_P^* \ltimes C/\beta_C^* \\
Q/\beta_Q^* \xrightarrow{\iota^*} P/\beta_P^*
\]

The target, source and embedding of $B$ are respectively given by

\[
\iota^*(\beta_Q^*(q'), \beta_C^*(q, c)) = \beta_Q^*(q'),
\]

\[
h^*(\beta_Q^*(q'), \beta_C^*(q, c)) = \mathcal{D}^*(\beta_C^*(q, c) \circ \beta_Q^*(q') = \beta_Q^*(q) \circ \beta_Q^*(q') = \beta_Q^*(q \cdot q'),
\]

\[
k^*(\beta_Q^*(q)) = (\beta_Q^*(q), \omega^* C).
\]

We then define an isomorphism of cat\(^1\)-groups $(\lambda, Id) : B \to \iota^{**} A,$

\[
\begin{array}{c}
\xymatrix{Q/\beta_Q^* \ltimes \iota^* C/\beta_C^* \ar[r]^-{\lambda} & \iota^{**}(P/\beta_P^* \ltimes C/\beta_C^*) \\
Q/\beta_Q^* \ar[r]_-{Id} \ar[u]^-{k^*} & Q/\beta_Q^* \ar[u]_-{k^{**}} \\
Q/\beta_Q^* \ar[u]_-{\iota^* h^*} & Q/\beta_Q^* \ar[u]_-{\iota^{**} h^{**}}}
\end{array}
\]

where

\[
\lambda\left(\beta_Q^*(q'), \beta_C^*(q, c)\right) = \left(\beta_Q^*(q'), (\beta_P^*(\iota(q')), \beta_C^*(c), \beta_Q^*(q \cdot q')\right)
\]

First note that $\lambda(\beta_Q^*(q'), \beta_C^*(q, c)) \in \iota^{**}(P/\beta_P^* \ltimes C/\beta_C^*)$ because

\[
t(\beta_P^*(\iota(q')), \beta_C^*(c)) = \beta_P^*(\iota(q')) = \iota^*(\beta_Q^*(q'))
\]
and
\[ h(\beta_P(q'), \beta_C(c)) = \mathcal{D}(\beta_C^*(c)) \circ \iota^*(\beta_Q^*(q')) \]
\[ = \iota^*(\beta_Q^*(q)) \circ \iota^*(\beta_Q^*(q')) \]
\[ = \iota^*(\beta_Q^*(q) \circ \beta_Q^*(q')) \]
\[ = \iota^*(\beta_Q^*(q \cdot q')). \]

We verify that \( \lambda \) is a homomorphism as follows:

\[ \lambda\left( (\beta_Q^*(q_1'), \beta\cdot \beta_I \cdot \beta(q_1 \cdot c_1)) \left( \beta_Q^*(q_2'), \beta\cdot \beta_I \cdot \beta(q_2 \cdot c_2) \right) \right) \]
\[ = \left( \beta_Q^*(q_1 \cdot q_2'), \left( \iota^*(\beta_Q^*(q_1 \cdot q_2')) \right) \left[ \iota^*(\beta_C^*(c_1)) \right] \left( \beta_Q^*(q_1 \cdot q \cdot q_2 \cdot q_2') \right) \right) \]

and

\[ \lambda\left( \beta_Q^*(q_1'), \beta\cdot \beta_I \cdot \beta(q_1 \cdot c_1) \right) \lambda\left( \beta_Q^*(q_2'), \beta\cdot \beta_I \cdot \beta(q_2 \cdot c_2) \right) \]
\[ = \left( \beta_Q^*(q_1'), \left( \beta_P^*(\iota(q_1')) \right) \left( \beta_C^*(c_1) \right) \left( \beta_Q^*(q_1 \cdot q_1) \right) \left( \beta_Q^*(q_2') \right) \right) \left( \beta_Q^*(q_2 \cdot q_2') \right) \]
\[ = \left( \beta_Q^*(q_1) \circ \beta_Q^*(q_2), \left( \beta_P^*(\iota(q_1)) \right) \left( \beta_C^*(c_1) \right) \left( \beta_Q^*(q_1 \cdot q_1) \right) \left( \beta_Q^*(q_2') \right) \right) \]
\[ = \left( \beta_Q^*(q_1 \cdot q_2), \left( \iota^*(\beta_Q^*(q_1)) \right) \left( \beta_C^*(c_1) \right) \left( \beta_Q^*(q_2') \right) \right) \]
\[ = \left( \beta_Q^*(q_1 \cdot q_2), \left( \iota^*(\beta_Q^*(q_1)) \right) \left( \beta_C^*(c_1) \right) \left( \beta_Q^*(q_1 \cdot q_1 \cdot q_2 \cdot q_2') \right) \right) \]
\[ = \left( \beta_Q^*(q_1 \cdot q_2), \left( \iota^*(\beta_Q^*(q_1)) \right) \left( \beta_C^*(c_1) \right) \left( \beta_Q^*(q_1 \cdot q_1 \cdot q_2 \cdot q_2') \right) \right) \]

The inverse of \( \lambda \) is given by

\[ \lambda^{-1}\left( \beta_Q(q_1), \left( \beta_P(p), \beta_C^*(c) \right) \right) = \left( \beta_Q(q_1), \beta_Q^*(q_1^{-1} \cdot q_2), \beta_C^*(c) \right). \]
Then,
\[ t^* (\beta^*_Q(q'), \beta^*_{\mathcal{C}}(q, c)) = t^* (\beta^*_Q(q'), (\beta^*_P(\epsilon(q')), \beta^*_C(c)), \beta^*_Q(q \cdot q')) = \beta^*_Q(q') = t^* (\beta^*_Q(q'), \beta^*_{\mathcal{C}}(q, c)), \]

\[ k^* \lambda (\beta^*_{\mathcal{Q}}(q), \beta^*_{\mathcal{C}}(q, c)) = k^* \lambda (\beta^*_Q(q'), (\beta^*_P(\epsilon(q')), \beta^*_C(c)), \beta^*_Q(q \cdot q')) = \beta^*_Q(q \cdot q') = k^* (\beta^*_Q(q'), \beta^*_{\mathcal{C}}(q, c)) \]

\[ \lambda k^* (\beta^*_Q(q)) = \lambda (\beta^*_Q(q), (\omega_Q, \omega_C)) = \left( \beta^*_Q(q), (t^*(\beta^*_Q(q)), \omega_C) \right) = k^* (\beta^*_Q(q)). \]

Therefore, the diagram commutes. \( \Box \)

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