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ASYMPTOTICS FOR THE INFINITE TIME RUIN PROBABILITY OF A DEPENDENT RISK MODEL WITH A CONSTANT INTEREST RATE AND DOMINATEDLY VARYING-TAILED CLAIM SIZES

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ABSTRACT. This paper mainly considers a nonstandard risk model with a constant interest rate, where both the claim sizes and the inter-arrival times follow some certain dependence structures. When the claim sizes are dominatedly varying-tailed, asymptotics for the infinite time ruin probability of the above dependent risk model have been given.

Keywords: Asymptotics, infinite time ruin probability, constant interest rate, dominatedly varying tail.

MSC(2010): Primary: 62P05; Secondary: 62E10, 60F05.

1. Introduction

In this paper, we consider the infinite time ruin probability of a dependent risk model with a constant interest rate. In this model, the claim sizes, $X_n, n \geq 1$, form a sequence of nonnegative and dependent identically distributed random variables (r.v.s) with common distribution F , and the inter-arrival times, $Y_n, n \geq 1$, form another sequence of nonnegative and dependent identically distributed r.v.s, which are not

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degenerate at zero. The times of successive claims, $\tau_n = \sum_{k=1}^n Y_k$, $n \geq 1$, constitute a quasi renewal counting process

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}_{\{\tau_n \leq t\}}, \quad t \geq 0,$$

where $\mathbf{1}_A$ is the indicator function of the set A . Let $\lambda(t) = EN(t)$, $t \geq 0$, and assume that $\lambda(t) < \infty$ for all $0 < t < \infty$. The total amount of premium accumulated up to time $t \geq 0$, denoted by $C(t)$ with $C(0) = 0$ and $C(t) < \infty$ almost surely (a.s.) for every $t \geq 0$, is a nonnegative and nondecreasing stochastic process. Assume that $\{X_n, n \geq 1\}$, $\{Y_n, n \geq 1\}$ and $\{C(t), t \geq 0\}$ are mutually independent. Let $x \geq 0$ be the initial capital reserve of the insurance company, and $\delta > 0$ be the constant interest rate, that is to say, after time t a capital y becomes $ye^{\delta t}$. Then the total reserve up to time $t \geq 0$, denoted by $U(x, t)$, satisfies

$$U(x, t) = xe^{\delta t} + \int_{0-}^t e^{\delta(t-y)} C(dy) - \int_{0-}^t e^{\delta(t-y)} S(dy),$$

where $S(t) = \sum_{k=1}^{N(t)} X_k$ is the total amount of claims up to time $t \geq 0$ with $S(t) = 0$ when $N(t) = 0$. Hence, the infinite time ruin probability is defined by

$$\psi(x) = P(U(x, t) < 0 \text{ for some } t \geq 0).$$

This paper mainly investigates the asymptotics of the infinite time ruin probability $\psi(x)$ as $x \rightarrow \infty$. Assume that $\tilde{C}(\infty) = \int_{0-}^{\infty} e^{-\delta y} C(dy) < \infty$ a.s.. Then

$$(1.1) \quad P\left(\sum_{k=1}^{\infty} X_k e^{-\delta \tau_k} > x + \tilde{C}(\infty)\right) \leq \psi(x) \leq P\left(\sum_{k=1}^{\infty} X_k e^{-\delta \tau_k} > x\right).$$

We will assume that the distributions of the claim sizes are heavy tailed. In the following, some heavy-tailed distribution classes will be given. In this paper, all limit relationships are for x tending to ∞ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$, write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$, write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$, and write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$.

For a proper distribution V on $(-\infty, \infty)$, let $\bar{V}(x) = 1 - V(x)$, $x \in (-\infty, \infty)$. Denote the upper and lower Matuszewska index of V , respectively, by

$$J_V^+ = - \lim_{y \rightarrow \infty} \frac{\log \bar{V}_*(y)}{\log y} \quad \text{and} \quad J_V^- = - \lim_{y \rightarrow \infty} \frac{\log \bar{V}^*(y)}{\log y},$$

where for any $y > 1$,

$$\bar{V}_*(y) = \liminf_{x \rightarrow \infty} \frac{\bar{V}(xy)}{\bar{V}(x)} \quad \text{and} \quad \bar{V}^*(y) = \limsup_{x \rightarrow \infty} \frac{\bar{V}(xy)}{\bar{V}(x)}.$$

Let

$$L_V = \lim_{y \downarrow 1} \bar{V}_*(y).$$

Say that a distribution V on $(-\infty, \infty)$ belongs to the extended regular variation class, if there are some $0 \leq \alpha \leq \beta < \infty$ such that for all $y > 1$,

$$y^{-\beta} \leq \bar{V}_*(y) \leq \bar{V}^*(y) \leq y^{-\alpha},$$

denoted by $V \in \mathcal{ERV}(-\alpha, -\beta)$. Say that a distribution V on $(-\infty, \infty)$ belongs to the consistent variation class, denoted by $V \in \mathcal{C}$, if $L_V = 1$. A larger class is the dominated variation class, denoted by \mathcal{D} . Say that a distribution V on $(-\infty, \infty)$ belongs to the class \mathcal{D} , if for all $y > 1$, $\bar{V}_*(y) > 0$. A related distribution class is the long-tailed distribution class, denoted by \mathcal{L} . Say that a distribution V on $(-\infty, \infty)$ belongs to the class \mathcal{L} , if for any $y > 0$,

$$\bar{V}(x+y) \sim \bar{V}(x).$$

It is well known that the above distribution classes have the following proper relations: for any $0 \leq \alpha \leq \beta < \infty$,

$$\mathcal{ERV}(-\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{D},$$

(see, e.g. [1], [4] and [6]).

In this paper, we will assume that both the claim sizes and the inter-arrival times follow some certain dependence structures. In the following, we will introduce some dependence structures. [18] introduced the widely orthant dependence structure, which can allow some common negatively dependent r.v.s and also allow some positively dependent r.v.s (see examples in [18]). The widely orthant dependence structure has been investigated extensively in the literature, such as [19], [20] and [22].

Definition 1.1. For the r.v.s $\{\xi_n, n \geq 1\}$, if there exists a finite real sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$(1.2) \quad P\left(\bigcap_{i=1}^n \{\xi_i > x_i\}\right) \leq g_U(n) \prod_{i=1}^n P(\xi_i > x_i),$$

then we say that the r.v.s $\{\xi_n, n \geq 1\}$ are widely upper orthant dependent (WUOD) with dominating coefficients $\{g_U(n), n \geq 1\}$; if there exists a finite real sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$(1.3) \quad P\left(\bigcap_{i=1}^n \{\xi_i \leq x_i\}\right) \leq g_L(n) \prod_{i=1}^n P(\xi_i \leq x_i),$$

then we say that the r.v.s $\{\xi_n, n \geq 1\}$ are widely lower orthant dependent (WLOD) with dominating coefficients $\{g_L(n), n \geq 1\}$. Further, we say that the r.v.s $\{\xi_n, n \geq 1\}$ are widely orthant dependent (WOD) if $\{\xi_n, n \geq 1\}$ are both WUOD and WLOD.

Recall that when $g_L(n) = g_U(n) \equiv 1$ for any $n \geq 1$ in (1.2) and (1.3), the r.v.s $\{\xi_n, n \geq 1\}$ are called negatively upper orthant dependent (NUOD) and negatively lower orthant dependent (NLOD), respectively, and say that the r.v.s $\{\xi_n, n \geq 1\}$ are negatively orthant dependent (NOD) if $\{\xi_n, n \geq 1\}$ are both NUOD and NLOD (see, e.g. [5] or [2]). Say that the r.v.s $\{\xi_n, n \geq 1\}$ are pairwise negatively quadrant dependent (NQD) or pairwise NOD, if for all positive integers $i \neq j$, the r.v.s ξ_i and ξ_j are NOD (see, e.g. [13]).

Geluk and Tang (see [7]) introduced a pairwise dependence structure. **Assumption A** For the real-valued r.v.s $\{\xi_n, n \geq 1\}$, it holds that for any $1 \leq i \neq j < \infty$,

$$(1.4) \quad \lim_{\min\{x_i, x_j\} \rightarrow \infty} P(|\xi_i| > x_i | \xi_j > x_j) = 0.$$

For the nonnegative r.v.s, it can be verified that the WUOD r.v.s and pairwise NQD r.v.s satisfy Assumption A.

Before giving the main results of this paper, we briefly review some existing results of the infinite time ruin probability $\psi(x)$. When the claim sizes, $X_n, n \geq 1$, and the inter-arrival times, $Y_n, n \geq 1$, are independent and identically distributed (i.i.d.) r.v.s, respectively, the ruin probability has been investigated by [10]-[12], [15], [16], among others.

When the claim sizes, $X_n, n \geq 1$, have a certain dependence structure, there are some results about the ruin probability. When the claim sizes, $X_n, n \geq 1$, are pairwise NQD with common distribution $F \in \mathcal{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$, the inter-arrival times, $Y_n, n \geq 1$, are i.i.d. r.v.s, and the process $\{C(t), t \geq 0\}$ is a deterministic linear function, [3] obtained the asymptotics for $\psi(x)$:

$$(1.5) \quad \psi(x) \sim \int_{0-}^{\infty} \bar{F}(xe^{\delta y})\lambda(dy).$$

[14] extended the above result and considered the case that the claim sizes, $X_n, n \geq 1$, are pairwise NQD with common distribution $F \in \mathcal{D}$ and $J_{\bar{F}} > 0$, the inter-arrival times, $Y_n, n \geq 1$, are NLOD, and the process $\{C(t), t \geq 0\}$ is a deterministic linear function. They got the following result:

$$(1.6) \quad L_F \int_{0-}^{\infty} \bar{F}(xe^{\delta y})\lambda(dy) \lesssim \psi(x) \lesssim L_F^{-2} \int_{0-}^{\infty} \bar{F}(xe^{\delta y})\lambda(dy).$$

For the general stochastic process $\{C(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ is a delayed renewal counting process, [21] also obtained the above results. when the claim sizes, $X_n, n \geq 1$, satisfy Assumption A with common distribution $F \in \mathcal{L} \cap \mathcal{D}$ and $J_{\bar{F}} > 0$, and the inter-arrival times, $Y_n, n \geq 1$, are WLOD, the asymptotics of the infinite time ruin probability have been investigated by [22].

Recently, the risk model with a constant interest rate has been extended by some researchers. Hao and Tang (see [9]) gave a bivariate Lévy-driven risk model, in which they used two independent Lévy processes to represent, respectively, a loss process in a world without economic factors and a process describing return on investgates in real terms. When the Lévy measure of the loss process belongs to the class $\mathcal{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$, the infinite time ruin probability has been investigated. Guo and Wang (see [8]) introduced a risk model with risk-free and risk assets, where they used an adapted càdlàg process to model the log investment returns. When the claim sizes, $X_n, n \geq 1$, are bivariate upper tail independent with common distribution $F \in \mathcal{C}$ and the inter-arrival times, $Y_n, n \geq 1$, are i.i.d., they investigated the finite time and infinite time ruin probabilities.

In this paper, we still consider the risk model with a constant interest rate. Since the dependence structure of Assumption A and the WLOD structure are wider than the pairwise NQD structure and the NLOD structure, respectively, in this paper, we will consider the case that the

claim sizes satisfy Assumption A and the inter-arrival times are WLOD r.v.s. The following is the main result.

Theorem 1.2. *Consider the above risk model. Assume that the claim sizes, $X_n, n \geq 1$, satisfy Assumption A with common distribution $F \in \mathcal{D}$ and $J_F^- > 0$, and the inter-arrival times, $Y_n, n \geq 1$, are WLOD r.v.s with dominating coefficients $\{g_L(n), n \geq 1\}$ satisfying for any $\varepsilon > 0$,*

$$(1.7) \quad \lim_{n \rightarrow \infty} g_L(n)e^{-\varepsilon n} = 0.$$

Then (1.6) holds.

Particularly, if $F \in \mathcal{C}$ then (1.5) holds.

Remark 1.3. (1) *Theorem 1.2 has extended the corresponding results of [3] and [14]. When the counting process $\{N(t), t \geq 0\}$ has no delays in [21], their result about the infinite time ruin probability has been extended by Theorem 1.2.*

(2) *Compared with Theorem 2 of [22], Theorem 1.2 has extended the scopes of the distributions of the claim sizes from the class $\mathcal{L} \cap \mathcal{D}$ to the class \mathcal{D} .*

These results will be shown in Section 2.

2. Proof of Theorem 1.2

We first give some lemmas, which will be used in the proof of Theorem 1.2. The first lemma is Proposition 1.1 of [18], which gives some properties about WUOD and WLOD r.v.s.

Lemma 2.1. (1) *Let $\{\xi_n, n \geq 1\}$ be WLOD (or WUOD) r.v.s with dominating coefficients $\{g_L(n), n \geq 1\}$ (or $\{g_U(n), n \geq 1\}$). If $\{f_n(\cdot), n \geq 1\}$ are nondecreasing, then $\{f_n(\xi_n), n \geq 1\}$ are still WLOD (or WUOD) r.v.s with dominating coefficients $\{g_L(n), n \geq 1\}$ (or $\{g_U(n), n \geq 1\}$); If $\{f_n(\cdot), n \geq 1\}$ are nonincreasing, then $\{f_n(\xi_n), n \geq 1\}$ are WUOD (or WLOD) r.v.s with dominating coefficients $\{g_L(n), n \geq 1\}$ (or $\{g_U(n), n \geq 1\}$).*

(2) *If $\{\xi_n, n \geq 1\}$ are nonnegative and WUOD r.v.s with dominating coefficients $\{g_U(n), n \geq 1\}$, then for each $n \geq 1$,*

$$E \prod_{i=1}^n \xi_i \leq g_U(n) \prod_{i=1}^n E \xi_i.$$

The following lemma is obtained from Proposition 2.2.1 of [1] and Lemma 3.5 of [17].

Lemma 2.2. *If a distribution V on $(-\infty, \infty)$ belongs to the class \mathcal{D} , then for any $0 < \alpha < J_V^- \leq J_V^+ < \beta$, there exist positive constants C_i and $D_i, i = 1, 2$, such that for all $x \geq y \geq D_1$,*

$$\frac{\bar{V}(y)}{\bar{V}(x)} \geq C_1 \left(\frac{y}{x}\right)^{-\alpha}$$

and for all $x \geq y \geq D_2$,

$$\frac{\bar{V}(y)}{\bar{V}(x)} \leq C_2 \left(\frac{y}{x}\right)^{-\beta}.$$

Furthermore, for any $\beta > J_V^+$,

$$x^{-\beta} = o(\bar{V}(x)).$$

The following two lemmas will deal with the weighted sums of r.v.s satisfying Assumption A.

Lemma 2.3. *Suppose that $\xi_k, 1 \leq k \leq n$, are n real-valued r.v.s and for any $1 \leq i \neq j \leq n$, relation (1.4) holds. Then for each nonempty set $I \subsetneq \{1, 2, \dots, n\}$ and each $j \in \{1, 2, \dots, n\}/I$,*

$$\lim_{\min\{x,y\} \rightarrow \infty} P \left(\sum_{i \in I} |\xi_i| > x \mid \xi_j > y \right) = 0.$$

Proof. Since

$$P \left(\sum_{i \in I} |\xi_i| > x \mid \xi_j > y \right) \leq \sum_{i \in I} P \left(|\xi_i| > xn^{-1} \mid \xi_j > y \right),$$

by using Assumption A we know that the result of Lemma 2.3 holds. \square

Before giving the next lemma, we first introduce a notation. For n real-valued numbers $c_i, 1 \leq i \leq n$, let $\underline{c}_n = (c_1, \dots, c_n)$.

Lemma 2.4. *Suppose that $\xi_k, 1 \leq k \leq n$, are n r.v.s and for any $1 \leq i \neq j \leq n$, relation (1.4) holds.*

(1) *If $\xi_k, 1 \leq k \leq n$, are nonnegative r.v.s then for any fixed constant $b > 0$,*

$$(2.1) \quad \liminf_{x \rightarrow \infty} \inf_{\underline{c}_n \in (0,b]^n} \frac{P(\sum_{k=1}^n c_k \xi_k > x)}{\sum_{k=1}^n P(c_k \xi_k > x)} \geq 1.$$

(2) If $\xi_k, 1 \leq k \leq n$, are identically distributed and real-valued r.v.s with common distribution $V \in \mathcal{D}$, then for any fixed constant $b > 0$,

$$(2.2) \quad \liminf_{x \rightarrow \infty} \inf_{\underline{c}_n \in (0, b]^n} \frac{P(\sum_{k=1}^n c_k \xi_k > x)}{\sum_{k=1}^n P(c_k \xi_k > x)} \geq L_V$$

and

$$(2.3) \quad \limsup_{x \rightarrow \infty} \sup_{\underline{c}_n \in (0, b]^n} \frac{P(\sum_{k=1}^n c_k \xi_k > x)}{\sum_{k=1}^n P(c_k \xi_k > x)} \leq L_V^{-1}.$$

Proof. It follows from (1.4) that for any $\varepsilon > 0$ there exists $y_0 > 0$ such that for any $1 \leq i \neq j \leq n$, when $x_i > y_0$ and $x_j > y_0$,

$$(2.4) \quad P(|\xi_i| > x_i | \xi_j > x_j) < \varepsilon.$$

(1) We first prove (2.1). By (2.4), for any $1 \leq i \neq j \leq n$, when $x > by_0$ it holds uniformly for $\underline{c}_n \in (0, b]^n$ that

$$\begin{aligned} P(c_i \xi_i > x, c_j \xi_j > x) &= P(c_i \xi_i > x | c_j \xi_j > x) P(c_j \xi_j > x) \\ &\leq \varepsilon \sum_{k=1}^n P(c_k \xi_k > x). \end{aligned}$$

Since $\xi_k, 1 \leq k \leq n$, are nonnegative r.v.s, for any $\underline{c}_n \in (0, b]^n$ and $x > 0$,

$$\begin{aligned} P\left(\sum_{k=1}^n c_k \xi_k > x\right) &\geq P\left(\bigcup_{k=1}^n \{c_k \xi_k > x\}\right) \\ &\geq \sum_{k=1}^n P(c_k \xi_k > x) - \sum_{1 \leq i \neq j \leq n} P(c_i \xi_i > x, c_j \xi_j > x). \end{aligned}$$

Thus, by the above two formulae we know that (2.1) holds.

(2) We will prove (2.2). It follows from Lemma 2.3 that for any $\varepsilon > 0$ there exists $y_1 > 0$ such that for any $1 \leq j \leq n$, when $x_i > y_1$ and $x_j > y_1$,

$$(2.5) \quad P\left(\sum_{i=1, i \neq j}^n |\xi_i| > x_i \mid \xi_j > x_j\right) < \varepsilon.$$

For any $x > 0$ and any fixed constant $L_1 > by_1$,

$$\begin{aligned}
 P\left(\sum_{k=1}^n c_k \xi_k > x\right) &\geq \sum_{i=1}^n P\left(\sum_{k=1}^n c_k \xi_k > x, c_i \xi_i > x + L_1\right) \\
 &\quad - \sum_{1 \leq i < j \leq n} P(c_i \xi_i > x + L_1, c_j \xi_j > x + L_1) \\
 (2.6) \qquad \qquad \qquad &= K_1 - K_2.
 \end{aligned}$$

Letting x be sufficiently large such that $x + L_1 > by_0$, by (2.4) and the arbitrariness of ε , it holds uniformly for $\underline{c}_n \in (0, b]^n$ that

$$(2.7) \quad K_2 = o\left(\sum_{k=1}^n P(c_k \xi_k > x + L_1)\right) = o\left(\sum_{k=1}^n P(c_k \xi_k > x)\right).$$

For K_1 ,

$$\begin{aligned}
 K_1 &\geq \sum_{i=1}^n P\left(\sum_{k=1, k \neq i}^n c_k \xi_k > -L_1, c_i \xi_i > x + L_1\right) \\
 &= \sum_{i=1}^n P(c_i \xi_i > x + L_1) \\
 &\quad - \sum_{i=1}^n P\left(\sum_{k=1, k \neq i}^n c_k \xi_k \leq -L_1, c_i \xi_i > x + L_1\right) \\
 (2.8) \qquad \qquad \qquad &= K_{11} - K_{12}.
 \end{aligned}$$

Since $x + L_1 > L_1 > by_1$, by (2.5) it holds uniformly for $\underline{c}_n \in (0, b]^n$ that

$$\begin{aligned}
 K_{12} &\leq \sum_{i=1}^n P\left(\left|\sum_{k=1, k \neq i}^n c_k \xi_k\right| \geq L_1 \mid c_i \xi_i > x + L_1\right) \\
 &\qquad \qquad \qquad P(c_i \xi_i > x + L_1) \\
 &\leq \sum_{i=1}^n P\left(\sum_{k=1, k \neq i}^n |\xi_k| \geq b^{-1} L_1 \mid \xi_i > (x + L_1) c_i^{-1}\right) \\
 &\qquad \qquad \qquad P(c_i \xi_i > x + L_1) \\
 (2.9) \qquad \qquad \qquad &\leq \varepsilon \sum_{i=1}^n P(c_i \xi_i > x + L_1) = \varepsilon K_{11}.
 \end{aligned}$$

Hence, by (2.8), (2.9) since ε is arbitrary, it holds uniformly for $\underline{c}_n \in (0, b]^n$ that

$$K_1 \gtrsim \sum_{i=1}^n P(c_i \xi_i > x + L_1).$$

Since $V \in \mathcal{D}$, for any $\theta_1 > 1$, when x is sufficiently large, it holds uniformly for $\underline{c}_n \in (0, b]^n$ that

$$\begin{aligned} K_1 &\gtrsim \sum_{i=1}^n P(c_i \xi_i > x \theta_1) \\ &\gtrsim \bar{V}_*(\theta_1) \sum_{i=1}^n P(c_i \xi_i > x). \end{aligned}$$

Therefore,

$$\liminf_{x \rightarrow \infty} \inf_{\underline{c}_n \in (0, b]^n} \frac{K_1}{\sum_{i=1}^n P(c_i \xi_i > x)} \geq \lim_{\theta_1 \downarrow 1} \bar{V}_*(\theta_1) = L_V.$$

Combining with (2.6) and (2.7) we get that (2.2) holds.

Now we prove (2.3). For any $x > 0$ and any fixed constant $L_2 > bny_0$,

$$\begin{aligned} &P\left(\sum_{k=1}^n c_k \xi_k > x\right) \\ &\leq \sum_{k=1}^n P(c_k \xi_k > x - L_2) \\ &+ P\left(\sum_{k=1}^n c_k \xi_k > x, \bigcup_{k=1}^n \{c_k \xi_k > xn^{-1}\}, \bigcap_{k=1}^n \{c_k \xi_k \leq x - L_2\}\right) \\ (2.10) \quad &= J_1 + J_2. \end{aligned}$$

For J_1 , since $V \in \mathcal{D}$, for any $0 < \theta_2 < 1$, when x is sufficiently large, it holds uniformly for $\underline{c}_n \in (0, b]^n$ that

$$\begin{aligned} J_1 &\leq \sum_{k=1}^n P(c_k \xi_k > x \theta_2) \\ &\lesssim (\bar{V}_*(\theta_2^{-1}))^{-1} \sum_{k=1}^n P(c_k \xi_k > x). \end{aligned}$$

Thus,

$$\begin{aligned}
 & \limsup_{x \rightarrow \infty} \sup_{\underline{c}_n \in (0, b]^n} \frac{J_1}{\sum_{k=1}^n P(c_k \xi_k > x)} \\
 (2.11) \quad & \leq \lim_{\theta_2 \uparrow 1} (\bar{V}_*(\theta_2^{-1}))^{-1} = L_V^{-1}.
 \end{aligned}$$

For J_2 , since $L_2 > bny_0$, by (2.4) when x is sufficiently large, it holds uniformly for $\underline{c}_n \in (0, b]^n$ that

$$\begin{aligned}
 J_2 & \leq \sum_{i=1}^n P \left(\sum_{k=1}^n c_k \xi_k - c_i \xi_i > L_2, c_i \xi_i > xn^{-1} \right) \\
 & \leq \sum_{i=1}^n \sum_{k=1, k \neq i}^n P(c_k \xi_k > L_2(n-1)^{-1}, c_i \xi_i > xn^{-1}) \\
 & = \sum_{i=1}^n \sum_{k=1, k \neq i}^n P(\xi_k > L_2 c_k^{-1} (n-1)^{-1} | \xi_i > xc_i^{-1} n^{-1}) P(c_i \xi_i > xn^{-1}) \\
 & \leq \varepsilon(n-1) \sum_{i=1}^n P(c_i \xi_i > xn^{-1}).
 \end{aligned}$$

Hence, by $V \in \mathcal{D}$,

$$\begin{aligned}
 & \limsup_{x \rightarrow \infty} \sup_{\underline{c}_n \in (0, b]^n} \frac{J_2}{\sum_{k=1}^n P(c_k \xi_k > x)} \\
 & \leq \lim_{\varepsilon \downarrow 0} \limsup_{x \rightarrow \infty} \sup_{\underline{c}_n \in (0, b]^n} \frac{\varepsilon(n-1) \sum_{k=1}^n P(c_k \xi_k > xn^{-1})}{\sum_{k=1}^n P(c_k \xi_k > x)} \\
 & = 0.
 \end{aligned}$$

Combining with (2.10) and (2.11), we get (2.3). □

Now we prove Theorem 1.2.

Proof. We first prove that

$$(2.12) \quad \liminf_{x \rightarrow \infty} \frac{P(\sum_{k=1}^{\infty} X_k e^{-\delta \tau_k} > x)}{\sum_{k=1}^{\infty} P(X_k e^{-\delta \tau_k} > x)} \geq 1$$

and

$$(2.13) \quad \limsup_{x \rightarrow \infty} \frac{P(\sum_{k=1}^{\infty} X_k e^{-\delta \tau_k} > x)}{\sum_{k=1}^{\infty} P(X_k e^{-\delta \tau_k} > x)} \leq L_F^{-2}.$$

For any positive integer n , it follows from Lemma 2.4 that

$$\begin{aligned}
 & P\left(\sum_{k=1}^{\infty} X_k e^{-\delta\tau_k} > x\right) \geq P\left(\sum_{k=1}^n X_k e^{-\delta\tau_k} > x\right) \\
 &= \int_{\{0 \leq t_1 \leq \dots \leq t_n < \infty\}} P\left(\sum_{k=1}^n X_k e^{-\delta t_k} > x\right) \\
 &\quad P(\tau_1 \in dt_1, \dots, \tau_n \in dt_n) \\
 &\gtrsim \sum_{k=1}^n \int_{\{0 \leq t_1 \leq \dots \leq t_n < \infty\}} P\left(X_k e^{-\delta t_k} > x\right) \\
 &\quad P(\tau_1 \in dt_1, \dots, \tau_n \in dt_n) \\
 &= \sum_{k=1}^n P\left(X_k e^{-\delta\tau_k} > x\right) \\
 (2.14) \quad &= \left(\sum_{k=1}^{\infty} - \sum_{k=n+1}^{\infty}\right) P\left(X_k e^{-\delta\tau_k} > x\right).
 \end{aligned}$$

By Lemma 2.2, when x is sufficiently large,

$$\begin{aligned}
 \sum_{k=n+1}^{\infty} P\left(X_k e^{-\delta\tau_k} > x\right) &= \sum_{k=n+1}^{\infty} \int_{0-}^{\infty} \bar{F}(xe^{\delta u}) P(\tau_k \in du) \\
 &\leq \sum_{k=n+1}^{\infty} \int_{0-}^{\infty} C_1^{-1} e^{-\delta\alpha u} \bar{F}(x) P(\tau_k \in du) \\
 &= C_1^{-1} \bar{F}(x) \sum_{k=n+1}^{\infty} E e^{-\delta\alpha\tau_k}.
 \end{aligned}$$

On the other hand, when x is sufficiently large,

$$\begin{aligned}
 \sum_{k=1}^{\infty} P\left(X_k e^{-\delta\tau_k} > x\right) &\geq \int_{0-}^{\infty} \bar{F}(xe^{\delta u}) P(\tau_1 \in du) \\
 (2.15) \quad &\geq C_2^{-1} \bar{F}(x) E e^{-\delta\beta\tau_1}.
 \end{aligned}$$

Hence, by Lemma 2.1 and (1.7),

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{\sum_{k=n+1}^{\infty} P(X_k e^{-\delta \tau_k} > x)}{\sum_{k=1}^{\infty} P(X_k e^{-\delta \tau_k} > x)} \\
 & \leq C_2 C_1^{-1} (E e^{-\delta \beta \tau_1})^{-1} \lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} g_L(k) (E e^{-\delta \alpha \tau_1})^k \\
 (2.16) \quad & = 0,
 \end{aligned}$$

which, combining with (2.14), yields that (2.12) holds.

We will prove (2.13). For any $0 < \theta < 1$ and any positive integer n ,

$$\begin{aligned}
 P\left(\sum_{k=1}^{\infty} X_k e^{-\delta \tau_k} > x\right) & \leq P\left(\sum_{k=1}^n X_k e^{-\delta \tau_k} > (1-\theta)x\right) \\
 (2.17) \quad & + P\left(\sum_{k=n+1}^{\infty} X_k e^{-\delta \tau_k} > \theta x\right).
 \end{aligned}$$

By Lemma 2.4,

$$\begin{aligned}
 & P\left(\sum_{k=1}^n X_k e^{-\delta \tau_k} > (1-\theta)x\right) \\
 & \lesssim L_F^{-1} \sum_{k=1}^n P\left(X_k e^{-\delta \tau_k} > (1-\theta)x\right) \\
 & = L_F^{-1} \sum_{k=1}^n \int_{0-}^{\infty} \bar{F}((1-\theta)x e^{\delta u}) P(\tau_k \in du) \\
 & \lesssim L_F^{-1} \sum_{k=1}^n \int_{0-}^{\infty} (\bar{F}_*((1-\theta)^{-1}))^{-1} \bar{F}(x e^{\delta u}) P(\tau_k \in du) \\
 & \leq L_F^{-1} (\bar{F}_*((1-\theta)^{-1}))^{-1} \sum_{k=1}^{\infty} P\left(X_k e^{-\delta \tau_k} > x\right).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 & \lim_{\theta \downarrow 0} \limsup_{x \rightarrow \infty} \frac{P(\sum_{k=1}^n X_k e^{-\delta \tau_k} > (1-\theta)x)}{\sum_{k=1}^{\infty} P(X_k e^{-\delta \tau_k} > x)} \\
 & \leq L_F^{-1} \lim_{\theta \downarrow 0} (\bar{F}_*((1-\theta)^{-1}))^{-1} \\
 (2.18) \quad & = L_F^{-2}.
 \end{aligned}$$

In order to prove (2.13), we need to prove that

$$(2.19) \quad \lim_{n \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{P\left(\sum_{k=n+1}^{\infty} X_k e^{-\delta\tau_k} > \theta x\right)}{\sum_{k=1}^{\infty} P(X_k e^{-\delta\tau_k} > x)} = 0.$$

Since $F \in \mathcal{D}$, by (2.15) we know that (2.19) holds if the following relation is satisfied:

$$(2.20) \quad \lim_{n \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{P\left(\sum_{k=n+1}^{\infty} X_k e^{-\delta\tau_k} > \theta x\right)}{\bar{F}(\theta x)} = 0.$$

We will prove (2.20) along the line of the proof of (4.5) in [3]. For any positive integer n satisfying $\sum_{k=n+1}^{\infty} k^{-2} < 1$, we have

$$P\left(\sum_{k=n+1}^{\infty} X_k e^{-\delta\tau_k} > \theta x\right) \leq \sum_{k=n+1}^{\infty} P(X_k e^{-\delta\tau_k} > \theta x k^{-2}).$$

Since $F \in \mathcal{D}$ and $J_F^- > 0$, for some $0 < \alpha < J_F^- \leq J_F^+ < \beta < \infty$, the results of Lemma 2.2 hold for $V = F$. Let $A_1(k, x) = \{k^{-2} e^{\delta\tau_k} \leq D_2(\theta x)^{-1}\}$, $A_2(k, x) = \{D_2(\theta x)^{-1} < k^{-2} e^{\delta\tau_k} \leq 1\}$ and $A_3(k, x) = \{k^{-2} e^{\delta\tau_k} > 1\}$. Hence,

$$(2.21) \quad \begin{aligned} P\left(\sum_{k=n+1}^{\infty} X_k e^{-\delta\tau_k} > \theta x\right) &\leq \sum_{j=1}^3 \sum_{k=n+1}^{\infty} P(X_k e^{-\delta\tau_k} > \theta x k^{-2}, A_j(k, x)) \\ &= \sum_{j=1}^3 I_j(n, x). \end{aligned}$$

By Markov's inequality, Lemmas 2.1 and 2.2 and (1.7), we have

$$(2.22) \quad \begin{aligned} I_1(n, x) &\leq D_2^\beta(\theta x)^{-\beta} \sum_{k=n+1}^{\infty} k^{2\beta} g_L(k) (E e^{-\beta\delta\tau_1})^k \\ &= o(\bar{F}(\theta x)). \end{aligned}$$

Still by Markov's inequality, Lemmas 2.1 and 2.2, when x is sufficiently large,

$$I_2(n, x) \leq C_2 \bar{F}(\theta x) \sum_{k=n+1}^{\infty} k^{2\beta} g_L(k) (E e^{-\beta\delta\tau_1})^k$$

and

$$I_3(n, x) \leq C_1^{-1} \bar{F}(\theta x) \sum_{k=n+1}^{\infty} k^{2\alpha} g_L(k) (E e^{-\alpha\delta\tau_1})^k.$$

Hence, by (1.7),

$$(2.23) \quad \lim_{n \rightarrow \infty} \limsup_{x \rightarrow \infty} \frac{I_j(n, x)}{\bar{F}(\theta x)} = 0, j = 2, 3.$$

By (2.21)-(2.23) we know that (2.20) holds.

Thus, it follows from (2.17)-(2.19) that (2.13) holds.

Now we prove (1.6). Since

$$\sum_{k=1}^{\infty} P\left(X_k e^{-\delta\tau_k} > x\right) = \int_{0-}^{\infty} \bar{F}(xe^{\delta u})\lambda(du),$$

by (1.1) and (2.13) we get

$$\lim_{x \rightarrow \infty} \frac{\psi(x)}{\int_{0-}^{\infty} \bar{F}(xe^{\delta u})\lambda(du)} \leq L_F^{-2}.$$

On the other hand, let $Z = \tilde{C}(\infty)$, for any fixed $T > 0$, it follows from (1.1) and (2.12) that

$$\begin{aligned} \psi(x) &\geq P\left(\sum_{k=1}^{\infty} X_k e^{-\delta\tau_k} > x + Z\right) \\ &\geq \int_0^T P\left(\sum_{k=1}^{\infty} X_k e^{-\delta\tau_k} > x + z\right) P(Z \in dz) \\ &\gtrsim \int_0^T \sum_{k=1}^{\infty} P\left(X_k e^{-\delta\tau_k} > x + z\right) P(Z \in dz) \\ &\geq \sum_{k=1}^{\infty} P\left(X_k e^{-\delta\tau_k} > x + T\right) P(Z \leq T). \end{aligned}$$

For any $r > 1$, when x is sufficiently large, it holds uniformly for $k \geq 1$ that

$$\begin{aligned} P\left(X_k e^{-\delta\tau_k} > x + T\right) &\geq P(X_k e^{-\delta\tau_k} > rx) \\ &= \int_{0-}^{\infty} \bar{F}(rx e^{\delta u}) P(\tau_k \in du) \\ &\gtrsim \bar{F}_*(r) P(X_k e^{-\delta\tau_k} > x). \end{aligned}$$

Hence,

$$\begin{aligned} \liminf_{x \rightarrow \infty} \frac{\psi(x)}{\int_{0-}^{\infty} \overline{F}(xe^{\delta u}) \lambda(du)} &\geq \lim_{T \rightarrow \infty} \lim_{r \downarrow 1} \overline{F}_*(r) P(Z \leq T) \\ &= L_F. \end{aligned}$$

This completes the proof of (1.6). □

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