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OD-CHARACTERIZATION OF ALMOST SIMPLE GROUPS RELATED TO $L_3(25)$

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Abstract. Let G be a finite group and $\pi(G)$ be the set of all the prime divisors of $|G|$. The prime graph of G is a simple graph $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if G has an element of order pq, and in this case we will write $p \sim q$. The degree of p is the number of vertices adjacent to p and is denoted by $deg(p)$. If $|G| = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, p_i's different primes, $p_1 < p_2 < ... < p_k$, then $D(G) = (deg(p_1), deg(p_2), ..., deg(p_k))$ is called the degree pattern of G. A finite group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups S with $|G| = |S|$ and $D(G) = D(S)$. In this paper, we characterize groups with the same order and degree pattern as an almost simple groups related to $L_3(25)$.
Keywords: OD-characterizable group, degree pattern, prime graph.
MSC(2010): Received: 30 April 2009, Accepted: 21 June 2010.

1. Introduction

Throughout this article, all groups under consideration are finite. For any group G, we denote by $\pi(G)$ the set of all prime divisors of $|G|$ and the set of orders of the elements of G is denoted by $\pi_e(G)$. The prime graph $\Gamma(G)$ of a group G is a simple graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (written $p \sim q$) if and only if G contains an element of order pq. For $p \in \pi(G)$, we put $deg(p) := |\{q \in \pi(G) | p \sim q\}|$, which is called the degree of p. If $|G| = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$,

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p_i's different primes, we define $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, where $p_1 < p_2 < ... < p_k$, which is called the degree pattern of G.

Definition 1.1. *The group G is called k-fold* OD-*characterizable if there exist exactly k non-isomorphic groups H satisfying conditions $|G| = |H|$ and $D(G) = D(H)$. In particular, a 1-fold* OD-*characterizable group is simply called* OD-*characterizable.*

The interest in characterizing finite groups by degree pattern started in [4] by M.R. Darafsheh,et.al., in which the authors proved that if G is a finite group such that $|G| = |M|$ and $D(G) = D(M)$, where M is one of these simple groups: (1) sporadic simple groups, (2) alternating A_p with p and $p-2$ primes, (3) some simple groups of Lie type, then $G \cong M$.

A group G is an almost simple group, if $S \trianglelefteq G \lesssim \mathrm{Aut}(S)$, for some non-abelian group S. In many articles it has been shown that many finite almost simple groups are OD-characterizable or k-fold OD-characterizable for certain $k \geq 2$.

Let A and B be two groups then a split extension is denoted by $A : B$. If L is a finite simple group and $\mathrm{Aut}(L) \cong L : A$, then if B is a cyclic subgroup of A of order n, we will write $L : n$ for the split extension $L : B$. Moreover if there are more than one subgroup of order n in A, then we will denote them by $L : n_1$, $L : n_2$, etc.

In [3], for $p = 23, 31, 43$ and 47, OD-characterizability of A_{p+3} has been proved. Also the authors have shown that the automorphism groups of these groups are 3-fold OD-characterizable.

In [7], for $L := L_2(49)$, it is shown that finite almost simple groups L, $L : 2_1$, $L : 2_2$ and $L : 2_3$ are OD-characterizable; $L : 2^2$ is 9-fold OD-characterizable(2^2 is the Klein's four group) and in [9], for $L := U_6(2)$, it is shown that finite almost simple groups L and $L : 2$ are OD-characterizable, $L : 3$ is 3-fold OD-characterizable, and $L : \mathbb{S}_3$ is 5-fold OD-characterizable. Also in [8], it is shown that all simple K_4-groups except A_{10} are OD-characterizable (we recall that a finite group possessing exactly n prime divisors is called K_n-group).

We denote the socle of G by $\mathrm{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of G. For $p \in \pi(G)$, we denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of G and the set of all Sylow p-subgroups of G respectively, also for a prime t, $|G|_t$ denotes the t-part of $|G|$, i.e., $|G|_t = t^r$ such that $t^r \| |G|$ and $t^{r+1} \nmid |G|$. All further unexplained notations are standard and can be found in [5].

In this article our main aim is to show the characterizability of the almost simple groups related to $L := L_3(25)$ by the degree pattern in the prime graph and the order of the group. In fact, we will prove the following Theorem.

Main Theorem Let M be an almost simple group related to $L = L_3(25)$. If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, then the following assertions hold:

(a) If $M = L$, then $G \cong L$.

(b) If $M = L : 2_1$, then $G \cong L : 2_1$.

(c) If $M = L : 2_2$, then $G \cong L : 2_2$.

(d) If $M = L : 2_3$, then $G \cong L : 2_3$.

(e) If $M = L : 3$, then $G \cong L : 3$, $\mathbb{Z}_3 \times L$ or $\mathbb{Z}_3.L$.

(f) If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2.(L : 2_1)$, $\mathbb{Z}_2.(L : 2_2)$, $\mathbb{Z}_2.(L : 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

(g) If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, $\mathbb{Z}_3 \times (L : 2_1)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ or $(\mathbb{Z}_3.L).\mathbb{Z}_2$.

(h) If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2$, $\mathbb{Z}_3.(L : 2_2)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ or $(\mathbb{Z}_3.L).\mathbb{Z}_2$.

(i) If $M = L : 6$, then $G \cong L : 6$, $\mathbb{Z}_3 \times (L : 2_3)$, $\mathbb{Z}_3.(L : 2_3)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ or $(\mathbb{Z}_3.L).\mathbb{Z}_2$.

(j) If $M = L : D_{12}$, then $G \cong L : D_{12}$, $\mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2)$, $\mathbb{Z}_2 \times (L : 6)$, $\mathbb{Z}_3.(L : 2^2)$, $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$, $(\mathbb{Z}_3.(L : 2_1)).\mathbb{Z}_2$, $(\mathbb{Z}_3.(L : 2_2)).\mathbb{Z}_2$, $(\mathbb{Z}_3.(L : 2_3)).\mathbb{Z}_2$, $\mathbb{Z}_4 \times (L : 3)$, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$, $(\mathbb{Z}_4 \times L).\mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$, $\mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$, $\mathbb{Z}_6 \times (L : 2_3)$, $(\mathbb{Z}_6 \times L).\mathbb{Z}_2$, $D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$, $D_6 \times (L : 2_3)$, $\mathbb{Z}_{12} \times L$, $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, $D_{12} \times L$, $A_4 \times L$, $T \times L$.

2. **Preliminary lemmas**

It is well-known that $\mathrm{Aut}(L_3(25)) \cong L_3(25) : D_{12}$ where D_{12} denotes the dihedral group of order 12. We remark that D_{12} has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to $D_6 \cong \mathbb{S}_3$ and one subgroup of order 4 isomorphic to the Klein's four group denoted by 2^2. The field and the duality automorphisms of $L_3(25)$ are denoted by 2_1 and 2_2 respectively, and we set $2_3 = 2_1.2_2$(duality$*$field which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to $L_3(25)$.

Lemma 2.1. *If G is an almost simple group related to $L = L_3(25)$, then G is isomorphic to one of the following groups: $L, L:2_1, L:2_2, L:2_3, L:3, L:2^2, L:(D_6)_1, L:(D_6)_2, L:6, L:D_{12}$.*

A completely reducible group will be called a CR-group. A CR-group has trivial center if and only if it is a direct product of non-abelian simple groups and in this case, it is called a centerless CR-group. The following Lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.2. [5], Theorem 3.3.20 *Let R be a finite centerless CR-group and write $R = R_1 \times R_2 \times ... \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i, and H_i and H_j are not isomorphic if $i \neq j$. Then $\mathrm{Aut}(R) = \mathrm{Aut}(R_1) \times \mathrm{Aut}(R_2) \times ... \times \mathrm{Aut}(R_k)$ and $\mathrm{Aut}(R_i) \cong \mathrm{Aut}(H_i) \wr \mathbb{S}_{n_i}$, where in this wreath product $\mathrm{Aut}(H_i)$ appears in its right regular representation and the symmetric group $\mathbb{S}_{n_i}$ in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms $\mathrm{Out}(R) \cong \mathrm{Out}(R_1) \times \mathrm{Out}(R_2) \times ... \times \mathrm{Out}(R_k)$ and $\mathrm{Out}(R_i) \cong \mathrm{Out}(H_i) \wr \mathbb{S}_{n_i}$.*

Lemma 2.3. [2], Theorem 10.3.1 *Let G be a Frobenius group with kernel K and complement H. Then:*
(a)*K is a nilpotent group.*
(b)*$|K| \equiv 1(\mathrm{mod}|H|)$.*

Let $p \geq 5$ be a prime. We denote by $\mathfrak{S}_p$ the set of all simple groups with prime divisors at most p. Clearly, if $q \leq p$, then $\mathfrak{S}_q \subseteq \mathfrak{S}_p$. We list all simple groups S in class $\mathfrak{S}_{31}$ with their order and the order of their outer automorphisms $o = |\mathrm{Out}(S)|$ in TABLE 1, taken from [6].**TABLE 1: Simple groups in $\mathfrak{S}_p$, $p \leq 31$.**

S	$\|S\|$	o	S	$\|S\|$	o
A_5	$2^2 \cdot 3 \cdot 5$	2	$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
A_6	$2^3 \cdot 3^2 \cdot 5$	4	$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	4
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_3(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	A_{13}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2

(Continued)

S	$\lvert S\rvert$	o	S	$\lvert S\rvert$	o
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	4
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6	$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	$U_4(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2
A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2	$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$	4
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6	$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$	2
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2	$L_3(16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4	$S_6(4)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$O_8^+(4)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	12
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4	A_{17}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	A_{18}	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	$L_2(19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	2
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6	$L_3(7)$	$2^5 \cdot 3^2 \cdot 7^3 \cdot 19$	6
$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2	$U_3(2^3)$	$2^9 \cdot 3^4 \cdot 7 \cdot 19$	18
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3	$U_3(19)$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7^3 \cdot 19$	2
$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3	$L_4(7)$	$2^9 \cdot 3^4 \cdot 5^2 \cdot 7^6 \cdot 19$	4
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	2	A_{29}	$2^{26} \cdot 3^{13} \cdot 5^6 \cdot 7^4$ $.11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29$	2
J_1	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	1	A_{30}	$2^{27} \cdot 3^{14} \cdot 5^7 \cdot 7^4$ $.11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29$	2
$L_3(11)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$	2	$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	2
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	2	$L_3(5)$	$2^5 \cdot 3 \cdot 5^3 \cdot 31$	2
$U_4(2^3)$	$2^{18} \cdot 3^7 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	6	$L_2(2^5)$	$2^5 \cdot 3 \cdot 11 \cdot 31$	5
A_{19}	$2^{16} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	2	$L_2(5^3)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$	6
A_{20}	$2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	2	$G_2(5)$	$2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$	1
A_{21}	$2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	2	$L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	2

(Continued)

S	$\lvert S\rvert$	o	S	$\lvert S\rvert$	o
A_{22}	$2^{20}\cdot 3^9\cdot 5^4\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19$	2	$L_6(2)$	$2^{15}\cdot 3^4\cdot 5\cdot 7^2\cdot 31$	2
${}^2E_6(2)$	$2^{36}\cdot 3^9\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19$	6	$L_4(5)$	$2^7\cdot 3^2\cdot 5^6\cdot 13\cdot 31$	8
$L_2(23)$	$2^3\cdot 3\cdot 11\cdot 23$	2	$L_3(5^2)$	$2^7\cdot 3^2\cdot 5^6\cdot 7\cdot 13\cdot 31$	12
$U_3(23)$	$2^7\cdot 3^2\cdot 11\cdot 13^2\cdot 23^3$	6	$O_7(5)$	$2^9\cdot 3^4\cdot 5^9\cdot 7\cdot 13\cdot 31$	2
M_{23}	$2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$	1	$S_6(5)$	$2^9\cdot 3^4\cdot 5^9\cdot 7\cdot 13\cdot 31$	2
M_{24}	$2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$	1	$O_8^+(5)$	$2^{12}\cdot 3^5\cdot 5^{12}\cdot 7\cdot 13^2\cdot 31$	24
Co_3	$2^{10}\cdot 3^7\cdot 5^3\cdot 7\cdot 11\cdot 23$	1	$O_{10}^+(2)$	$2^{20}\cdot 3^5\cdot 5^2\cdot 7\cdot 17\cdot 31$	2
Co_2	$2^{18}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23$	1	$U_3(31)$	$2^{11}\cdot 3\cdot 5\cdot 7^2\cdot 19\cdot 31^3$	2
Co_1	$2^{21}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 23$	1	$L_5(2^2)$	$2^{20}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 17\cdot 31$	4
Fi_{23}	$2^{18}\cdot 3^{13}\cdot 5^2\cdot 7$ $.11\cdot 13\cdot 17\cdot 23$	1	$S_{10}(2)$	$2^{25}\cdot 3^6\cdot 5^2\cdot 7\cdot 11\cdot 17\cdot 31$	1
A_{23}	$2^{20}\cdot 3^9\cdot 5^4\cdot 7^3$ $.11^2\cdot 13\cdot 17\cdot 19\cdot 23$	2	$O_{12}^+(2)$	$2^{30}\cdot 3^8\cdot 5^2\cdot 7^2\cdot 11\cdot 17\cdot 31$	2
A_{24}	$2^{23}\cdot 3^{10}\cdot 5^4\cdot 7^3$ $.11^2\cdot 13\cdot 17\cdot 19\cdot 23$	2	$O'N$	$2^9\cdot 3^4\cdot 5\cdot 7^3\cdot 11\cdot 19\cdot 31$	2
A_{25}	$2^{23}\cdot 3^{10}\cdot 5^6\cdot 7^3$ $.11^2\cdot 13\cdot 17\cdot 19\cdot 23$	2	Th	$2^{15}\cdot 3^{10}\cdot 5^3\cdot 7^2\cdot 13\cdot 19\cdot 31$	1
A_{26}	$2^{24}\cdot 3^{10}\cdot 5^6\cdot 7^3$ $.11^2\cdot 13^2\cdot 17\cdot 19\cdot 23$	2	$O_{12}^-(2)$	$2^{30}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 13\cdot 17\cdot 31$	2
A_{27}	$2^{24}\cdot 3^{13}\cdot 5^6\cdot 7^3$ $.11^2\cdot 13^2\cdot 17\cdot 19\cdot 23$	2	$L_6(2^2)$	$2^{30}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 31$	12
A_{28}	$2^{26}\cdot 3^{13}\cdot 5^6\cdot 7^4$ $.11^2\cdot 13^2\cdot 17\cdot 19\cdot 23$	2	$S_{12}(2)$	$2^{36}\cdot 3^8\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 31$	1
$L_2(29)$	$2^2\cdot 3\cdot 5\cdot 7\cdot 29$	2	A_{31}	$2^{24}\cdot 3^{13}\cdot 5^6\cdot 7^4\cdot 11^2$ $.13^2\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31$	2
$L_2(17^2)$	$2^5\cdot 3^2\cdot 5\cdot 17^2\cdot 29$	4	A_{32}	$2^{29}\cdot 3^{13}\cdot 5^6\cdot 7^4\cdot 11^2$ $.13^2\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31$	2
$S_4(17)$	$2^{10}\cdot 3^4\cdot 5\cdot 17^4\cdot 29$	2	A_{33}	$2^{29}\cdot 3^{14}\cdot 5^6\cdot 7^4\cdot 11^3$ $.13^2\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31$	2
Ru	$2^{14}\cdot 3^3\cdot 5^3\cdot 7\cdot 13\cdot 29$	1	A_{34}	$2^{30}\cdot 3^{14}\cdot 5^6\cdot 7^4\cdot 11^3$ $.13^2\cdot 17^2\cdot 19\cdot 23\cdot 29\cdot 31$	2
$U_4(17)$	$2^{11}\cdot 3^7\cdot 5\cdot 7\cdot 13\cdot 17^6\cdot 29$	4	A_{35}	$2^{30}\cdot 3^{14}\cdot 5^7\cdot 7^5\cdot 11^3$ $.13^2\cdot 17^2\cdot 19\cdot 23\cdot 29\cdot 31$	2
Fi_{24}'	$2^{21}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11$ $.13\cdot 17\cdot 23\cdot 29$	2	A_{36}	$2^{32}\cdot 3^{16}\cdot 5^7\cdot 7^5\cdot 11^3$ $.13^2\cdot 17^2\cdot 19\cdot 23\cdot 29\cdot 31$	2

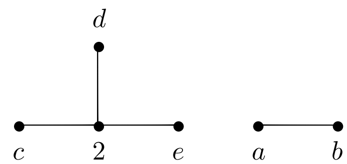
3. **Proof of the main theorem**

We assume M is an almost simple group related to $L = L_3(25)$ and G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$. We break the proof into a number of separate propositions. In each proposition, under assumptions we diagram all possibilities for $\Gamma(G)$ by use of the variables for some vertices. Also since in some propositions we need to know the structure of $\Gamma(M)$ to determine the possibilities for G, we diagram the prime graph of all extensions of L in pages 23 to 25. Note that the set of order elements in each of the following propositions and also the Schur multiplier of all extensions are calculated using Magma.

Proposition 3.1. *If $M = L$, then $G \cong L$.*

Proof. By TABLE 1, $|L| = 2^7.3^2.5^6.7.13.31$. $\pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 16, 20, 24, 26, 31, 40, 52, 104, 208, 217\}$, so $D(L) = (3, 1, 1, 1, 1, 1)$. Now under assumptions $|G| = |L|$ and $D(G) = D(L)$, we conclude that $\Gamma(G)$ has following forms:

Figure 3.1:



where $\{a, b, c, d, e\} = \pi(G) - \{2\}$.

To simplify, in every proposition, we break the proof into several steps.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.

We consider these two parts depending on $\{a, b\}$.

Part A. $\{a, b\} \neq \{13, 31\}$. First, we show that K is a $31^{'}$-group. Assume the contrary and let $31 \in \pi(K)$. We claim 13 does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{13, 31\}$-subgroup of K. It is seen that T is a nilpotent subgroup of order 13.31. Thus, $13.31 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus, $\{31\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$. Let $K_{31} \in \mathrm{Syl}_{31}(K)$. By Frattini argument, $G = KN_G(K_{31})$. Therefore, $N_G(K_{31})$ contains an element x of order 13. Since G has no element of order 13.31, $\langle x\rangle$ should act fixed point freely on K_{31}, that is implying $\langle x\rangle K_{31}$ is a Frobenius group. By Lemma 2.3(b), $|\langle x\rangle| | (|K_{31}| - 1)$. It follows that $13 | 31 - 1$, which is a contradiction. Now we show that K is a $p^{'}$-group where $p \in \{a, b, c, d, e\} - \{3, 5, 31\} = \{7, 13\}$. First suppose that $p \in \{c, d, e\} - \{3, 5, 31\}$. Assume the contrary and let x be an element of K of order p. According to $\Gamma(G)$, $C_G(x)$ is a $\{2, p\}$-group. Since $\frac{N_G(\langle x\rangle)}{C_G(x)} \lesssim \mathrm{Aut}(\langle x\rangle) \cong \mathbb{Z}_{p-1}$, $\pi(N_G(\langle x\rangle)) \subseteq \{2, 3, p\}$. By Frattini argument, $G = KN_G(\langle x\rangle)$, so 31 must divide the order of K, which is a contradiction. It is enough to show that K is a $p^{'}$-group where $p \in \{a, b\} - \{3, 5, 31\}$. Assume the contrary, so we may suppose that x is an element of K of order p. Then by $\Gamma(G)$, $C_G(x)$ is a $\{a, b\}$-group. Using similar argument as before, we see that $\pi(N_G(\langle x\rangle)) \subseteq \{2, 3, a, b\}$, therefore $\{c, d, e\} \subseteq \pi(K)$, that is a contradiction.

So K is a $\{2,3,5\}$-group.

Part B. $\{a,b\}=\{13,31\}$. First, we show that K is a $31^{'}$-group. Assume the contrary that $31\in\pi(K)$. By the same argument in Part A and considering 7 instead of 13, we get a contradiction. Now we show that K is a $p^{'}$-group for $p=7$ and 13. Let $p\in\pi(K)$ and x be an element of K of order p. First, put $p=7$ then by $\Gamma(G)$, $C_G(x)$ is a $\{2,7\}$-group. Since $\frac{N_G(\langle x\rangle)}{C_G(x)}\lesssim \mathrm{Aut}(\langle x\rangle)\cong\mathbb{Z}_6$, $N_G(\langle x\rangle)$ is a $\{2,3,7\}$-group. Now by Frattini argument, $G=KN_G(\langle x\rangle)$, so 31 must divide the order of K and that is a contradiction. Next we put $p=13$, then by $\Gamma(G)$ we see that $C_G(x)$ is a $\{13,31\}$-group. By the same argument as before, $N_G(\langle x\rangle)$ is a $\{2,3,13,31\}$-group, so 7 must divide the order of K which is a contradiction. Therefore K is a $\{2,3,5\}$-group.

In addition since $K\neq G$, G is non-solvable, and this completes the proof of Step 1.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S\trianglelefteq\frac{G}{K}\lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group.

Let $\overline{G}=\frac{G}{K}$. $S:=\mathrm{Soc}(\overline{G})=P_1\times P_2\times ...\times P_m$, where P_i's are finite non-abelian simple groups (by Step 1, we conclude that every minimal normal subgroup of $\frac{G}{K}$ is non-abelian). Also $C_{\bar{G}}(S)=1$. Because if $1\neq\frac{T}{K}=:C_{\bar{G}}(S)$, then by Zorn's Lemma, there exists a normal minimal subgroup M of $\bar{G}$ such that $M\leq\frac{T}{K}=C_{\bar{G}}(S)\leq C_{\bar{G}}(M)$. So $M\subseteq C_{\bar{G}}(M)\cap M=Z(M)$ and it implies that M is abelian, a contradiction. Now since $\frac{N_{\overline{G}}(S)}{C_{\overline{G}}(S)}\lesssim\mathrm{Aut}(S)$, we have $S\trianglelefteq\frac{G}{K}\lesssim\mathrm{Aut}(S)$. If we show that $m=1$, the proof of Step 2 is complete. Suppose that $m\geq 2$. We claim 31 does not divide $|S|$. Assume the contrary and let $31\mid|S|$, on the other hand, $\{2,3\}\subset\pi(P_i)$ for every i (by TABLE 1), hence $2\sim 31$ and $3\sim 31$, which is a contradiction. Now, by Step 1, we observe that $31\in\pi(\overline{G})\subseteq\pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S)=\mathrm{Aut}(S_1)\times\mathrm{Aut}(S_2)\times ...\times\mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S=S_1\times S_2\times ...\times S_r$. Therefore, for some j, 31 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i\in\mathfrak{S}_{31}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 31 (see TABLE 1), so 31 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)|=|\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t\geq 31$ and so 2^{62} must divide the order of G, which is a contradiction. Therefore, $m=1$ and $S=P_1$, so the proof of Step 2 is complete.

Step 3. G is isomorphic to $L_3(25)$.

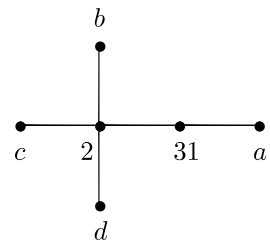
By TABLE 1 and Step 1, it is evident that $|S|=2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.31$, where $2\leq\alpha\leq 7$, $1\leq\beta\leq 2$ and $0\leq\gamma\leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S\cong L_3(25)$ and by Step 2, $L\trianglelefteq\frac{G}{K}\lesssim\mathrm{Aut}(L)$. As $|G|=|L|$, we deduce $K=1$, so $G\cong L$, and the proof is complete. □

Proposition 3.2. *If $M=L:2_1$, then $G\cong L:2_1$.*

Proof. As $|L:2_1| = 2^8.3^2.5^6.7.13.31$ and $\pi_e(L:2_1) = \{1,2,3,4,5,6,7,8,10,12,13,16,20,24,26,31,40,48,52,62,104,208,217\}$ then $D(L:2_1) = (4,1,1,1,1,2)$. Since $|G| = |L:2_1|$ and $D(G) = D(L:2_1)$, the prime graph of G has several possibilities are shown in the following figure:

Figure 3.2:

where $\{a,b,c,d\} = \{3,5,7,13\}$.



Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$-group. In particular, G is non-solvable.

We consider separate parts depending on a:

Part A. Let $a=3$. We choose one of the primes in $\{b,c,d\}$ that is unequal to 5 as r. First, we show that K is a $31^{'}$-group. Assume the contrary and let $31 \in \pi(K)$. Then r doesn't divide the order of K. Otherwise, there exists a Hall $\{r,31\}$-subgroup T of K and it is seen that T is a nilpotent subgroup of order $r.31$. Thus, $r.31 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus, $\{31\} \subseteq \pi(K) \subseteq \pi(G) - \{r\}$. Let $K_{31} \in \mathrm{Syl}_{31}(K)$. By Frattini argument, $G = KN_G(K_{31})$. Therefore, $N_G(K_{31})$ contains an element x of order r. Since G has no element of order $r.31$, $\langle x\rangle$ should act fixed point freely on K_{31}, which implies that $\langle x\rangle K_{31}$ is a Frobenius group. By Lemma 2.3(b), $|\langle x\rangle| | (|K_{31}|-1)$. It follows that $r|31-1$, which is a contradiction, because we know that $r \neq 3,5$. Now we show that K is a $p^{'}$-group for $p=7$ and 13. Assume the contrary: $p||K|$ and x is an element of K of order p. According to $\Gamma(G)$, $C_G(x)$ is a $\{2,p\}$-group. Since $\frac{N_G(\langle x\rangle)}{C_G(x)} \lesssim \mathrm{Aut}(\langle x\rangle) \cong \mathbb{Z}_{p-1}$, $N_G(\langle x\rangle)$ is a $\{2,3,p\}$-group. As by Frattini argument, $G = KN_G(\langle x\rangle)$, then 31 must divide the order of K, which is a contradiction. Therefore K is a $\{2,3,5\}$-group.

Part B. Let $a \neq 3$. In this part we choose one of the primes in $\{b,c,d\}$ that is unequal to 3 and 5 as r. By a similar way in Part A, it is seen that K is a $31^{'}$-group. We prove that K is a $p^{'}$-group where $p \in \{b,c,d\} - \{3,5\}$. Assume the contrary, let $p||K|$ and x be an element of K of order p. By the exact way in Part A for $p=7$ and 13, we get a contradiction. It is enough to show that K is a $a^{'}$-group if $a \neq 5$. Let $a \in \pi(K)$, and x be an element of K of order a. By $\Gamma(G)$, $C_G(x)$ is a $\{a,31\}$-group, therefore $N_G(\langle x\rangle)$ is a $\{2,3,a,31\}$-group, and since by Frattini argument, $G = KN_G(\langle x\rangle)$, r must divide the order of K, which is a contradiction.

In addition since $K \neq G$, G is non-solvable and this completes the proof of Step 1.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. If we set $S := \mathrm{Soc}(\overline{G})$, $S = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups. In Step 2 of Proposition 3.1, we showed that $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, then it is enough to prove that $m = 1$ to complete the proof of Step 2. Suppose that $m \geq 2$. We claim 13 does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2, 3\} \subset \pi(P_i)$ for every i (by TABLE 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by Step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 13 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{31}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 13 (see TABLE 1). Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq 13$ and so 2^{26} must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.31$, where $2 \leq \alpha \leq 8$, $1 \leq \beta \leq 2$ and $0 \leq \gamma \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong L_3(25)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2_1| = 2|L|$, we deduce $|K| = 1$ or 2.

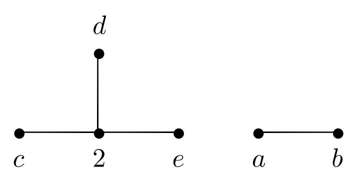
If $|K| = 1$, then $G \cong L : 2_1$ because $|G| = 2|L|$. Obviously, G can not be isomorphic to $L : 2_2$ or $L : 2_3$, because $deg(31) = 1$ in $\Gamma(L : 2_2)$ and $\Gamma(L : 2_3)$, (see page 24).

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction. $\square$

Proposition 3.3. *If $M = L : 2_2$, then $G \cong L : 2_2$.*

Proof. By TABLE 1, $|L : 2_2| = 2^8.3^2.5^6.7.13.31$. $\pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 16, 20, 24, 26, 31, 40, 48, 52, 104, 208, 217\}$, so $D(L : 2_2) = (3, 1, 1, 1, 1, 1)$. Since $|G| = |L : 2_2|$ and $D(G) = D(L : 2_2)$, we conclude that $\Gamma(G)$ has the possibilities like Proposition 3.1.

Figure 3.3:



where $\{a, b, c, d, e\} = \pi(G) - \{2\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.

Similarly to those in Proposition 3.1, We can prove these assertions.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group.

The proof is similar to Step 2, in Proposition 3.1.

Now by TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.31$, where $2 \leq \alpha \leq 8$, $1 \leq \beta \leq 2$ and $0 \leq \gamma \leq 6$. By using collected results contained in TABLE 1, we deduce that $S \cong L_3(25)$ and by Step 2 we conclude that $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2_2| = 2|L|$, hence $|K| = 1$ or 2.

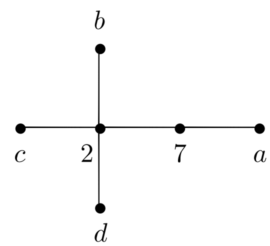
If $|K| = 1$, $G \cong L : 2_2$ because $|G| = 2|L|$. As $deg(2) = 4$ in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_3)$ (see pages 23 and 24), we have only one possibility for G.

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction. $\square$

Proposition 3.4. *If $M = L : 2_3$, then $G \cong L : 2_3$.*

Proof. By TABLE 1, $|L : 2_3| = 2^8.3^2.5^6.7.13.31$. $\pi_e(L : 2_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 16, 20, 24, 26, 31, 40, 52, 104, 208, 217\}$, so $D(L : 2_3) = (4, 1, 1, 2, 1, 1)$. Since $|G| = |L : 2_3|$ and $D(G) = D(L : 2_3)$, we conclude that there exist some possibilities for $\Gamma(G)$ are as follows:

Figure 3.4:



where $\{a, b, c, d\} = \{3, 5, 13, 31\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.

We consider different parts depending on a like Proposition 3.2:

Part A. Let $a = 3$. We choose one of the primes in $\{b, c, d\}$ that is unequal to 5 as r. We show that K is a $7'$-group. Assume the contrary and let $7 \in \pi(K)$. Then r doesn't divide the order of K. Otherwise, there exists a Hall $\{r, 7\}$-subgroup T of K and it is seen that T is a nilpotent subgroup of order $r.7$. Thus, $r.7 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus, $\{7\} \subseteq \pi(K) \subseteq \pi(G) - \{r\}$. Let $K_7 \in \mathrm{Syl}_7(K)$. By Frattini argument, $G = KN_G(K_7)$. Therefore, $N_G(K_7)$ contains an element x of order r. Since G has no element of order $r.7$, $\langle x \rangle$ should act fixed point freely on K_7, implying $\langle x \rangle K_7$ is a Frobenius group. By Lemma 2.3(b), $|\langle x \rangle| | (|K_7| - 1)$. It follows that $r | 7 - 1$, which is a contradiction, because we know that $r \neq 2, 3$. Now we show that K is a p'-group for $p = 13$ and 31. Assume the contrary: $p||K|$ and x is an element of K of order p. According to $\Gamma(G)$, $C_G(x)$ is a $\{2, p\}$-group. Since $\frac{N_G(\langle x \rangle)}{C_G(x)} \lesssim \mathrm{Aut}(\langle x \rangle) \cong \mathbb{Z}_{p-1}$, $N_G(\langle x \rangle)$ is a $\{2, 3, 5, p\}$-group. By Frattini argument, $G = KN_G(\langle x \rangle)$ then 7 must divide the order of K, which is a contradiction. Therefore K is a $\{2,3,5\}$-group.

Part B. Let $a \neq 3$. We choose one of the primes in $\{b,c,d\}$ that is unequal to 5 and 3 as r. By a similar way in Part A, it is seen that K is a $7'$-group. Next we prove that K is a p'-group where $p \in \{b,c,d\} - \{3,5\}$. Assume the contrary, let $p||K|$ and x be an element of K of order p. By the structure of $\Gamma(G)$, we conclude that $C_G(x)$ is a $\{2,p\}$-group, and by the same argument as in Part A, we see that $N_G(\langle x\rangle)$ is a $\{2,3,5,p\}$-group. As $G = KN_G(\langle x\rangle)$, then 7 must divide the order of K, which is a contradiction. It is enough to show that K is a a'-group if $a \neq 5$. Let $a \in \pi(K)$, and x be an element of K of order a. By $\Gamma(G)$, $C_G(x)$ is a $\{a,7\}$-group. We use the same technique as before and we can easily see that $N_G(\langle x\rangle)$ is a $\{2,3,5,7,a\}$-group. Since by Frattini argument, $G = KN_G(\langle x\rangle)$, r must divide the order of K, which is a contradiction. Therefore K is a $\{2,3,5\}$-group.

In addition since $K \neq G$, G is non-solvable and this completes the proof of Step 1.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. If we set $S := \mathrm{Soc}(\overline{G})$, $S = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups. In Step 2 of Proposition 3.1, we showed that $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, then it is enough to prove that $m = 1$ to complete the proof of Step 2. Suppose that $m \geq 2$. We claim 31 does not divide $|S|$. Assume the contrary and let $31 \mid |S|$, on the other hand, $\{2,3\} \subset \pi(P_i)$ (by TABLE 1), hence $2 \sim 31$ and $3 \sim 31$, which is a contradiction. Now, by Step 1, we observe that $31 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 31 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{31}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 31 (see TABLE 1). Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq 31$ and so 2^{62} must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.31$, where $2 \leq \alpha \leq 8$, $1 \leq \beta \leq 2$ and $0 \leq \gamma \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong L_3(25)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2_3| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, $G \cong L : 2_3$, because $|G| = 2|L|$. As $deg(7) = 1$ in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_2)$ (see pages 23 and 24), G can not be isomorphic to them.

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction. $\square$

Proposition 3.5. *If $M = L : 3$, then $G \cong L : 3$, $\mathbb{Z}_3 \times L$ or $\mathbb{Z}_3.L$.*

Proof. By TABLE 1, $|L:3| = 2^7.3^3.5^6.7.13.31$. $\pi_e(L:3) = \{1,2,3,4,5,6,7,8,10,12,13,15,16,20,21,24,26,30,31,39,40,48,52,60,78,93,104,120,156,208,217,312,624,651\}$, so $D(L:3) = (3,5,2,2,2,2)$. Since $|G| = |L:3|$ and $D(G) = D(L:3)$, we immediately conclude that $\Gamma(G)$ has several possibilities are as follows:

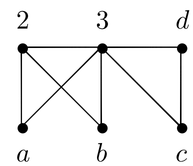


Figure 3.5:

where $\{a,b,c,d\} = \{5,7,13,31\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$-group. In particular, G is non-solvable.
We consider the following different parts:
Part A. Let $31 \in \{d,c\}$. First, we show that K is a $31^{'}$-group. Assume the contrary and let $31 \in \pi(K)$. Since there is no difference in the proof between choosing d as 31 and c as 31, we put $d = 31$. We know that one of the primes in $\{a,b\}$ is unequal to 5, we put it r. So r does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{31,r\}$-subgroup of K. It is easy to see that T is a nilpotent subgroup of order $r.31$. Thus, $r.31 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus, $\{31\} \subseteq \pi(K) \subseteq \pi(G) - \{r\}$. Let $K_{31} \in \mathrm{Syl}_{31}(K)$, by Frattini argument $G = KN_G(K_{31})$. Therefore, $N_G(K_{31})$ has an element x of order r. Since G has no element of order $r.31$, $\langle x \rangle$ should act fixed point freely on K_{31}, implying $\langle x \rangle K_{31}$ is a Frobenius group. By Lemma 2.3(b), $|\langle x \rangle|| (|K_{31}|-1)$. It follows that $r|31-1$, which is a contradiction. So K is a $31^{'}$-group. Next, we prove that K is a $p^{'}$-group for $p \in \{a,b,c\} - \{5\}$. If we assume $p||K|$ and x is an element of K of order p, we get a contradiction, because if $p \in \{a,b\} - \{5\}$, then by $\Gamma(G)$, $N_G(\langle x \rangle)$ is a $\{2,3,p\}$-group and since by Frattini argument $G = KN_G(\langle x \rangle)$, 31 must divide the order of K, which is impossible. Also if $p \in \{c\} - \{5\}$, we see that $N_G(\langle x \rangle)$ is a $\{2,3,31,c\}$-group and by the same argument as before, r must divide $|K|$, which is a contradiction. Therefore K is a $\{2,3,5\}$-group.
Part B. Let $\{31\} \nsubseteq \{d,c\}$. To show that K is a $31^{'}$-group, we assume on the contrary that $31||K|$. If we put $r = 13$ and follow the same technique in Part A, we get a contradiction. Thus K is a $31^{'}$-group. Next, we prove that K is a $p^{'}$-group where $p \in \{7,13\}$. If we assume $p||K|$ and x is an element of K of order p, we get a contradiction, because $\pi(C_G(x)) \subseteq \pi(G) - \{31\}$ and so is $\pi(N_G(\langle x \rangle))$. Since by Frattini argument $G = KN_G(\langle x \rangle)$, 31 must divide the order of K, which is impossible. Therefore K is a $\{2,3,5\}$-group.

In addition since $K \neq G$, G is non-solvable and the proof of this step is complete.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. If we set $S := \mathrm{Soc}(\overline{G})$, $S = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups. In Step 2 of Proposition 3.1, we showed that $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, then it is enough to prove that $m = 1$ to complete the proof of Step 2. Suppose that $m \geq 2$. We choose one of the primes in $\{d, c\}$ that is unequal to 5 as r. We claim r does not divide $|S|$. Assume the contrary and let $r \mid |S|$. Since $2 \in \pi(P_i)$, hence $2 \sim r$, which is a contradiction. Now, by Step 1, we observe that $r \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, r divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{31}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by r (note that $r \in \{7, 13, 31\}$ and see TABLE 1). Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq r \geq 7$ and so 2^{14} must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^\gamma.7.13.31$, where $2 \leq \alpha \leq 7$, $1 \leq \beta \leq 3$ and $0 \leq \gamma \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong L_3(25)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 3| = 3|L|$, we deduce $|K| = 1$ or 3.

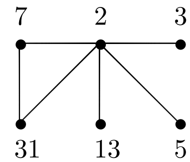
If $|K| = 1$, then by assumption, $G \cong L : 3$.

If $|K| = 3$, $K \leq C_G(K)$ and therefore, $\frac{C_G(K)}{K} \trianglelefteq \frac{G}{K} \cong L$ implies that $\frac{C_G(K)}{K} = 1$ or $\frac{C_G(K)}{K} \cong L$ because L is simple. If $\frac{C_G(K)}{K} = 1$, then $|L| = |\frac{G}{K}| = |\frac{G}{C_G(K)}|||\mathrm{Aut}(K)| = 2$, a contradiction. So, $G = C_G(K)$. Therefore $K \leq Z(G)$, that is, G is a central extension of K by L. If G splits over K, we obtain $G \cong \mathbb{Z}_3 \times L$, otherwise, we have $G \cong \mathbb{Z}_3.L$. □

Proposition 3.6. *If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.*

Proof. As $|L : 2^2| = 2^9.3^2.5^6.7.13.31$ and $\pi_e(L : 2^2) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 16, 20, 24, 26, 31, 40, 48, 52, 62, 104, 208, 217\}$, then $D(L : 2^2) = (5, 1, 1, 2, 1, 2)$. By assumptions $|G| = |L : 2^2|$ and $D(G) = D(L : 2^2)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : 2^2)$, see page 24):

Figure 3.6:



Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$-group. In particular, G is non-solvable.

First, we claim K is a $31^{'}$-group. Similarly to the former propositions, we assume the contrary that $31 \in \pi(K)$. we easily see 13 does not divide the order of K. Because otherwise, we may suppose that T is a Hall $\{13,31\}$-subgroup of K such that this subgroup is a nilpotent of order 13.31. Thus, $13.31 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus, $\{31\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$. Now let $K_{31} \in \mathrm{Syl}_{31}(K)$, by Frattini argument $G = KN_G(K_{31})$. Therefore, $N_G(K_{31})$ has an element x of order 13. Since G has no element of order 13.31, $\langle x\rangle$ should act fixed point freely on K_{31}, implying $\langle x\rangle K_{31}$ is a Frobenius group. By Lemma 2.3(b), $|\langle x\rangle|\,|(|K_{31}|-1)$. It follows that $13|31-1$, which is a contradiction, therefore K is a $31^{'}$-group. To show that K is a $p^{'}$-group for $p=7$ and 13, we assume the contrary that there exists an element x of K of order p. First put $p=13$. By the structure of $\Gamma(G)$, we see that $C_G(x)$ is a $\{2,13\}$-group, then $N_G(\langle x\rangle)$ is a $\{2,3,13\}$-group. So since by Frattini argument, $G = KN_G(\langle x\rangle)$ then 31 must divide the order of K, which is a contradiction. But for $p=7$, we see that $N_G(\langle x\rangle)$ is a $\{2,3,7,31\}$-group and then 13 must divide the order of K, which is impossible. Therefore K is a $\{2,3,5\}$-group. In addition since $K \neq G$, G is non-solvable and the proof of Step 1 is complete.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.

To get the proof, follow the method in the proof of Step 2 in Proposition 3.2.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.31$, where $2 \leq \alpha \leq 9$, $1 \leq \beta \leq 2$ and $0 \leq \gamma \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong L_3(25)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L:2^2| = 4|L|$, we deduce $|K| = 1, 2$ or 4.

If $|K|=1$, then by assumption, $G \cong L:2^2$.

If $|K|=2$, then $K \leq Z(G)$. In this case G is a central extension of $\mathbb{Z}_2$ by $L:2_1$, $L:2_2$ or $L:2_3$. If G splits over K then $G \cong \mathbb{Z}_2 \times (L:2_1)$, $\mathbb{Z}_2 \times (L:2_2)$ or $\mathbb{Z}_2 \times (L:2_3)$, otherwise $|K|$ must divide the Schur multiplier of $L:2_1$, $L:2_2$ or $L:2_3$, which are 1, 1 and 3 respectively, and it is impossible.

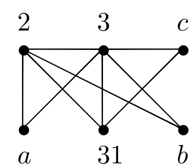
If $|K| = 4$, $K \leq C_G(K)$ and therefore, $\frac{C_G(K)}{K} \trianglelefteq \frac{G}{K} \cong L$ implies that $\frac{C_G(K)}{K} = 1$ or $\frac{C_G(K)}{K} \cong L$ because L is simple. If $\frac{C_G(K)}{K} = 1$, then $|L| =$

$|\frac{G}{K}| = |\frac{G}{C_G(K)}|||\mathrm{Aut}(K)| = 2$ or 6, a contradiction. So, $G = C_G(K)$. Therefore $K \leq Z(G)$, that is, G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schur multiplier of L, which is 3 (see [1]), but this is a contradiction. Therefore G splits over K. Hence $G \cong K \times L$. So we have $G \cong \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$. □

Proposition 3.7. *If* $M = L : (D_6)_1$*, then* $G \cong L : (D_6)_1$, $\mathbb{Z}_3.(L : 2_1)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ *or* $(\mathbb{Z}_3.L).\mathbb{Z}_2$*.*

Proof. As $|L : (D_6)_1| = 6|L| = 2^8.3^3.5^6.7.13.31$ and $\pi_e(L : (D_6)_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 15, 16, 20, 21, 24, 26, 30, 31, 39, 40, 48, 52, 60, 62, 78, 93, 104, 120, 156, 208, 217, 312, 624, 651\}$, then $D(L : (D_6)_1) = (4, 5, 2, 2, 2, 3)$. Since $|G| = |L : (D_6)_1|$ and $D(G) = D(L : (D_6)_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

Figure 3.7:



where $\{a, b, c\} = \{5, 7, 13\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
First, we show that K is a $31'$-group. Assume the contrary that $31||K|$. We know one of the primes in $\{a, b\}$ is unequal to 5, put it r. Therefore by a similar proof as in Proposition 3.5 (Step 1, Part A) we can get a contradiction and therefore K is a $31'$-group. Now we prove that K is a p'-group for $p \in \{a, b, c\} - \{5\} = \{7, 13\}$. Assume the contrary: $p||K|$ and x is an element of K of order p. First, let $p \in \{a, b\} - \{5\}$, then by $\Gamma(G)$ we conclude that $C_G(x)$ is a $\{2, 3, p\}$-group and therefore so is $N_G(\langle x \rangle)$. As $G = KN_G(\langle x \rangle)$, then 31 must divide the order of K, which is a contradiction. For $p \in \{c\} - \{5\}$, $N_G(\langle x \rangle)$ is a $\{2, 3, 31, c\}$-group. Similar to the above discussion, as $G = KN_G(\langle x \rangle)$, r have to divide $|K|$, which is impossible. So K is a $\{2, 3, 5\}$-group. In addition since $K \neq G$, G is non-solvable and the proof of Step 1 is complete.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. If we set $S := \mathrm{Soc}(\overline{G})$, $S = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups. In Step 2 of Proposition 3.1, we showed that $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, then it is enough to prove that $m = 1$ to complete the proof of Step 2. Suppose that $m \geq 2$. We claim r (r is one of the primes of a and b that is unequal to 5) does not divide $|S|$. Assume the contrary and let $r \mid |S|$, so by considering $|G|_r = r$ (note that $r \in \{7, 13\}$), we conclude that r just divides the order of one of the P_i's. Without losing generality, we assume that $r||P_1|$.

Then the rest of the P_i's must be $\{2,3\}$-group (because only 2 and 3 are adjacent to r in $\Gamma(G)$), this is a contradiction because P_i's are finite non-abelian simple groups. Now, by Step 1, we observe that $r \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, r divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{31}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by r (see TABLE 1), so r does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq r \geq 7$ and so 2^{14} must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.31$, where $2 \leq \alpha \leq 8$, $1 \leq \beta \leq 3$ and $0 \leq \gamma \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong L_3(25)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : (D_6)_1| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L : (D_6)_1$, because $|G| = 6|L|$. Obviously, G can not be isomorphic to $L : (D_6)_2$ and $L : 6$, because $deg(31) = 2$ in $\Gamma(L : (D_6)_2)$ and $\Gamma(L : 6)$.

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction.

If $|K| = 3$, then $\frac{G}{K} \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $\frac{G}{C_G(K)} \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|\frac{G}{C_G(K)}| = 1$ or 2. If $|\frac{G}{C_G(K)}| = 1$, then $K \leq Z(G)$, that is, G is a central extension of $\mathbb{Z}_3$ by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K, then only $\mathbb{Z}_3 \times (L : 2_1)$ is possible for G because in $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$ and $\Gamma(\mathbb{Z}_3 \times (L : 2_3))$, $deg(31) = 2$. Otherwise we get a contradiction, because 3 does not divide the Schur multiplier of $L : 2_1$ and $L : 2_2$ and$deg(31) \neq 3$ in $\Gamma(\mathbb{Z}_3.(L : 2_3))$. If $|\frac{G}{C_G(K)}| = 2$, then $K < C_G(K)$, hence by $1 \neq \frac{C_G(K)}{K} \trianglelefteq \frac{G}{K} \cong L : 2_1, L : 2_2$ or $L : 2_3$ and simplicity of L, we obtain $\frac{C_G(K)}{K} \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. Thus $C_G(K) \cong \mathbb{Z}_3 \times L$ or $\mathbb{Z}_3.L$. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$ or $(\mathbb{Z}_3.L).\mathbb{Z}_2$.

If $|K| = 6$, then $\frac{C_G(K)K}{K} \trianglelefteq \frac{G}{K} \cong L$ implies that $\frac{C_G(K)K}{K} = 1$ or $\frac{C_G(K)K}{K} \cong L$, because L is simple. If $\frac{C_G(K)K}{K} = 1$, then $C_G(K) \leq K$ and hence, $|L| = |\frac{G}{K}|||\frac{G}{C_G(K)}|||\mathrm{Aut}(K)| = 2$ or 6, a contradiction. Therefore, $G = C_G(K)K$. Now we consider following cases:

(1) If $K \cong \mathbb{Z}_6$, then $G = C_G(K)$. Therefore $K \leq Z(G)$ and it follows that $deg(2) = 5$, a contradiction.

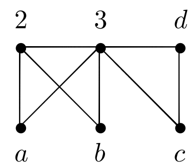
(2) If $K \cong D_6$, then $C_G(K) \cap K = 1$. So $C_G(K) \cong \frac{C_G(K)K}{K} = \frac{G}{K} \cong L$ and therefore $G \cong D_6 \times L$, which implies that $deg(2) = 5$, a contradiction.

□

Proposition 3.8. *If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2$, $\mathbb{Z}_3 \times (L : 2_2)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ or $(\mathbb{Z}_3.L).\mathbb{Z}_2$.*

Proof. As $|L : (D_6)_2| = 6|L| = 2^8.3^3.5^6.7.13.31$ and $\pi_e(L : (D_6)_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 15, 16, 20, 21, 24, 26, 30, 31, 39, 40, 48, 52, 60, 78, 93, 104, 120, 156, 208, 217, 312, 624, 651\}$, then $D(L : (D_6)_2) = (3, 5, 2, 2, 2, 2)$. Since $|G| = |L : (D_6)_2|$ and $D(G) = D(L : (D_6)_2)$, there exist several possibilities for $\Gamma(G)$ similarly to Proposition 3.5:

Figure 3.8:



where $\{a, b, c, d\} = \{5, 7, 13, 31\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
The proof is similar to that of Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group.
Again we refer to Step 2 of Proposition 3.5 to get the proof.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.31$, where $2 \le \alpha \le 8$, $1 \le \beta \le 3$ and $0 \le \gamma \le 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong L_3(25)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : (D_6)_2| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L : (D_6)_2$, because $|G| = 6|L|$. Obviously G can't be isomorphic to $L : (D_6)_1$ and $L : 6$, because $deg(2) = 4$ in $\Gamma(L : (D_6)_1)$ and $\Gamma(L : 6)$.

If $|K| = 2$, then $K \le Z(G)$ and so $deg(2) = 5$, which is a contradiction.

If $|K| = 3$, then $\frac{G}{K} \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $\frac{G}{C_G(K)} \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|\frac{G}{C_G(K)}| = 1$ or 2. If $|\frac{G}{C_G(K)}| = 1$, then $K \le Z(G)$, that is, G is a central extension of $\mathbb{Z}_3$ by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K, then only $G \cong \mathbb{Z}_3 \times (L : 2_2)$ because in $\Gamma(\mathbb{Z}_3 \times (L : 2_1))$ and $\Gamma(\mathbb{Z}_3 \times (L : 2_3))$, $deg(2) = 4$. Otherwise we get a contradiction, because 3 does not divide the Schur multiplier of $L : 2_1$ and $L : 2_2$, and $deg(2) = 4$ in $(Z3.(L : 2_3))$. If $|\frac{G}{C_G(K)}| = 2$, then $K < C_G(K)$, hence by $1 \ne \frac{C_G(K)}{K} \trianglelefteq \frac{G}{K} \cong L : 2_1, L : 2_2$ or $L : 2_3$ and simplicity of L, we obtain $\frac{C_G(K)}{K} \cong L$. Since $K \le Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. Thus $C_G(K) \cong \mathbb{Z}_3 \times L$ or $\mathbb{Z}_3.L$. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$ or $(\mathbb{Z}_3.L).\mathbb{Z}_2$.

If $|K| = 6$, then $\frac{C_G(K)K}{K} \trianglelefteq \frac{G}{K} \cong L$ implies that $\frac{C_G(K)K}{K} = 1$ or $\frac{C_G(K)K}{K} \cong L$, because L is simple. If $\frac{C_G(K)K}{K} = 1$, then $C_G(K) \leq K$ and hence, $|L| = |\frac{G}{K}|||\frac{G}{C_G(K)}|||\mathrm{Aut}(K)| = 2$ or 6, a contradiction. Therefore, $G = C_G(K)K$. Now we consider following cases:

(1) If $K \cong \mathbb{Z}_6$, then $G = C_G(K)$. Therefore $K \leq Z(G)$ and it follows that $deg(2) = 5$, a contradiction.

(2) If $K \cong D_6$, then $C_G(K) \cap K = 1$. So $C_G(K) \cong \frac{C_G(K)K}{K} = \frac{G}{K} \cong L$ and therefore $G \cong D_6 \times L$, which implies that $deg(2) = 5$, a contradiction.

$\square$

Proposition 3.9. *If $M = L : 6$, then $G \cong L : 6$, $\mathbb{Z}_3 \times (L : 2_3)$, $\mathbb{Z}_3.(L : 2_3)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$ or $(\mathbb{Z}_3.L).\mathbb{Z}_2$.*

Proof. As $|L : 6| = 2^8.3^3.5^6.7.13.31$ and $\pi_e(L : 6) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 20, 21, 24, 26, 30, 31, 39, 40, 42, 48, 52, 60, 78, 93, 104, 120, 156, 208, 217, 312, 624, 651\}$, then $D(L : 6) = (4, 5, 2, 3, 2, 2)$ and since $|G| = |L : 6|$ and $D(G) = D(L : 6)$ then the prime graph of G has the following forms:

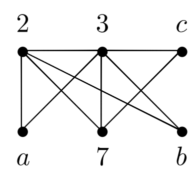


Figure 3.9:

where $\{a, b, c\} = \{5, 13, 31\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
First, we show that K is a $7^{'}$-group. Assume the contrary that $7||K|$. We know that one of the primes in $\{a, b\}$ is unequal to 5, we put it r. So r does not divide the order of K. Otherwise, we may suppose that T is a Hall $\{7, r\}$-subgroup of K. It is easy to see that T is a nilpotent subgroup of order $r.7$. Thus, $r.7 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus, $\{7\} \subseteq \pi(K) \subseteq \pi(G) - \{r\}$. Let $K_7 \in \mathrm{Syl}_7(K)$, by Frattini argument $G = KN_G(K_7)$. Therefore, $N_G(K_7)$ has an element x of order r. Since G has no element of order $r.7$, $\langle x \rangle$ should act fixed point freely on K_7, implying $\langle x \rangle K_7$ is a Frobenius group. By Lemma 2.3(b), $|\langle x \rangle||(|K_7| - 1)$. It follows that $r|7-1$, which is a contradiction, because $r = 13$ or 31. So K is a $7^{'}$-group. Next, we prove that K is a $p^{'}$-group for $p \in \{a, b, c\} - \{5\} = \{13, 31\}$. If we assume $p||K|$ and x is an element of K of order p, we get a contradiction, because if $p \in \{a, b\} - \{5\}$ then by $\Gamma(G)$, $N_G(\langle x \rangle)$ is a $\{2, 3, 5, p\}$-group and since by Frattini argument $G = KN_G(\langle x \rangle)$, 7 must divide the order of K, which is impossible. Also if $p \in \{c\} - \{5\}$, we see that $N_G(\langle x \rangle)$ is a $\{2, 3, 5, 7, c\}$-group, then we conclude that r must divide the

order of K, which is impossible. Therefore K is a $\{2,3,5\}$-group. In addition since $K \neq G$, G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

The proof is similar to those in Proposition 3.7. But the reader must replace $t \geq r \geq 7$ and 2^{14} with $t \geq r \geq 13$ and 2^{26} respectively.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.31$, where $2 \leq \alpha \leq 8$, $1 \leq \beta \leq 3$ and $0 \leq \gamma \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong L_3(25)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L:6| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L:6$ because $|G| = 6|L|$. Obviously G can not be isomorphic to $L:(D_6)_1$ or $L:(D_6)_2$, because $deg(7) = 2$ in $\Gamma(L:(D_6)_1)$ and $\Gamma(L:(D_6)_2)$.

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction.

If $|K| = 3$, then $\frac{G}{K} \cong L:2_1, L:2_2$ or $L:2_3$. But $\frac{G}{C_G(K)} \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|\frac{G}{C_G(K)}| = 1$ or 2. If $|\frac{G}{C_G(K)}| = 1$, then $K \leq Z(G)$, that is, G is a central extension of $\mathbb{Z}_3$ by $L:2_1, L:2_2$ or $L:2_3$. If G splits over K, then only $G \cong \mathbb{Z}_3 \times (L:2_3)$ because in $\Gamma(\mathbb{Z}_3 \times (L:2_1))$ we have $deg(31) = 3$ and in $\Gamma(\mathbb{Z}_3 \times (L:2_2))$ we have $deg(2) = 3$. Otherwise only $G \cong \mathbb{Z}_3.(L:2_3)$ because 3 does not divide the Schur multiplier of $L:2_1$ and $L:2_2$. If $|\frac{G}{C_G(K)}| = 2$, then $K < C_G(K)$, hence by $1 \neq \frac{C_G(K)}{K} \trianglelefteq \frac{G}{K} \cong L:2_1, L:2_2$ or $L:2_3$ and simplicity of L, we obtain $\frac{C_G(K)}{K} \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. Thus $C_G(K) \cong \mathbb{Z}_3 \times L$ or $\mathbb{Z}_3.L$. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$ or $(\mathbb{Z}_3.L).\mathbb{Z}_2$.

If $|K| = 6$, then $\frac{C_G(K)K}{K} \trianglelefteq \frac{G}{K} \cong L$ implies that $\frac{C_G(K)K}{K} = 1$ or $\frac{C_G(K)K}{K} \cong L$, because L is simple. If $\frac{C_G(K)K}{K} = 1$, then $C_G(K) \leq K$ and hence, $|L| = |\frac{G}{K}|||\frac{G}{C_G(K)}|||\mathrm{Aut}(K)| = 2$ or 6, a contradiction. Therefore, $G = C_G(K)K$. Now we consider following cases:

(1) If $K \cong \mathbb{Z}_6$, then $G = C_G(K)$. Therefore $K \leq Z(G)$ and it follows that $deg(2) = 5$, a contradiction.

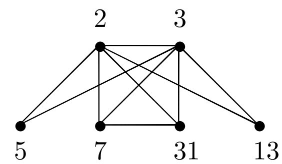
(2) If $K \cong D_6$, then $C_G(K) \cap K = 1$. So $C_G(K) \cong \frac{C_G(K)K}{K} = \frac{G}{K} \cong L$ and therefore $G \cong D_6 \times L$, which implies that $deg(2) = 5$, a contradiction.

□

Proposition 3.10. *If $M = L:D_{12}$, then $G \cong L:D_{12}$, $\mathbb{Z}_2 \times (L:(D_6)_1)$, $\mathbb{Z}_2 \times (L:(D_6)_2)$, $\mathbb{Z}_2 \times (L:6)$, $\mathbb{Z}_3.(L:2^2)$, $(\mathbb{Z}_3 \times (L:2_3)).\mathbb{Z}_2$, $(\mathbb{Z}_3.(L:2_1)).\mathbb{Z}_2$, $(\mathbb{Z}_3.(L:2_2)).\mathbb{Z}_2$, $(\mathbb{Z}_3.(L:2_3)).\mathbb{Z}_2$, $\mathbb{Z}_4 \times (L:3)$, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L:3)$, $(\mathbb{Z}_4 \times L).\mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$, $\mathbb{Z}_6 \times (L:2_1)$, $\mathbb{Z}_6 \times (L:2_2)$, $\mathbb{Z}_6 \times (L:2_3)$, $(\mathbb{Z}_6 \times L).\mathbb{Z}_2$, $D_6 \times (L:2_1)$, $D_6 \times (L:2_2)$, $D_6 \times (L:2_3)$, $\mathbb{Z}_{12} \times L$, $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, $D_{12} \times L$, $A_4 \times L$, $T \times L$.*

Proof. As $|L : D_{12}| = 2^9.3^3.5^6.7.13.31$ and $\pi_e(L : D_{12}) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 20, 21, 24, 26, 30, 31, 39, 40, 42, 48, 52, 60, 62, 78, 93, 104, 120, 156, 208, 217, 312, 624, 651\}$, then $D(L : D_{12}) = (5, 5, 2, 3, 2, 3)$. By assumptions $|G| = |L : D_{12}|$ and $D(G) = D(L : D_{12})$, so the prime graph of G has the following form (like $\Gamma(L : D_{12})$, see page 25):

Figure 3.10:



Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.

First, by the same way in Proposition 3.6 we can prove that K is a $31^{'}$-group. Now we show that K is a $p^{'}$-group for $p = 7$ and 13. Suppose the contrary: $p||K|$ and x is an element of K of order p. First put $p = 13$. According to $\Gamma(G)$, $C_G(x)$ is a $\{2, 3, 13\}$-group and since $\frac{N_G(\langle x\rangle)}{C_G(x)} \lesssim \mathrm{Aut}(\langle x\rangle) \cong \mathbb{Z}_{12}$, so is $N_G(\langle x\rangle)$. As by Frattini argument, $G = KN_G(\langle x\rangle)$ then 31 must divide the order of K, which is a contradiction. Now put $p = 7$, then by the same argument as before we see that $N_G(\langle x\rangle)$ is a $\{2, 3, 7, 31\}$-group. Since $G = KN_G(\langle x\rangle)$, 13 must divide $|K|$, which is a contradiction. Therefore K is a $\{2, 3, 5\}$-group. In addition since $K \neq G$, G is non-solvable and the proof of this step is completed.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.

The proof is similar to Step 2 of Proposition 3.7, by replacing r with 13.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^\gamma.7.13.31$, where $2 \leq \alpha \leq 9$, $1 \leq \beta \leq 3$ and $0 \leq \gamma \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong L_3(25)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : D_{12}| = 12|L|$, we deduce $|K| = 1, 2, 3, 4, 6$ or 12.

If $|K| = 1$, then by assumption $G \cong L : D_{12}$.

If $|K| = 2$, then $\frac{G}{K} \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ and $K \leq Z(G)$. It follows that G is a central extension of K by $L : (D_6)_1, L : (D_6)_2$ or $L : 6$. If G splits over K, then $G \cong \mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2)$ or $\mathbb{Z}_2 \times (L : 6)$. Otherwise 2 must divide the Schur multiplier of $L : (D_6)_1, L : (D_6)_2$ or $L : 6$, which is impossible.

If $|K| = 3$, then $\frac{G}{K} \cong L : 2^2$. But $\frac{G}{C_G(K)} \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|\frac{G}{C_G(K)}| = 1$ or 2. If $|\frac{G}{C_G(K)}| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2^2$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L : 2^2)$. We get a contradiction, because 3 does not divide the Schur multiplier of $L : 2^2$, which is 1. If $|\frac{G}{C_G(K)})| = 2$, then $K < C_G(K)$, hence by $1 \neq \frac{C_G(K)}{K} \trianglelefteq \frac{G}{K} \cong L : 2^2$ and simplicity of L, we obtain $\frac{C_G(K)}{K} \cong L : 2_1, L : 2_2$ or $L : 2_3$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If $C_G(K)$ splits over K then $C_G(K) \cong \mathbb{Z}_3 \times (L : 2_1)$, Therefore, $G \cong (\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$, or $(\mathbb{Z}_3.(L : 2_3)).\mathbb{Z}_2$.

If $|K| = 4$, then $\frac{G}{K} \cong L : 3$ and $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we have $\frac{G}{C_G(K)} \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$ or D_6. Thus $|\frac{G}{C_G(K)}| = 1, 2, 3$ or 6. If $|\frac{G}{C_G(K)}| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 3$. If G splits over K, then $G \cong \mathbb{Z}_4 \times (L : 3)$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$. Otherwise we get a contradiction because 4 must divide the Schur multiplier of $L : 3$, which is 3. If $|\frac{G}{C_G(K)}| \neq 1$, as $|\frac{G}{C_G(K)}| = 2, 3$ or 6, then $K < C_G(K)$, and by simplicity of L, we conclude that $1 \neq \frac{C_G(K)}{K}$ must be an extension of L. So only $|\frac{G}{C_G(K)}| = 3$ is acceptable and therefore $\frac{C_G(K)}{K} \cong L$. Now since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, $C_G(K) \cong \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$, otherwise $|K|$ must divide the Schur multiplier of L, which is impossible. Therefore $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3$ or $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$.

If $|K| = 6$, then $\frac{G}{K} \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ and $K \cong \mathbb{Z}_6$ or D_6.
If $K \cong \mathbb{Z}_6$, then $\frac{G}{C_G(K)} \lesssim \mathbb{Z}_2$ and so $|\frac{G}{C_G(K)}| = 1$ or 2. If $|\frac{G}{C_G(K)}| = 1$, then $K \leq Z(G)$, that is G is a central extension of $\mathbb{Z}_6$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If G splits over K, we obtain $G \cong \mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$ or $\mathbb{Z}_6 \times (L : 2_3)$, otherwise we get a contradiction because 6 must divide the Schur multiplier of $L : 2_1$, $L : 2_2$ or $L : 2_3$, and it is impossible. If $|\frac{G}{C_G(K)}| = 2$, then $K < C_G(K)$, hence by $1 \neq \frac{C_G(K)}{K} \trianglelefteq \frac{G}{K} \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ and simplicity of L, we obtain $\frac{C_G(K)}{K} \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, $C_G(K) \cong \mathbb{Z}_6 \times L$ and if $C_G(K)$ is a non-split extension of K by L, $|K|$ must divide the Schur multiplier of L, which is impossible. Therefore $G \cong (\mathbb{Z}_6 \times L).\mathbb{Z}_2$.
If $K \cong D_6$, then $\frac{G}{C_G(K)} \lesssim D_6$ and so $|\frac{G}{C_G(K)}| = 1, 2, 3$ or 6. If $|\frac{G}{C_G(K)}| = 1$, then $K \leq Z(G)$, a contradiction. If $|\frac{G}{C_G(K)}| = 2$, then we have $|KC_G(K)| = 6.|G|/2 = 3|G|$ because $K \cap C_G(K) = 1$, that is a contradiction. If $|\frac{G}{C_G(K)}| = 3$, then we have $|KC_G(K)| = 6.|G|/3 = 2|G|$ because $K \cap C_G(K) = 1$, that is a contradiction. If $|\frac{G}{C_G(K)}| = 6$, then $\frac{G}{C_G(K)} \cong D_6$. Thus $C_G(K) \neq 1$. Then, $1 \neq C_G(K) \cong \frac{C_G(K)K}{K} \trianglelefteq \frac{G}{K} \cong L : 2_1, L : 2_2$ or $L : 2_3$, follows that $C_G(K) \cong L : 2_1, L : 2_2$ or $L : 2_3$. Therefore $G \cong D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$ or $D_6 \times (L : 2_3)$.

Before processing the last case, we recall the following fact.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group A_4, dihedral group D_{12} and the dicyclic group T with generators a and b, subject to the relations $a^6 = 1$, $a^3 = b^2$ and $b^{-1}ab = a^{-1}$.

If $|K| = 12$, then $\frac{C_G(K)K}{K} \trianglelefteq \frac{G}{K} \cong L$ implies that $\frac{C_G(K)K}{K} = 1$ or $\frac{C_G(K)K}{K} \cong L$, because L is simple. If $\frac{C_G(K)K}{K} = 1$, then $C_G(K) \leq K$ and hence, $|L| = |\frac{G}{K}|||\frac{G}{C_G(K)}|||\mathrm{Aut}(K)| \leq 12^{[log_2 12]}$, a contradiction. Therefore, $G = C_G(K)K$. Now we consider following cases:

(1) If $K \cong \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$, then $G = C_G(K)$. Therefore $K \leq Z(G)$, that is G is a central extension of K by L. If G splits over K, we obtain $G \cong \mathbb{Z}_{12} \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, and otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of L, which is 3.

(2) If $K \cong D_{12}$, then $G = K.L$ and $\frac{G}{C_G(K)} \cong \frac{K}{Z(K)} \cong D_6$. Since $\frac{C_G(K)}{Z(K)} \cong \frac{C_G(K)K}{K} = \frac{G}{K} \cong L$ and $Z(K) \leqslant Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $Z(K) \cong \mathbb{Z}_2$ by L. If $C_G(K)$ is a non-split extension, then 2 must divide the Schur multiplier of L, which is 3 and it is impossible. Thus $C_G(K) \cong \mathbb{Z}_2 \times L$ and hence G is a split extension of K by L. Now, since $Hom(L, \mathrm{Aut}(D_{12}))$ is trivial, we have $G \cong D_{12} \times L$.

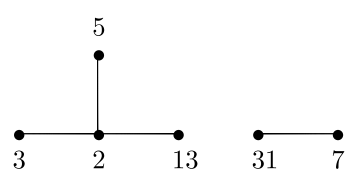
(3) If $K \cong A_4$, then $\frac{G}{C_G(K)} \cong \frac{K}{Z(K)} \cong A_4$, then $C_G(K) \neq 1$. Thus $1 \neq C_G(K) \cong \frac{C_G(K)K}{K} \trianglelefteq \frac{G}{K} \cong L$. Hence $L \cong C_G(K)$ because L is simple. Therefore $G \cong A_4 \times L$, because $Z(A_4) = 1$.

(4) If $K \cong T$, then by the similar way in case $K \cong D_{12}$, we can conclude that G is a split extension of T by L. Also, since $Hom(L, \mathrm{Aut}(T))$ is trivial, we have $G \cong T \times L$.

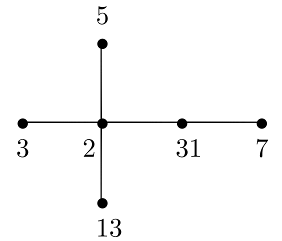
The proof of our main Theorem is complete. $\square$

According to what we said before the proof, here we diagram $\Gamma(M)$ by $|M|$ and $\pi_e(M)$, where M is an almost simple group related to $L = L_3(25)$.

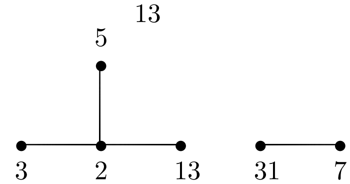
$\Gamma(L)$:



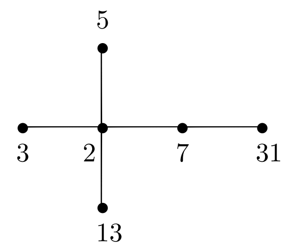
$\Gamma(L:2_1)$:



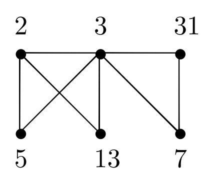
$\Gamma(L:2_2)$:



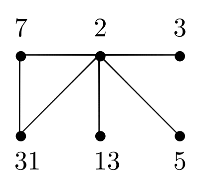
$\Gamma(L:2_3)$:



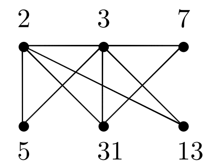
$\Gamma(L:3)$:



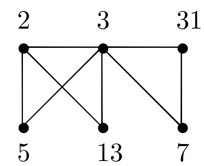
$\Gamma(L:2^2)$:



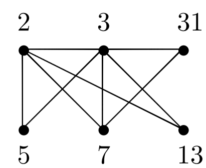
$\Gamma(L:(D_6)_1)$:



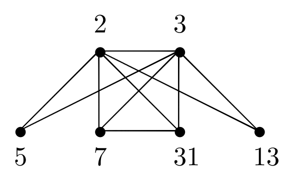
$\Gamma(L:(D_6)_2)$:



$\Gamma(L:6)$:



$\Gamma(L:D_{12})$:



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