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k-TUPLE TOTAL RESTRAINED DOMINATION/DOMATIC IN GRAPHS

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ABSTRACT. For any integer $k \geq 1$, a set S of vertices in a graph $G = (V, E)$ is a k-tuple total dominating set of G if any vertex of G is adjacent to at least k vertices in S, and any vertex of $V - S$ is adjacent to at least k vertices in $V - S$. The minimum number of vertices of such a set in G we call the k-tuple total restrained domination number of G. The maximum number of classes of a partition of V such that its all classes are k-tuple total restrained dominating sets in G we call the k-tuple total restrained domatic number of G.

In this paper, we give some sharp bounds for the k-tuple total restrained domination number of a graph, and also calculate it for some of the known graphs. Next, we mainly present basic properties of the k-tuple total restrained domatic number of a graph.
Keywords: k-tuple total domination number, k-tuple total domatic number, k-tuple total restrained domination number, k-tuple total restrained domatic number.
MSC(2010): Primary: 05C69.

1. Introduction

1.1. **Preliminary definitions.** Let $G = (V, E)$ be a graph with *vertex set* V of *order* $n(G)$ and *edge set* E of *size* $m(G)$. The *open neighborhood* and the *closed neighborhood* of a subset $X \subseteq V(G)$ are $N_G(X) = \{u \in V | uv \in E$, for some $v \in X\}$ and $N_G[X] = N_G(X) \cup X$, respectively. If $X = \{v\}$, we write $N_G(v)$ and $N_G[v]$ in stead of $N_G\{v\}$ and $N_G[\{v\}]$, respectively. The *degree* of a vertex v is also $deg_G(v) =| N_G(v) |$. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and

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$\Delta = \Delta(G)$, respectively. If every vertex of G has degree k, then G is called *k-regular*. We write K_n, P_n and C_n for the *complete graph*, the *path* and the *cycle* of order n, respectively, while $K_{n_1,n_2,\ldots,n_p}$ and $G[S]$ denote the *complete p-partite graph* and the *induced subgraph* of G by the vertex set S. The *complement* of a graph G is denoted by $\overline{G}$ and is a graph with the vertex set $V(G)$ and for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$. For notation and graph theory terminology, which is not appeared here, we in general follow [5].

For each integer $k \geq 1$, the *k-join* $G \circ_k H$ of a graph G to a graph H of order at least k is the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H [7]. Also, $G \circ_{*k} H$ denotes the k-join $G \circ_k H$ such that each vertex of G is joined to exactly k vertices of H.

1.2. **k-tuple total restrained domination/ domatic.** The research of domination in graphs has been an evergreen of the graph theory. Its basic concept is the dominating set and the domination number. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [5, 6]. A numerical invariant of a graph which is in a certain sense dual to it is the domatic number of a graph. And many variants of the dominating set were introduced and the corresponding numerical invariants were defined for them. The k-tuple total domination number is one of them, which is first introduced in 1991 by V. R. Kulli, in [10].

Let $k \geq 1$ be an integer and let G be a graph with $\delta(G) \geq k$. A subset $S \subseteq V(G)$ is called a *k-tuple total dominating set*, briefly kTDS, of G if for each $x \in V(G)$, $\mid N(x) \cap S \mid \geq k$. The minimum number of vertices of a k-tuple total dominating set in a graph G is called the *k-tuple total domination number* of G and denoted by $\gamma_{\times k,t}(G)$. We recall that 1-tuple total dominating set and 1-tuple total domination number are known as *total dominating set* and *total domination number*, respectively. For more information see [1, 8, 9].

Here, we begin to study two new concepts: k-tuple total restrained domination number, and k-tuple total restrained domatic number.

Definition 1.1. In a graph G with $\delta(G) \geq k \geq 1$, a *k-tuple total restrained dominating set S,* briefly kTRDS, of G is a k-tuple total dominating set of G such that each vertex of $V(G) - S$ is adjacent to at least k vertices of $V(G) - S$. The *k-tuple total restrained domination number* $\gamma^r_{\times k,t}(G)$ of G is the minimum cardinality of a kTRDS.

The domatic number $d(G)$ and the total domatic number $d_t(G)$ of a graph were introduced in [3] and [2], respectively. Sheikholeslami and Volkmann extended the last definition to the k-tuple total domatic number.

Definition 1.2. [11] The *k-tuple total domatic partition,* briefly kTDP, of G is a partition $\mathbb{D}$ of the vertex set of G such that all classes of $\mathbb{D}$ are k-tuple total dominating sets in G. The maximum number of classes of a k-tuple total domatic partition of G is called the *k-tuple total domatic number* $d_{\times k,t}(G)$ of G.

We define the *star k-tuple total domatic number* $d^*_{\times k,t}(G)$ of G as the maximum number of classes of a kTDP of G such that at least one of the k-tuple total dominating sets in it has cardinality $\gamma_{\times k,t}(G)$.

In an analogous way, we define k-tuple total restrained domatic number and star k-tuple total restrained domatic number.

Definition 1.3. The *k-tuple total restrained domatic partition,* briefly kTRDP, of G is a partition $\mathbb{D}$ of the vertex set of G such that all classes of $\mathbb{D}$ are k-tuple total restrained dominating sets in G. The maximum number of classes of a k-tuple total restrained domatic partition of G is the *k-tuple total restrained domatic number* $d^r_{\times k,t}(G)$ of G. Similarly, the *star k-tuple total restrained domatic number* $d^{r*}_{\times k,t}(G)$ of G is the maximum number of classes of a kTRDP of G such that at least one of the k-tuple total restrained dominating sets in it has cardinality $\gamma^r_{\times k,t}(G)$.

Since every kTRDS is also a kTDS, we have $\gamma_{\times k,t}(G) \leq \gamma^r_{\times k,t}(G)$, and so $d^r_{\times k,t}(G) \leq d_{\times k,t}(G)$. Also $\gamma^r_t(G) = \gamma^r_{\times 1,t}(G)$ and $d^r_t(G) = d^r_{\times 1,t}(G)$.

1.3. **Our goal.** In this paper, we give some sharp bounds for the k-tuple total restrained domination number of a graph, and also calculate it for some of the known graphs. Next, we mainly present basic properties of the k-tuple total restrained domatic number of a graph. We begin our discussion with the following trivial observation. The proof follows readily from the definitions and is omitted.

Observation 1.4. *Let G be a graph of order n in which $\delta(G) \geq k$. Then*

i. *every vertex of degree at most $2k-1$ of G and at least k of its neighbors belong to every* kTRDS*,*

ii. $d^r_{\times k,t}(G) = 1$ *if* $\delta(G) \leq 2k-1$*,*

iii. $\gamma^r_{\times k,t}(G) < n-k-1$ *if* $\gamma^r_{\times k,t}(G) < n$*,*

iv. $\Delta(G) \geq 2k$ *if* $\gamma^r_{\times k,t}(G) < n$*. Hence* $n \geq 2k+2$*.*

Through this paper, we assume that k is a positive integer, and

$$V(C_n) = V(\overline{C_n}) = V(P_n) = V(\overline{P_n}) = \{i \mid 1 \le i \le n\},$$
$$E(C_n) = E(P_n) \cup \{1n\} = \{ij \mid 1 \le i \le n-1 \text{ and } j = i+1\} \cup \{1n\}.$$

2. k-tuple total restrained domination in some graphs

Here, we calculate the k-tuple total restrained domination number of the complete graph, the cycle, the bipartite graph and the complement of a path or a cycle.

Proposition 2.1. *Let $k < n$ be positive integers. Then*

$$\gamma^r_{\times k,t}(K_n) = \begin{cases} n & \text{if } n \le 2k+1, \\ k+1 & \text{otherwise.} \end{cases}$$

Proof. Observation 1.4 (iv) implies $\gamma^r_{\times k,t}(K_n) = n$ if $n \le 2k+1$. Since also every $(k+1)$-subset of vertices is a kTRDS of K_n when $n > 2k+1$, we obtain $\gamma^r_{\times k,t}(K_n) = k+1$ if $n > 2k+1$. □

Proposition 2.2. *Let $n \ge 4$. Then*

$$\gamma^r_t(C_n) = \begin{cases} 2\lceil n/4 \rceil - 1 & \text{if } n \equiv 1 \pmod 4, \\ 2\lceil n/4 \rceil + 1 & \text{if } n \equiv 3 \pmod 4, \\ 2\lceil n/4 \rceil & \text{otherwise.} \end{cases}$$

Proof. Observation 1.4 (iv) implies $\gamma^r_{\times 2,t}(C_n) = n$. Also in many references, for example [5], it can be seen that

$$\gamma_t(C_n) = \begin{cases} 2\lceil n/4 \rceil - 1 & \text{if if } n \equiv 1 \pmod 4, \\ 2\lceil n/4 \rceil & \text{otherwise.} \end{cases}$$

Since the sets $S_0 = \{2+4i, 3+4i \mid 0 \le i \le \lfloor n/4 \rfloor - 1\}$, $S_1 = S_0 \cup \{n-1\}$ and $S_2 = S_0 \cup \{1, n-2\}$ are total restrained dominating sets of cardinality $\gamma_t(C_n)$ for $n \equiv 0, 1, 2 \pmod 4$, respectively, we have nothing to prove when $n \not\equiv 3 \pmod 4$. Now let $n \equiv 3 \pmod 4$. Then it can be easily verify that $\gamma^r_t(C_n) \ge \gamma_t(C_n) + 1$. Since $S_3 = S_0 \cup \{1, n-3, n\}$ is a total restrained dominating set of C_n of cardinality $\gamma_t(C_n) + 1$, we obtain $\gamma^r_t(C_n) = 2\lceil n/4 \rceil + 1$. □

Proposition 2.3. *Let $n \ge k+3 \ge 4$. Then*

$$\gamma^r_{\times k,t}(\overline{C_n}) = \begin{cases} n & \text{if } n \le 2k+2, \\ k+2 & \text{if } 2k+3 \le n \le 3k+2, \\ k+1 & \text{if } n \ge 3k+3. \end{cases}$$

Proof. We first prove that $\gamma^r_{\times k,t}(\overline{C_n}) = k+1$ if and only if $n \geq 3k+3$. Let S be a kTRDS of $\overline{C_n}$ of cardinality $k+1$. Then for any two arbitrary vertices i and j in S, $\mid i-j \mid \geq 3$. Hence $n \geq 3k+3$. On the other hand, if $n \geq 3k+3$, then $\{3i+1 \mid 0 \leq i \leq k\}$ is a kTRDS of $\overline{C_n}$, and so $\gamma^r_{\times k,t}(\overline{C_n}) = k+1$.

Observation 1.4 (iv) implies that $\gamma^r_{\times k,t}(\overline{C_n}) = n$ if and only if $k+3 \leq n \leq 2k+1$. Now let $n = 2k+2$. Then $\delta(G) = \Delta(G) = n-3 = 2k-1$. Let S be a kTRDS of $\overline{C_n}$, and let $i \in V-S$. Since $\mid N(i) \cap S \mid \geq k$ and $\mid N(i) \cap (V-S) \mid \geq k$, we have $deg(i) \geq 2k$, a contradiction. Therefore $S = V(\overline{C_n})$ and so $\gamma^r_{\times k,t}(\overline{C_n}) = n$. For the remained case, obviously $S = \{2i+1 \mid 0 \leq i \leq k+1\}$ is a kTRDS of $\overline{C_n}$ and so $\gamma^r_{\times k,t}(\overline{C_n}) = k+2$. □

Proposition 2.4. *Let $n \geq k+3 \geq 4$. Then*

$$\gamma^r_t(\overline{P_n}) = \begin{cases} 4 & \text{if } n = 4, \\ 2 & \text{if } n \geq 5, \end{cases}$$

and if $k \geq 2$, then

$$\gamma^r_{\times k,t}(\overline{P_n}) = \begin{cases} n & \text{if } n \leq 2k+2, \\ k+2 & \text{if } 2k+3 \leq n \leq 3k, \\ k+1 & \text{if } n \geq 3k+1. \end{cases}$$

Proof. One can easily verify that $\gamma^r_t(\overline{P_n})$ is 2 if and only if $n \geq 5$, and is n otherwise. Now let $k \geq 2$. It can be easily verify that $\gamma^r_{\times k,t}(\overline{P_n}) = k+1$ if and only if there exists a kTRDS S of $\overline{P_n}$ such that for every two disjoint vertices i and j in S, the difference between i and j is at least 3, to modulo n, or $\{i,j\} = \{1,n\}$. This implies $n \geq 3k+1$. Since $S = \{3i+1 \mid 0 \leq i \leq k-1\} \cup \{n\}$ is a kTRDS of $\overline{P_n}$ for $n \geq 3k+1$, we obtain $\gamma^r_{\times k,t}(\overline{P_n}) = k+1$. Now let $n = k+i \leq 3k$ and let S be a kTRDS of $\overline{P_n}$. For any vertex $x \in V-S$, we have

$$deg(x) \geq n-1- \mid S \mid \geq n-k-3 = i-3.$$

Since also $deg(x) \geq k$, we obtain $i \geq k+3$. Hence $\gamma^r_{\times k,t}(\overline{P_n}) = n$ if $n \leq 2k+2$. Now let $2k+3 \leq n \leq 3k$. Since $\overline{C_n}$ is a spanning subgraph of $\overline{P_n}$, we have $\gamma^r_{\times k,t}(\overline{P_n}) \leq \gamma^r_{\times k,t}(\overline{C_n})$. Now $\gamma^r_{\times k,t}(\overline{C_n}) = k+2$ (by Proposition 2.3) and $\gamma^r_{\times k,t}(\overline{P_n}) > k+1$ imply $\gamma^r_{\times k,t}(\overline{P_n}) = k+2$. □

3. Complete multipartite graphs

Here, we present some lower and upper bounds for the k-tuple total restrained domination number of a complete multipartite graph.

Proposition 3.1. *Let G be a bipartite graph with $\delta(G) \geq k \geq 1$. Then $2k \leq \gamma^r_{\times k,t}(G) \leq n$. Moreover, if X and Y are the bipartite sets of $V(G)$, then $\gamma^r_{\times k,t}(G) = 2k$ if and only if there exist two k-subsets $S \subseteq X$ and $T \subseteq Y$ such that*

1. *for each vertex $x \in X$, $T \subseteq N(x)$, and*
2. *for each vertex $y \in Y$, $S \subseteq N(y)$, and*
3. *$\delta(G[(X - S) \cup (Y - T)]) \geq k$.*

Proof. Let D be a $\gamma^r_{\times k,t}(G)$-set, and let $w \in X$ and $z \in Y$ be two arbitrary vertices. The definition of k-tuple total restrained dominating set implies that $\mid D \cap N(w) \mid \geq k$ and $\mid D \cap N(z) \mid \geq k$. Since $N(w) \cap N(z) = \emptyset$, we deduce $\mid D \mid \geq 2k$ and thus $2k \leq \gamma^r_{\times k,t}(G) \leq n$. If there exist two k-subsets $S \subseteq X$ and $T \subseteq Y$ satisfying the above three conditions, then obviously $S \cup T$ is a k-tuple total restrained dominating set of G. This implies $\gamma^r_{\times k,t}(G) \leq 2k$ and so $\gamma^r_{\times k,t}(G) = 2k$.

Conversely, we assume $\gamma^r_{\times k,t}(G) = 2k$, and D is a $\gamma^r_{\times k,t}(G)$-set. Then

$$\mid D \cap X \mid = \mid D \cap Y \mid = k.$$

Now let $S = D \cap X$ and $T = D \cap Y$. Then $T \subseteq N(x)$ for each vertex $x \in X$ and $S \subseteq N(y)$ for each vertex $y \in Y$. If $\mid X \mid > k$ and $\mid Y \mid > k$, then $\delta(G[(X - S) \cup (Y - T)]) \geq k$, by the definition, and this completes our proof. □

Corollary 3.2. *Let $G = K_{n,m}$ be a complete bipartite graph with $n \geq m \geq k \geq 1$. Then*

$$\gamma^r_{\times k,t}(G) = \begin{cases} 2k & \text{if } n \geq m \geq 2k, \\ n + m & \text{otherwise.} \end{cases}$$

Now we present some bounds for $\gamma^r_{\times k,t}(G)$, where G is the complete p-partite graph $K_{n_1,\ldots,n_p}$ with $p \geq 3$. First a lower bound.

Proposition 3.3. *Let G be a complete p-partite graph of order n with $p \geq 3$. Then*

$$\gamma^r_{\times k,t}(G) \geq \lceil \frac{pk}{p-1} \rceil.$$

Proof. We assume that $G = K_{n_1,\dots,n_p}$ has the vertex partition $V = X_1 \cup \dots \cup X_p$ such that $\mid X_i \mid = n_i$ and $n = n_1 + \dots + n_p$. Let S be an arbitrary kTRDS of G and let $S_i = X_i \cap S$ has cardinality s_i. Since every vertex of X_i is adjacent to at least k vertices of $S - X_i = \bigcup_{i \neq j=1}^{p} S_j$, we have

$$\sum_{j=1}^{p} s_j - s_i \geq k,$$

for each $1 \leq i \leq p$. Hence $(p-1) \mid S \mid \geq pk$, and so $\mid S \mid \geq \lceil \frac{pk}{p-1} \rceil$. Since S was arbitrary, we get $\gamma^r_{\times k,t}(G) \geq \lceil \frac{pk}{p-1} \rceil$. □

For giving an upper bound for the k-tuple total restrained domination number of a complete p-partite graph G with $p \geq 3$, we use the following definitions and notations. We assume that $G = K_{n_1,\dots,n_p}$ is a complete p-partite graph with the vertex partition $V = X_1 \cup \dots \cup X_p$ such that $\mid X_i \mid = n_i$ and $n = n_1 + \dots + n_p$. Let S be a kTRDS of $G = K_{n_1,\dots,n_p}$ and let $S_i = X_i \cap S$, $S'_i = X_i - S$, $\mid S_i \mid = s_i$. Let also $t(S)$ be the number of is that $s_i < n_i$ and

$$t_0 = \min\{t(S) \mid S \text{ is a kTRDS of } G\}.$$

We may assume $t(S) \geq 1$. Because $t_0 = 0$ if and only if $\gamma^r_{\times k,t}(G) = n$. Then obviously $t(S) \geq 2$. Without loss of generality, we may assume $s_i < n_i$ if and only if $1 \leq i \leq t(S)$. Let $w_j \in X_j - S = X_j - S_j$ for each $1 \leq j \leq t(S)$. Since S is a kTRDS, we get $\mid N(w_j) \cap (V - S) \mid \geq k$. Hence for each $1 \leq j \leq t$,

$$\begin{aligned} k &\leq \mid N(w_j) \cap (V - S) \mid \\ &= \sum_{i=1, i \neq j}^{t(S)} \mid N(w_j) \cap S'_i \mid \\ &= \sum_{i=1, i \neq j}^{t(S)} \mid S'_i \mid \\ &= \sum_{i=1}^{t(S)} \mid S'_i \mid - \mid S'_j \mid . \end{aligned}$$

Because $N(w_j)\cap(V-S) = \bigcup_{j\neq i=1}^{t(S)} N(w_j)\cap S'_i$. By summing the inequalities we obtain

$$\begin{aligned} t(S)k &\leq (t(S)-1)\sum_{i=1}^{t(S)}(n_i - s_i) \\ &= (t(S)-1)\sum_{i=1}^{p}(n_i - s_i) \\ &= (t(S)-1)(n- \mid S \mid). \end{aligned}$$

Hence $\mid S \mid \leq n - k - \lceil \frac{k}{t(S)-1} \rceil$. Since S was arbitrary, we obtain

$$\gamma^r_{\times k,t}(G) \leq n - k - \lceil \frac{k}{t_0 - 1} \rceil.$$

Therefore, we have proved the next result.

Proposition 3.4. *Let G be a complete p-partite graph of order n with $p \geq 3$. If $\gamma^r_{\times k,t}(G) < n$, then $\gamma^r_{\times k,t}(G) \leq n - k - \lceil \frac{k}{t_0-1} \rceil$.*

4. **Some Bounds**

Before giving some bounds for the k-tuple total restrained domination number of a graph, we characterize the structure of a graph whose k-tuple total restrained domination number is equal m, for any $m \geq k+1$.

Theorem 4.1. *Let G be a graph with $\delta(G) \geq k$, and let $m \geq k+1$ be an integer. Then $\gamma^r_{\times k,t}(G) = m$ if and only if $G = K'_m$ or $G = F \circ_k K'_m$, where m is the minimum of the set*

$$\top = \{t \mid G = F' \circ_k K'_t, \text{ for some graph } F' \text{ with } \delta(F') \geq k, \text{ and some spanning subgraph } K'_t \text{ of } K_t \text{ with } \delta(K'_t) \geq k\},$$

and $F = G - K'_m$ with $\delta(F') \geq k$.

Proof. Let S be a $\gamma^r_{\times k,t}(G)$-set and let $\gamma^r_{\times k,t}(G) = m$, for some $m \geq k+1$. Then $\mid S \mid = m$, and every vertex has at least k neighbours in S, and also every vertex in $V-S$ has at least k neighbours in $V-S$. Then $G[S]$ is a spanning subgraph, say K'_m, of K_m with $\delta(K'_m) \geq k$. If $\mid V \mid = m$, then $G = K'_m$. If not, let F be the induced subgraph $G[V-S]$. Then $\delta(F) \geq k$ and $G = F \circ_k K'_m$. Also the definition of the k-tuple total restrained domination number implies that m is the minimum of the set $\top$.

Conversely, let $G = K'_m$ or $G = F \circ_k K'_m$, where m is the minimum of $\top$, K'_m is a spanning subgraph of K_m with $\delta(K'_m) \geq k$, and $F = G - K'_m$

with $\delta(F') \geq k$. Then $\gamma^r_{\times k,t}(G) \leq m$. Because $V(K'_m)$ is a kTRDS of G of cardinality m. If $\gamma^r_{\times k,t}(G) = m' < m$, then the previous paragraph implies $G = F' \circ_k K'_{m'}$, for some spanning subgraph $K'_{m'}$ of $K_{m'}$ with $\delta(K'_{m'}) \geq k$, and $F' = G - K'_{m'}$ with $\delta(F') \geq k$, that contradicts the minimality of m. Therefore $\gamma^r_{\times k,t}(G) = m$. □

Corollary 4.2. *Let G be a graph with $\delta(G) \geq k$. Then $\gamma^r_{\times k,t}(G) = k+1$ if and only if $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$, for some graph F with $\delta(F) \geq k$.*

In the next two theorems we will present a lower bound and an upper bound for the k-tuple total restrained domination number of a graph.

Theorem 4.3. *If G is a graph on n vertices, with m edges and minimum degree at least k, then*

$$\gamma^r_{\times k,t}(G) \geq \frac{3n}{2} - \frac{m}{k}, \tag{4.1}$$

with equality if and only if there exist k-regular graphs H and F of orders $\gamma^r_{\times k,t}(G)$ and $n - \gamma^r_{\times k,t}(G)$, respectively, such that G is isomorphic to $F \circ_{*k} H$.

Proof. Let S be a kTRDS of G with minimum cardinality. Since $\delta(G[S]) \geq k$, $\delta(G[V-S]) \geq k$ and S is a kTDS, we have the following inequalities:

$$\begin{array}{rcl} m_1 & \geq & \frac{k\gamma^r_{\times k,t}(G)}{2}, \\ m_2 & \geq & \frac{k(n-\gamma^r_{\times k,t}(G))}{2}, \\ m_3 & \geq & k(n - \gamma^r_{\times k,t}(G)), \end{array}$$

where m_1 and m_2 are respectively the number of edges in the induced subgraphs $G[S]$ and $G[V-S]$ and m_3 is the number of edges connecting vertices in $V-S$ to the vertices in S. By summing the inequalities, we obtain

$$m = m_1 + m_2 + m_3 \geq \frac{3kn}{2} - k\gamma^r_{\times k,t}(G)).$$

Hence $\gamma^r_{\times k,t}(G) \geq \frac{3n}{2} - \frac{m}{k}$.

We know that the equality holds in (4.1) if and only if the inequality occurring in the proof becomes equality, that is,

$$\begin{array}{rcl} m_1 & = & \frac{k\gamma^r_{\times k,t}(G)}{2}, \\ m_2 & = & \frac{k(n-\gamma^r_{\times k,t}(G))}{2}, \\ m_3 & = & k(n - \gamma^r_{\times k,t}(G)). \end{array}$$

The first and the second equalities is equivalent to the existence of k-regular graphs $H = G[S]$ and $F = G[V(G) - S]$ of orders $\gamma^r_{\times k,t}(G)$ and $n - \gamma^r_{\times k,t}(G)$, respectively, while the third equality gives that every vertex of F is adjacent to exactly k vertices of H. Hence equality holds in (4.1) if and only if G is isomorphic to $F \circ_{*k} H$. □

As an example, if G is the graph obtained by the complete graph K_{2k+2} minus a perfect matching, then $\gamma^r_{\times k,t}(G) = \frac{3n}{2} - \frac{m}{k} = k + 1$.

Corollary 4.4. [4] *If G is a graph without isolated vertices of order n and size m, then*

$$\gamma^r_t(G) \geq \frac{3}{2}n - m.$$

Theorem 4.5. *Let G be a graph with $\delta(G) \geq a + k$, for some finite number a. If $\gamma_{\times k,t}(G) \leq a$, then $\gamma^r_{\times k,t}(G) \leq a$.*

Proof. Let us consider a kTDS S such that $\mid S \mid \leq a$. For every $v \in V(G) - S$, we have

$$\deg(v) \geq \delta(G) \geq a + k \geq \mid S \mid + k.$$

Then $\mid N(v) \cap (V(G) - S) \mid \geq k$. This means S is a kTRDS of G and so $\gamma^r_{\times k,t}(G) \leq a$. □

5. k-tuple total restrained domatic in graphs

In this section we mainly present basic properties of $d^r_{\times k,t}(G)$ and some other bounds on the k-tuple total restrained domatic number of a graph. First we give the next proposition which its proof is clear and we have left it to the reader.

Proposition 5.1. *If $n > k \geq 1$ be two integers, then $d^r_{\times k,t}(K_n) = \lfloor \frac{n}{k+1} \rfloor$.*

Theorem 5.2. *If G is a graph of order n with $\delta(G) \geq k$, then*

$$\gamma^r_{\times k,t}(G) \cdot d^r_{\times k,t}(G) \leq n.$$

Moreover, if $\gamma^r_{\times k,t}(G) \cdot d^r_{\times k,t}(G) = n$, then for each kTRDP $\{V_1, V_2, ..., V_d\}$ of $V(G)$ with $d = d^r_{\times k,t}(G)$, each set V_i is a $\gamma^r_{\times k,t}(G)$-set.

Proof. Let $\{V_1, V_2, ..., V_d\}$ be a kTRDP of $V(G)$ such that $d = d^r_{\times k,t}(G)$. Then

$$\begin{aligned} d \cdot \gamma^r_{\times k,t}(G) &= \sum_{i=1}^{d} \gamma^r_{\times k,t}(G) \\ &\leq \sum_{i=1}^{d} \mid V_i \mid \\ &= n. \end{aligned}$$

If $\gamma^r_{\times k,t}(G) \cdot d^r_{\times k,t}(G) = n$, then the inequality occurring in the proof becomes equality. Hence for the kTRDP $\{V_1, V_2, ..., V_d\}$ of G and for each i, $\mid V_i \mid = \gamma^r_{\times k,t}(G)$. Thus each set V_i is a $\gamma^r_{\times k,t}(G)$-set. □

An immediate consequence of Theorem 5.2 and Corollary 4.2 now follows.

Corollary 5.3. *If G is a graph of order n with $\delta(G) \geq k$, then*

$$d^r_{\times k,t}(G) \leq \frac{n}{k+1},$$

with equality if and only if $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$, for some graph F with $\delta(F) \geq k$.

For bipartite graphs, Proposition 3.1 improves the bound given in Corollary 5.3.

Corollary 5.4. *If G is a bipartite graph of order n with the vertex partition $V(G) = X \cup Y$ and $\delta(G) \geq k$, then*

$$d^r_{\times k,t}(G) \leq \frac{n}{2k},$$

with equality if and only if there exist two k-subsets $S \subseteq X$ and $T \subseteq Y$ such that

1. *for each vertex $x \in X$, $T \subseteq N(x)$, and*
2. *for each vertex $y \in Y$, $S \subseteq N(y)$, and*
3. $\delta(G[(X - S) \cup (Y - T)]) \geq k$.

Now, we show that the k-tuple total restrained domatic number of a graph is equal to the k-tuple total domatic number of it.

Theorem 5.5. *Let G be a graph with $\delta(G) \geq k \geq 1$. Then*

$$d^r_{\times k,t}(G) = d_{\times k,t}(G).$$

Proof. Each k-tuple total restrained dominating set in G is a k-tuple total dominating set in G, therefore each k-tuple total restrained domatic partition of $V(G)$ is a k-tuple total domatic partition of $V(G)$ and

$d^r_{\times k,t}(G) \leq d_{\times k,t}(G)$. Now let $d = d_{\times k,t}(G) \geq 2$ and let $\mathbb{D} = \{D_1, ..., D_d\}$ be a k-tuple total domatic partition of $V(G)$. Choose D_1 as an arbitrary class of $\mathbb{D}$. Let $x \in V(G)$. As D_1 is a k-tuple total dominating set of G, there exists k-subset $S^1_x \subseteq N(x) \cap D_1$. Now suppose $x \in V(G) - D_1$, then $x \in D_i$ for some $2 \leq i \leq d$. The set D_i is also a k-tuple total dominating set of G, therefore there exists k-subset $S^i_x \subseteq N(x) \cap D_i$ and evidently $S^i_x \subseteq V(G) - D_1$, because $D_1 \cap D_i = \emptyset$. So, we have proved that D_1 is a k-tuple total restrained dominating set in G. The set D_1 was chosen arbitrarily, hence $\mathbb{D}$ is a k-tuple total restrained domatic partition of G and $d_{\times k,t}(G) \leq d^r_{\times k,t}(G)$, which together with the former inequality gives the required result. □

Corollary 5.6. [12] *Let G be a graph without isolated vertices. Then*

$$d^r_t(G) = d_t(G).$$

Now, we give a sufficient condition for $\gamma^r_{\times k,t}(G) = \gamma_{\times k,t}(G)$.

Theorem 5.7. *Let G be a graph with minimum degree at least k. If $d^*_{\times k,t}(G) \geq 2$, then $\gamma^r_{\times k,t}(G) = \gamma_{\times k,t}(G)$.*

Proof. Every k-tuple total restrained dominating set in G is also k-tuple total dominating set in G, therefore $\gamma_{\times k,t}(G) \leq \gamma^r_{\times k,t}(G)$. The condition $d^*_{\times k,t}(G) \geq 2$ implies that there exist two disjoint k-tuple total dominating sets S and S' in G such that $| S |= \gamma_{\times k,t}(G)$. Let $x \in V(G) - S$. Then x is adjacent to at least k vertices of S', since S' is a k-tuple total dominating set of G. This implies that x is adjacent to at least k vertices of $V(G) - S$. Therefore, S is a k-tuple total restrained dominating set of G and so $\gamma^r_{\times k,t}(G) \leq| S |= \gamma_{\times k,t}(G)$. The previous two inequalities give $\gamma^r_{\times k,t}(G) = \gamma_{\times k,t}(G)$. □

Corollary 5.8. *Let G be a graph without isolated vertices. If $d^*_t(G) \geq 2$, then $\gamma^r_t(G) = \gamma_t(G)$.*

The converse of Theorem 5.7 does not hold. For example, if $G = K_{k+1}$, then $\gamma^r_{\times k,t}(G) = \gamma_{\times k,t}(G) = k + 1$ but $d^*_{\times k,t}(G) = 1$. Also as another example let $G = K_{n,m}$ be the complete bipartite graph with the conditions $k \leq n \leq m < 2k$ and $(n, m) \neq (k, k)$. Then $\gamma_{\times k,t}(G) = 2k < \gamma^r_{\times k,t}(G) = n + m$, but $d^*_{\times k,t}(G) = 1$.

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