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n-COCOHERENT RINGS, n-COSEMIHEREDITARY RINGS AND n-V-RINGS

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Abstract. Let R be a ring, and let n, d be non-negative integers. A right R-module M is called (n, d)-projective if $Ext_R^{d+1}(M, A) = 0$ for every n-copresented right R-module A. R is called right n-cocoherent if every n-copresented right R-module is $(n+1)$-copresented, it is called a right co-(n, d)-ring if every right R-module is (n, d)-projective. R is called right n-cosemihereditary if every submodule of a projective right R-module is $(n, 0)$-projective, it is called a right n-V-ring if it is a right co-$(n, 0)$-ring. Some properties of (n, d)-projective modules and (n, d)-projective dimensions of modules over n-cocoherent rings are studied. Certain characterizations of n-copresented modules, $(n, 0)$-projective modules, right n-cocoherent rings, right n-cosemihereditary rings, as well as right n-V-rings are given respectively.
Keywords: (n, d)-projective module; n-cocoherent ring; co-(n, d)-ring; n-cosemihereditary ring; n-V-ring.
MSC(2010): Primary: 16D40; Secondary: 16E10, 16E60.

1. Introduction and preliminaries

Throughout this paper, R is an associative ring with identity and all modules are unitary.

First we recall some known notions and facts needed in the sequel. Let R be a ring, n, d non-negative integers and M a right R-module. Then:

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(1) M is called *n-presented* [1] if there is an exact sequence of right R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ where each F_i is a finitely generated free, equivalently projective, right R-module.

(2) R is called *right n-coherent* [1] if every n-presented right R-module is $(n+1)$-presented.

(3) M is called *(n,d)-injective* [13] if $\mathrm{Ext}_R^{d+1}(A,M)=0$ for every n-presented right R-module A.

(4) R is called a *right (n,d)-ring* [13] if every n-presented right R-module has the projective dimension at most d, or equivalently, if every right R-module is (n,d)-injective. We note that a commutative right (n,d)-ring is called an (n,d)-ring in [1]. Right n-coherent rings and right (n,d)-rings have been studied by several authors (see, for example, [1, 2, 5, 6, 7, 13, 15]).

(5) M is said to be *cofree* [3] if it is isomorphic to a direct product of the injective hulls of some simple right R-modules.

(6) M is said to be *finitely corelated* [3] if there is a short exact sequence $0 \to M \to N \to A \to 0$ of right R-modules, where N is finitely cogenerated, cofree, and A is finitely cogenerated. It is easy to see that M is finitely corelated if and only if there exists a short exact sequence of right R-modules $0 \to M \to E_0 \to E_1$, where each E_i is a finitely cogenerated injective module. Finitely corelated modules are also called finitely copresented modules in some literatures such as [10].

(7) M is said to be *n-copresented* [12] if there is an exact sequence of right R-modules $0 \to M \to E_0 \to E_1 \to \cdots \to E_n$, where each E_i is a finitely cogenerated injective module.

(8) R is called right *co-semihereditary* [8, 11, 16] if every finitely cogenerated factor module of a finitely cogenerated injective right R-module is injective.

(9) R is called right *co-coherent* (cocoherent) [16] if every finitely cogenerated factor module of a finitely cogenerated injective right R-module is finitely copresented.

(10) R is called right *n-cocoherent* [12] in case every n-copresented right R-module is $(n+1)$-copresented. It is easy to see that R is right cocoherent if and only if it is right 1-cocoherent. Recall that a ring R is called *right co-noethrian* [3] if every factor module of a finitely cogenerated right R-module is finitely cogenerated. By [3, Proposition 17], a ring R is right co-noethrian if and only if it is right 0-cocoherent.

In this paper, we shall introduce the dual concepts of (n,d)-injective right R-modules and right (n,d)-rings, respectively. We shall call a right R-module M (n,d)-projective if $Ext_R^{d+1}(M,A)=0$ for every n-copresented right R-module A, and we shall call a ring R *right co-(n,d)-ring* if every right R-module is (n,d)-projective. Some characterizations and properties of (n,d)-projective modules will be provided and (n,d)-projective dimensions of right R-modules over right n-cocoherent rings will be discussed. Moreover, the concepts of right n-cosemihereditary rings and right n-V-rings will be introduced and right n-cosemihereditary rings and right n-V-rings will be characterized by $(n,0)$-projective right R-modules.

Lemma 1.1. *Let A, B be two right R-modules and let n be a non-negative integer. Then $A\oplus B$ is n-copresented if and only if both A and B are n-copresented.*

Proof. Assume that A and B are n-copresented. Then there exist two exact sequence of right R-modules

$$0\to A\overset{\alpha}{\to}E_0\overset{f_0}{\to}E_1\overset{f_1}{\to}\cdots\to E_{n-1}\overset{f_{n-1}}{\to}E_n$$

and

$$0\to B\overset{\beta}{\to}E_0'\overset{g_0}{\to}E_1'\overset{g_1}{\to}\cdots\to E_{n-1}'\overset{g_{n-1}}{\to}E_n'$$

where E_i and E_i' are finitely cogenerated injective modules for all i. Thus, we obtain an exact sequence of right R-modules

$$0\to A\oplus B\overset{\alpha\oplus\beta}{\to}E_0\oplus E_0'\overset{f_0\oplus g_0}{\to}E_1\oplus E_1'\overset{f_1\oplus g_1}{\to}\cdots$$
$$\to E_{n-1}\oplus E_{n-1}'\overset{f_{n-1}\oplus g_{n-1}}{\to}E_n\oplus E_n'$$

where each $E_i\oplus E_i'$ is a finitely cogenerated injective module. Thus $A\oplus B$ is n-copresented.

Conversely, suppose that $A\oplus B$ is n-copresented. Then there exists an exact sequence of right R-modules

$$0\to A\oplus B\overset{\varepsilon}{\to}E_0\overset{d_0}{\to}E_1\overset{d_1}{\to}\cdots\to E_{n-1}\overset{d_{n-1}}{\to}E_n$$

where each E_i is a finitely cogenerated injective module. Hence we have an exact sequence of right R-modules

$$0\to A\overset{\varepsilon}{\to}E(\varepsilon(A))\overset{d_0i_0}{\to}E(Im(d_0i_0))\overset{d_1i_1}{\to}E(Im(d_1i_1))\to\cdots$$

$$\to E(Im(d_{n-2}i_{n-2})) \overset{d_{n-1}i_{n-1}}{\to} E(Im(d_{n-1}i_{n-1}))$$

where $E(\varepsilon(A))$ is a direct summand of E_0, $E(Im(d_k i_k))$ is a direct summand of E_{k+1}, i_0 is the natural injection from $E(\varepsilon(A))$ to E_0 and i_k is the natural injection from $E(Im(d_k i_k))$ to E_{k+1} for each $k = 0, \cdots, n-1$. Therefore, A is n-copresented. □

Now, we give some characterizations of n-copresented modules.

Proposition 1.2. *Let n be a positive integer. Then the following statements are equivalent for a right R-module M:*

(1) M is n-copresented.

(2) There exists an exact sequence of right R-modules

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to L \to 0$$

where $E_0, \cdots, E_{n-1}$ are finitely cogenerated injective modules and L is finitely cogenerated.

(3) M is (n-1)-copresented and, if the sequence of right R-modules

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to L \to 0$$

is exact, where $E_0, \cdots, E_{n-1}$ are finitely cogenerated injective modules, then L is finitely cogenerated.

(4) There exists an exact sequence of right R-modules

$$0 \to M \to E \to L \to 0$$

where E is finitely cogenerated injective and L is $(n-1)$-copresented.

(5) M is finitely cogenerated and, if the sequence of right R-modules

$$0 \to M \to E \to L \to 0$$

is exact with E finitely cogenerated injective, then L is $(n-1)$-copresented.

Proof. (1) $\Rightarrow$ (2). Since M is n-copresented, there exists an exact sequence of right R-modules $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \xrightarrow{f} E_n$, where each E_i is finitely cogenerated injective. Let $L = Im(f)$. Then L is finitely cogenerated and the sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to L \to 0$ is exact.

(2) $\Rightarrow$ (3). Follows by the dual theorem of the generalization of Schanuel's Lemma [9, Exercise 3.37].

(3) $\Rightarrow$ (1). Since M is $(n-1)$-copresented, there exists an exact sequence of right R-modules $0 \to M \to E_0 \to E_1 \to \cdots \xrightarrow{g} E_{n-1}$, where $E_0, E_1, \cdots, E_{n-1}$ are finitely cogenerated injective modules. Let $L = E_{n-1}/Im(g)$. Then by (3), L is finitely cogenerated. Let $E_n = E(L)$.

Then we get an exact sequence of right R-modules $0 \to M \to E_0 \to E_1 \to \cdots \to E_n$ with each E_i finitely cogenerated injective. Therefore, M is n-copresented.

(1) $\Rightarrow$ (4). Since M is n-copresented, there exists an exact sequence of right R-modules $0 \to M \to E \xrightarrow{\alpha} E_1 \to \cdots \to E_{n-1} \to E_n$, where $E, E_1, \cdots, E_{n-1}$ are finitely cogenerated injective modules. Let $L = Im(\alpha)$. Then it is easy to see that L is $(n-1)$-copresented, and the sequence $0 \to M \to E \to L \to 0$ is exact.

(4) $\Rightarrow$ (5). Foloows by the dual theorem of Schanuel's Lemma and Lemma 1.1.

(5) $\Rightarrow$ (1). Since M is finitely cogenerated, $E(M)$ is finitely cogenerated injective. By (5), $E(M)/M$ is $(n-1)$-copresented, and so there exists an exact sequence of right R-modules $0 \to E(M)/M \xrightarrow{h} E_1 \to \cdots \to E_{n-1} \to E_n$ with each E_i finitely cogenerated injective. Thus we obtain an exact sequence of right R-modules $0 \to M \to E(M) \xrightarrow{h\pi} E_1 \to \cdots \to E_{n-1} \to E_n$, where π is the natural epimorphism of $E(M)$ onto $E(M)/M$, and hence M is n-copresented. $\square$

From Proposition 1.2(4), it is easy to see that right n-cocoherent ring is right $(n+1)$-cocoherent.

2. n-cocoherent rings and (n, d)-projective modules

We begin this section with some characterizations of right n-cocoherent rings.

Theorem 2.1. *The following statements are equivalent for a ring R:*

(1) R is right n-cocoherent.

(2) If the sequence

$$0 \to M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} E_{n-1} \xrightarrow{d_n} E_n \tag{2.1}$$

is exact, where each E_i is a finitely cogenerated injective right R-module, then there exists an exact sequence of right R-modules

$$0 \to M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} E_{n-1} \xrightarrow{d_n} E_n \xrightarrow{d_{n+1}} E_{n+1} \tag{2.2}$$

where each E_i is finitely cogenerated injective.

(3) Every $(n-1)$-copresented factor module of a finitely cogenerated injective right R-module is n-copresented.

Proof. (1) $\Rightarrow$ (2). By the exactness of (2.1), we have an exact sequence

$$0 \to M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} E_{n-1} \xrightarrow{d_n} E_n \to E_n/Im(d_n) \to 0.$$

Since R is right n-cocoherent, M is $(n+1)$-copresented. So by Proposition 1.2, $E_n/Im(d_n)$ is finitely cogenerated. Let $E_{n+1} = E(E_n/Im(d_n))$. Then (2.2) is exact.

(2) $\Rightarrow$ (1) is clear, and (1) $\Leftrightarrow$ (3) follows by Proposition 1.2. □

Definition 2.2. *Let n, d be non-negative integers. Then a right R-module M is called (n,d)-projective if $Ext_R^{d+1}(M,A) = 0$ for every n-copresented right R-module A.*

Recall that a module M_R is called *FCP-projective* [16] if $Ext_R^1(M,A) = 0$ for every finitely copresented right R-module A, and module M_R is called *FCG-projective* [14] if $Ext_R^1(M,A) = 0$ for every finitely cogenerated right R-module A. It is obvious that M is $(0,0)$-projective (respectively, $(1,0)$-projective) if and only if M is FCG-projective (respectively, FCP-projective). For a given d, every (m,d)-projective module is (n,d)-projective for every $m \leq n$.

Proposition 2.3. *Let $\{M_i\}_{i\in I}$ be a family of right R-modules. Then $\oplus_{i\in I} M_i$ is (n,d)-projective if and only if each M_i is (n,d)-projective.*

Proof. Follows by the isomorphism $Ext_R^{d+1}(\oplus_{i\in I} M_i, A) \cong \prod_{i\in I} Ext_R^{d+1}(M_i, A)$. □

Proposition 2.4. *Let P be a projective right R-module and let K be its submodule. If P/K is (n,d)-projective, then K is $(n+1,d)$-projective.*

Proof. Let A be an $(n+1)$-copresented right R-module. Then there exists an exact sequence $0 \to A \to E \to B \to 0$, where E is a finitely cogenerated injective module and B is n-copresented. Thus we get two exact sequences

$$0 = Ext_R^{d+1}(P,A) \to Ext_R^{d+1}(K,A) \to Ext_R^{d+2}(P/K,A) \to Ext_R^{d+2}(P,A) = 0$$

and

$$0 = Ext_R^{d+1}(P/K,E) \to Ext_R^{d+1}(P/K,B) \to Ext_R^{d+2}(P/K,A) \to Ext_R^{d+2}(P/K,E) = 0.$$

Hence $Ext_R^{d+1}(K,A) \cong Ext_R^{d+1}(P/K,B) = 0$, and it follows that K is $(n+1,d)$-projective. □

Recall that a short exact sequence of right R-modules $0 \to A \to B \to C \to 0$ is called *copure* [4] if every finitely copresented right R-module is injective with respect to the exact sequence, and a submodule A of a right R-module B is said to be copure in B if the exact sequence $0 \to A \to B \to B/A \to 0$ is copure.

Proposition 2.5. *Let $n \geq d+1$. Then every copure factor module of an (n,d)-projective module is (n,d)-projective. In particular, every copure factor module of an FCP-projective module is FCP-projective.*

Proof. Let N be a copure factor module of an (n,d)-projective module M. Then there exists a copure exact sequence of right R-modules $0 \to K \xrightarrow{f} M \to N \to 0$. For a given n-copresented module A with a finite n-coprentation $0 \to A \to E_0 \to E_1 \to \cdots \to E_n$, let $L = coker(E_{d-2} \to E_{d-1})$. Then since $n \geq d+1$, A is $(d+1)$-copresented, and so L is finitely copresented. Since $Ext_R^1(M,L) \cong Ext_R^{d+1}(M,A) = 0$, we have an exact sequence $Hom(M,L) \xrightarrow{f^*} Hom(K,L) \xrightarrow{\partial} Ext_R^1(N,L) \to 0$. Noting that f^* is epic because N is a copure factor module of M, we have that $\partial = 0$, and hence $Ext_R^1(N,L) = 0$. Thus, $Ext_R^{d+1}(N,A) \cong Ext_R^1(N,L) = 0$, as required. □

Definition 2.6. *A short exact sequence of right R-modules $0 \to A \to B \to C \to 0$ is called n-copure if every n-copresented right R-module is injective with respect to the exact sequence. A submodule A of a right R-module B is called n-copure in B if the exact sequence $0 \to A \to B \to B/A \to 0$ is n-copure.*

Next, we give some characterizations of $(n,0)$-projective modules.

Theorem 2.7. *Let n be a positive integer and let M be a right R-module. Then the following statements are equivalent:*

(1) M is $(n,0)$-projective.

(2) M is projective with respect to exact sequence $0 \to A \to B \to C \to 0$ of right R-modules with A n-copresented.

(3) If N is an $(n-1)$-copresented factor module of a finitely cogenerated injective right R-module E, then every right R-homomorphism f from M to N lifts to a homomorphism from M to E.

(4) Every exact sequence $0 \to M'' \to M' \to M \to 0$ is n-copure.

(5) There exists an n-copure exact sequence $0 \to K \to P \to M \to 0$ of right R-modules with P projective.

(6) There exists an n-copure exact sequence $0 \to K \to P \to M \to 0$ of right R-modules with P $(n,0)$-projective.

Proof. (1) $\Rightarrow$ (2). Follows by the exact sequence $Hom(M,B) \to Hom(M,C) \to Ext^1_R(M,A) = 0$.

(2) $\Rightarrow$ (3). Since the kernel of the natural epimorphism $E \to N$ is n-copresented, (3) follows immediately from (2).

(3) $\Rightarrow$ (1). For any n-copresented module A, there exists an exact sequence $0 \to A \to E \to N \to 0$, where E is a finitely cogenerated injective module and N is $(n-1)$-copresented. So we get an exact sequence $Hom(M,E) \to Hom(M,N) \to Ext^1_R(M,A) \to Ext^1_R(M,E) = 0$, and thus $Ext^1_R(M,A) = 0$ by (3).

(1) $\Rightarrow$ (4). Assume (1). Then we have an exact sequence

$$Hom(M',A) \to Hom(M'',A) \to Ext^1_R(M,A) = 0$$

for every n-copresented module A, and so (4) follows.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (1). By (6), we have an n-copure exact sequence $0 \to K \xrightarrow{f} P \to M \to 0$ of right R-modules with P $(n,0)$-projective, and so, for each n-copresented module A, we have an exact sequence $Hom(P,A) \xrightarrow{f^*} Hom(K,A) \to Ext^1_R(M,A) \to Ext^1_R(P,A) = 0$ with f^* epic. This implies that $Ext^1_R(M,A) = 0$, and (1) follows. □

Definition 2.8. *(1). The (n, d)-projective dimension of a module M_R is defined by*

$$(n,d)\text{-}pd(M_R) = inf\{k : Ext^{k+d+1}_R(M,A) = 0 \text{ for every } n\text{-copresented module } A\}$$

(2). The right (n, d)-projective global dimension of a ring R is defined by

$$r.(n,d)\text{-}PD(R) = sup\{(n,d)\text{-}pd(M) : M \text{ is a right } R\text{-module}\}$$

Lemma 2.9. *Let R be a right n-cocoherent ring and M a right R-module. Then the following statements are equivalent:*

(1) (n,d)-pd$(M) \leq k$.

(2) $Ext^{k+d+1}_R(M,A) = 0$ for every n-copresented right R-module A.

Proof. (1) $\Rightarrow$ (2). Use induction on k. Clear if (n,d)-$pd(M) = k$. Let (n,d)-$pd(M) \leq k-1$. Since A is n-copresented, there exists an exact sequence $0 \to A \to E \to N \to 0$, where E is a finitely cogenerated injective module and N is $(n-1)$-copresented. Since R is right n-cocoherent, by Theorem 2.1, N is n-copresented, and so $Ext^{k+d+1}_R(M,A) \cong Ext^{k+d}_R(M,N) = 0$ by induction hypothesis.

(2) $\Rightarrow$ (1) is clear. □

Corollary 2.10. *Let R be a right n-cocoherent ring and M_R $(n,\ d)$-projective. Then $Ext_R^{d+k}(M,A)=0$ for all n-copresented modules A and all positive integers k.*

Corollary 2.11. *Let R be a right n-cocoherent ring and M a right R-module. If the sequence $0\to P_k\overset{f_k}{\to}P_{k-1}\overset{f_{k-1}}{\to}\cdots\to P_0\overset{f_0}{\to}M\to 0$ is exact with $P_0,\cdots,P_{k-1}$ $(n,\ d)$-projective, then $Ext_R^{k+d+1}(M,A)\cong Ext_R^1(P_k,A)$ for any n-copresented right R-module A .*

Proof. Since R is right n-cocoherent and $P_0,P_1,\cdots,P_{k-1}$ are (n,d)-projective, by Corollary 2.10, we have

$$Ext_R^{k+d+1}(M,A)\cong Ext_R^{k+d}(ker(f_0),A)\cong Ext_R^{k+d-1}(ker(f_1),A)\cong\cdots\cong Ext_R^{d+1}(ker(f_{k-1},A))=Ext_R^{d+1}(P_k,A).$$

□

Theorem 2.12. *Let R be a right n-cocoherent ring, M a right R-module and k a non-negative integer. Then the following statements are equivalent:*

(1) (n,d)-$pd(M_R)\le k$.

(2) $Ext_R^{k+d+l}(M,A)=0$ for all n-copresented modules A and all positive integers l .

(3) $Ext_R^{k+d+1}(A,M)=0$ for all n-copresented modules A.

(4) If the sequence $0\to P_k\to P_{k-1}\to\cdots\to P_0\to M\to 0$ is exact with $P_0,\cdots,P_{k-1}$ (n,d)-projective, then P_k is also (n,d)-projective.

(5) There exists an exact sequence $0\to P_k\to P_{k-1}\to\cdots\to P_0\to M\to 0$ of right R-modules with $P_0,\cdots,P_{k-1},P_k$ (n,d)-projective.

Proof. (1) ⇒ (2). Assume (1), then (n,d)-$pd(M_R)\le k+l-1$, and so (2) follows from Lemma 2.9.

(2) ⇒ (3) and (4) ⇒ (5) are obvious. (3) ⇒ (4) and (5) ⇒ (1) follow by Corollary 2.11. □

3. n-cosemihereditary rings and n-V-rings

We set $\mu_R(M)$=sup$\{n\mid M$ has a finite n-copresentation$\}$, except that we set $\mu_R(M)=-1$ if M is not finitely cogenerated.

Definition 3.1. *Let R be a ring and n a non-negative integer. Then the right n-codimension of R is defined by*

$$r.n\text{-}codim(R)=sup\{id(M_R)\mid M\ \text{is an}\ n\text{-copresented right}\ R\text{-module}\}$$

Definition 3.2. *Let R be a ring and n, d non-negative integers. Then R is said to be a right co-(n,d)-ring if every right R-module is $(n,\ d)$-projective.*

It is easy to see that a ring R is a right co-(n,d)-ring if and only if every n-copresented right R-module has injective dimension at most d if and only if $r.n\text{-}codim(R) \leq d$. If $n \leq n'$ and $d \leq d'$, then every right co-(n,d)-ring is a right co-(n',d')-ring.

Lemma 3.3. *Let R be a ring and M an n-copresented right R-module. Then M is injective if and only if $Ext^1_R(A,M) = 0$ for all right R-modules A such that $\mu_R(A) \geq n-1$.*

Proof. The necessity is clear. To prove the sufficiency, let $0 \to M \to E \to N \to 0$ be exact with E finitely cogenerated injective module. Then N is $(n-1)$-copresented, so $Ext^1_R(N,M) = 0$ by hypothesis. It follows that $Hom_R(E,M) \to Hom_R(M,M)$ is surjective, so M is isomorphic to a direct summand of E, and hence M is injective. □

Lemma 3.4. *Let R be a ring and M an n-copresented right R-module. Then $id(M_R) \leq d$ if and only if $Ext^{d+1}_R(A,M) = 0$ for all right R-modules A such that $\mu_R(A) \geq n-(d+1)$.*

Proof. The necessity is clear. The sufficiency is obvious if $d \geq n$. If $d < n$, then since M is n-copresented, there exists an exact sequence of right R-modules $0 \to M \overset{\lambda}{\to} E_0 \overset{f_0}{\to} E_1 \overset{f_1}{\to} \cdots \overset{f_{d-2}}{\to} E_{d-1} \overset{f_{d-1}}{\to} im(f_{d-1}) \to 0$, where each E_i is a finitely cogenerated injective module and $im(f_{d-1})$ is $(n-d)$-copresented. Thus for each right R-module A such that $\mu_R(A) \geq n-(d+1)$, we have $Ext^1_R(A, im(f_{d-1})) \cong Ext^{d+1}_R(A,M) = 0$ by hypothesis. It follows that $im(f_{d-1})$ is injective Follows by Lemma 3.3, and therefore $id(M_R) \leq d$. □

Theorem 3.5. *Let $n, d \geq 1$. Then the following statements are equivalent for a ring R:*

(1) R is a right co-(n,d)-ring.

(2) $Ext^{d+1}_R(M,N) = 0$ for all right R-modules M, N such that $\mu_R(N) \geq n$ and $\mu_R(M) \geq n-(d+1)$.

Proof. (1) $\Rightarrow$ (2) is clear. (2) $\Rightarrow$ (1) Follows by Lemma 3.4. □

Definition 3.6. *A ring R is called right n-cosemihereditary, if every submodule of a projective right R-module is $(n,\ 0)$-projective.*

Clearly, a ring R is right cosemihereditary if and only if it is right 1-cosemihereditary, and right n-cosemihereditary ring is right $(n+1)$-cosemihe-reditary.

Theorem 3.7. *Let $n \geq 1$. Then the following statements are equivalent for a ring R:*

(1) R is a right n-cosemihereditary ring.

(2) R is a right co-(n,1)-ring.

(3) $Ext^2_R(M,N) = 0$ for all right R-modules M, N such that $\mu_R(N) \geq n$ and $\mu_R(M) \geq n-2$.

(4) Every (n-1)-copresented factor module of a finitely cogenerated injective right R-module is injective.

(5) R is right n-cocoherent and $r.(n,0)$-$PD(R) \leq 1$.

(6) Every submodule of an (n, 0)-projective right R-module is (n, 0)-projective.

Proof. (1) $\Rightarrow$ (2). Let A be any right R-module and M any n-copresented right R-module. Then there exists an exact sequence $0 \to K \to P \to A \to 0$ with P projective. By (1), K is $(n,0)$-projective, thus we have an exact sequence $0 = Ext^1_R(K,M) \to Ext^2_R(A,M) \to Ext^2_R(P,M) = 0$. And so $Ext^2_R(A,M) = 0$, as required.

(2) $\Leftrightarrow$ (3) Follows by Theorem 3.5.

(2) $\Rightarrow$ (4). Let N be an $(n-1)$-copresented factor module of a finitely cogenerated injective right R-module E. Then there exists an exact sequence of right R-modules $0 \to K \to E \to N \to 0$. Since K is n-copresented, by (2), $Ext^2_R(M,K) = 0$ for every right R-module M. And so $Ext^1_R(M,N) = 0$ for every right R-module M because the sequence $0 = Ext^1_R(M,E) \to Ext^1_R(M,N) \to Ext^2_R(M,K) = 0$ is exact, as requiried.

(4) $\Rightarrow$ (5). By (4), every $(n-1)$-copresented factor module of a finitely cogenerated injective right R-module is injective, and hence n-copresented, so R is right n-cocoherent by Theorem 2.1. Now let A be an n-copresented right R-module. Then we have an exact sequence $0 \to A \to E \to L \to 0$ of right R-modules, where E is finitely cogenerated and injective, L is $(n-1)$-copresented. By hypothesis, L is injective. So, for any right R-module M, the exact sequence $0 = Ext^1_R(M,L) \to Ext^2_R(M,A) \to Ext^2_R(M,E) = 0$ implies that $Ext^2_R(M,A) = 0$. It shows that $r.(n,0)$-$PD(R) \leq 1$.

(5) $\Rightarrow$ (6). Let M be an $(n,0)$-projective right R-module and K its submodule. Then for any n-copresented module A, we have an exact

sequence $0 = Ext^1_R(M, A) \to Ext^1_R(K, A) \to Ext^2_R(M/K, A) = 0$ by (5) and Lemma 2.9. It follows that $Ext^1_R(K, A) = 0$, and so K is $(n, 0)$-projective.

(6) $\Rightarrow$ (1). It is obvious. □

Next, we generalize the concept of right V-rings to right n-V-rings.

Definition 3.8. *A ring R is called right n-V-ring if it is a right co-(n,0)-ring.*

Clearly, R is a right V-ring if and only if it is a right 1-V-ring, and a right n-V-ring is a right $(n+1)$-V-ring.

Theorem 3.9. *The following conditions are equivalent for a ring R:*

(1) R is a right n-V-ring.

(2) Every right R-module is $(n, 0)$-projective.

(3) Every finitely cogenerated right R-module is $(n, 0)$-projective.

(4) R is right n-cosemihereditary and $E(S)$ is $(n, 0)$-projective for every simple right R-module S.

(5) R is right n-cocoherent and every n-copresented right R-module is $(n, 0)$-projective.

(6) Every n-copresented right R-module is injective.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Assume (3). Then clearly $E(S)$ is $(n, 0)$-projective. Let E be a finitely cogenerated injective module and N an $(n-1)$-copresented factor module of E. By (3), N is $(n, 0)$-projective, so by Theorem 2.7(3), N is isomorphic to a direct summand of E and hence N is injective. Therefore, R is right n-cosemihereditary by Theorem 3.7.

(4) $\Rightarrow$ (5). Assume (4). Since R is right n-cosemihereditary, it is right n-cocoherent by Theorem 3.7. Now let M be an n-copresented right R-module, then there exists an exact sequence of right R-modules $0 \to M \to E$, where E is finitely cogenerated injective module. Since $E \cong \oplus_{i=1}^{k} E(S_i)$ for some simple modules $E_i, i = 1, 2, \cdots, k$ and each E_i is $(n, 0)$-projective, by Proposition 2.3, E is $(n, 0)$-projective. Observing that R is right n-cosemihereditary, by Theorem 3.7, M is $(n, 0)$-projective.

(5) $\Rightarrow$ (6). Let M be an n-copresented right R-module. Since R is right n-cocoherent, and M is $(n+1)$-copresented, so there exists an exact sequence $0 \to M \to E \to N \to 0$ of right R-modules, where E is finitely cogenerated injective, and N is n-copresented. By hypothesis,

N is $(n, 0)$-projective, so N is projective with respect to this exact sequence. This follows that M is isomorphic to a direct summand of E, and therefore M is injective.

(6) $\Rightarrow$ (1). It is clear.

□

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