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**A modified Mann iterative scheme for a sequence of nonexpansive mappings and a monotone mapping with applications**

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## A MODIFIED MANN ITERATIVE SCHEME FOR A SEQUENCE OF NONEXPANSIVE MAPPINGS AND A MONOTONE MAPPING WITH APPLICATIONS

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**ABSTRACT.** In a real Hilbert space, an iterative scheme is considered to obtain strong convergence which is an essential tool to find a common fixed point for a countable family of nonexpansive mappings and the solution of a variational inequality problem governed by a monotone mapping. In this paper, we give a procedure which results in developing Shehu's result to solve equilibrium problem. Then, we state more applications of this procedure. Finally, we investigate some numerical examples which hold in our main results.

**Keywords:** Equilibrium problem, maximal monotone operator, strictly pseudocontractive mapping,  $W$ -mapping.

**MSC(2010):** Primary: 47H09; Secondary: 47J20, 49J30.

### 1. Introduction

Let  $H$  be a real Hilbert space and let  $C$  be a nonempty closed convex subset of  $H$ . The strong (weak) convergence of  $\{x_n\}$  to  $x$  is written by  $x_n \rightarrow x$  ( $x_n \rightharpoonup x$ ) as  $n \rightarrow \infty$ .

A mapping  $S$  from  $C$  into itself is called *nonexpansive* if  $\|Sx - Sy\| \leq \|x - y\|$ , for all  $x, y \in C$ .  $Fix(S) := \{x \in C : Sx = x\}$  is the set of fixed points of  $S$ . Note that  $Fix(S)$  is closed and convex if  $S$  is nonexpansive.

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It is known that  $H$  satisfies the Opial's condition [11]; for any sequence  $\{x_n\}$  with  $x_n \rightarrow x$ , the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every  $y \in H$  with  $x \neq y$ .

For any point  $x \in H$ , there exists a unique  $P_C x \in C$  such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

$P_C$  is called the *metric projection* of  $H$  onto  $C$ . We know that  $P_C$  is a nonexpansive mapping from  $H$  onto  $C$ . It is also known that  $P_C$  satisfies

$$(1.1) \quad \langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2,$$

for all  $x, y \in H$ . Furthermore,  $P_C x$  is characterized by the following properties:  $P_C x \in C$  and

$$\begin{aligned} \langle x - P_C x, P_C x - y \rangle &\geq 0, \\ \|x - y\|^2 &\geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \end{aligned}$$

for all  $y \in C$ .

Let  $A$  be a mapping of  $C$  into  $H$ . The variational inequality problem is to find a  $x \in C$  such that

$$(1.2) \quad \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C.$$

We shall denote the set of solutions of the variational inequality problem (1.2) by  $VI(C, A)$ . Then we have

$$(1.3) \quad x \in VI(C, A) \iff x = P_C(x - \lambda Ax), \quad \forall \lambda > 0.$$

A mapping  $A : C \rightarrow H$  is called *monotone* if  $\langle Ax - Ay, x - y \rangle \geq 0$  for all  $x, y \in C$ . It is called  *$\alpha$ -inverse strongly monotone* ( *$\alpha$ -strongly monotone*) if there exists a positive real number  $\alpha$  such that  $\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2$  ( $\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2$ ), for all  $x, y \in C$ . An  $\alpha$ -inverse strongly monotone mapping is sometimes called  *$\alpha$ -cocoercive*. A mapping  $A$  is said to be *relaxed  $\alpha$ -cocoercive* if there exists  $\alpha > 0$  such that

$$\langle Ax - Ay, x - y \rangle \geq -\alpha \|Ax - Ay\|^2,$$

for all  $x, y \in C$ . The mapping  $A$  is said to be *relaxed  $(\gamma, \delta)$ -cocoercive* if there exist  $\gamma, \delta > 0$  such that

$$\langle Ax - Ay, x - y \rangle \geq -\gamma \|Ax - Ay\|^2 + \delta \|x - y\|^2,$$

for all  $x, y \in C$ . A mapping  $A : H \rightarrow H$  is said to be  *$\mu$ -Lipschitzian* if there exists  $\mu \geq 0$  such that

$$\|Ax - Ay\| \leq \mu \|x - y\|,$$

for all  $x, y \in H$ . It is clear that each  $\alpha$ -inverse strongly monotone mapping is monotone and  $\frac{1}{\alpha}$ -Lipschitzian. Also, if  $A$  is an  $\alpha$ -inverse strongly monotone, then  $I - \lambda A$  is a nonexpansive mapping from  $C$  to  $H$ , provided that  $0 < \lambda \leq 2\alpha$ . In fact,

$$\begin{aligned}
 \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\
 (1.4) \qquad \qquad \qquad &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay \rangle \\
 &\quad + \lambda^2\|Ax - Ay\|^2 \\
 &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2,
 \end{aligned}$$

for all  $x, y \in C$  and  $\lambda > 0$ . In addition, each  $\mu$ -Lipschitzian, relaxed  $(\gamma, \delta)$ -cocoercive mapping is monotone, provided that  $\gamma\mu^2 \leq \delta$  (see [15]).

Let  $T$  be a set-valued operator with domain  $D(T) = \{x \in H : Tx \neq \emptyset\}$  and range  $R(T) = \{x \in H : x \in D(T)\}$ . An operator  $T$  is said to be *monotone* if  $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$  for each  $x_1, x_2 \in D(T)$  and  $y_1 \in Tx_1, y_2 \in Tx_2$ . A monotone operator  $T$  is said to be *maximal* if its graph  $G(T)$  is not properly contained in the graph of any other monotone operators, where  $G(T) := \{(x, y) \in H \times H : y \in Tx\}$ . It is known that  $T$  is maximal if and only if for  $(x, f) \in H \times H$  and for every  $(y, g) \in G(T)$  such that  $\langle x - y, f - g \rangle \geq 0$ , we would have  $f \in Tx$ . Let  $A$  be a monotone mapping from  $C$  into  $H$  and let  $N_C v$  be the normal cone to  $C$  at  $v \in C$ , i.e.,  $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$ , and define a mapping  $T$  by

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then  $T$  is maximal monotone and

$$(1.5) \qquad x \in VI(C, A) \iff x \in T^{-1}(0),$$

where  $T^{-1}(0) = \{x \in H : 0 \in Tx\}$  (see [14]).

In 1953, Mann [7] introduced the following iterative scheme for approximating a fixed point of  $S$ :

$x_1 \in C$  and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n,$$

for all  $n \in \mathbb{N}$ . He showed that the sequence  $\{x_n\}$  converges weakly to a fixed point of  $S$ . This iteration procedure is called a Mann type iteration. In 2003, Takahashi and Toyoda [22] suggested an iterative scheme, to find an element of  $Fix(S) \cap VI(C, A)$ , as follows:  $x_1 \in C$  and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n SP_C(I - \lambda_n A)x_n, \quad n \geq 1.$$

They obtained that the sequence  $\{x_n\}$  converges weakly to some  $\omega \in \text{Fix}(S) \cap \text{VI}(C, A)$ . Then, Wang and Hu [24] considered an iterative sequence  $\{x_n\}$  of  $C$  generated by  $x_1 \in C$  and

$$x_{n+1} := (1 - \alpha_n)x_n + \alpha_n S_n P_C(I - \lambda_n A)x_n, \quad n \geq 1,$$

for a countable family of nonexpansive mappings and a  $\mu$ -Lipschitzian and  $\alpha$ -strongly monotone operator. They proved that such a sequence converges strongly to a common fixed point of a countable family of nonexpansive mappings which solves the corresponding variational inequality.

On the other hand, in 2009, Yao et al. [27] showed that a new modified Mann iterative algorithm  $\{x_n\}$  generated iteratively by  $x_1 \in C$  and  $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S P_C(1 - \lambda_n)x_n$  ( $n \in \mathbb{N}$ ) strongly converges to a fixed point of  $S$ . Recently, Shehu [15] used an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of generalized mixed equilibrium problem and the set of solutions of the variational inequality problem of a cocoercive mapping. He gave a strong convergence theorem for this sequence.

It is worth pointing out that many authors extended the results in Hilbert space to the more general uniformly convex and uniformly smooth Banach space (see, for instance, [1, 2, 16, 28, 23]).

In this work, motivated and inspired by the above results, an iterative scheme generated by  $x_1 \in C$  and  $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S_n P_C[(1 - \lambda_n)(I - \lambda A)x_n]$  ( $n \in \mathbb{N}$ ) is utilized to find a common element of the set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of variational inequality for an  $\alpha$ -inverse strongly monotone mapping. Moreover, a strong convergence theorem is studied. As an application, an equilibrium problem is solved. Furthermore, it is shown that our results are improved and extended those of Shehu [15]. Then, the zeroes of a maximal monotone operator are investigated. In addition, a common fixed point for a family of strictly pseudocontractive mappings and  $W$ -mappings is obtained. Finally, some numerical examples are given.

The following lemmas will be useful in the sequel.

**Lemma 1.1.** (Suzuki [18]) *Let  $\{x_n\}$  and  $\{y_n\}$  be bounded sequences in a Banach space  $X$  and  $\{\alpha_n\}$  be a sequence in  $[0, 1]$  with  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ . Suppose  $x_{n+1} = (1 - \alpha_n)y_n + \alpha_n x_n$ , for all integers  $n \geq 1$ , and  $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$ . Then  $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$ .*

**Lemma 1.2.** (Xu [25]) Let  $\{a_n\}$  be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\gamma_n, \quad n \geq 0,$$

where

- (1)  $\{\alpha_n\}$  is a real sequence in  $(0, 1)$  such that  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (2)  $\gamma_n$  is a real sequence such that  $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$  or  $\sum_{n=1}^{\infty} |\alpha_n \gamma_n| < \infty$ .

Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 1.3.** (Aoyama et al. [2]) Let  $C$  be a nonempty closed subset of a Banach space and let  $\{S_n\}$  be a sequence of nonexpansive mappings from  $C$  into itself. Suppose that  $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_nx\| : x \in B\} < \infty$ , for any bounded subset  $B$  of  $C$ . Then, for each  $x \in C$ ,  $\{S_nx\}$  converges strongly to some points of  $C$ . If  $S$  is a mapping from  $C$  into itself which is defined by  $Sx := \lim_{n \rightarrow \infty} S_nx$ , for all  $x \in C$ , then  $\lim_{n \rightarrow \infty} \sup\{\|S_nx - Sx\| : x \in C\} = 0$ .

## 2. Iterative scheme and strong convergence

In this section, an iterative scheme is used to find a common element of the set of common fixed points of a countable family of nonexpansive mappings which is the solution of a variational inequality problem governed by an  $\alpha$ -inverse strongly monotone mapping.

**Theorem 2.1.** Suppose  $C$  is a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse strongly monotone mapping from  $C$  into  $H$  and  $\{S_n\}$  a sequence of nonexpansive mappings from  $C$  into itself such that  $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \text{VI}(C, A) \neq \emptyset$ . Suppose that  $\{\alpha_n\}$  and  $\{\lambda_n\}$  are two real sequences in  $(0, 1)$  and  $0 < \lambda \leq 2\alpha$ . Set  $x_1 \in C$  and let  $\{x_n\}$  be the iterative sequence defined by

$$\begin{cases} y_n := PC[(1 - \lambda_n)(I - \lambda A)x_n], \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n S_n y_n, \quad (n \geq 1), \end{cases}$$

satisfying the following conditions:

- (1)  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ ,
- (2)  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\sum_{n=1}^{\infty} \lambda_n = \infty$ ,

$$(3) \sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_nx\| : x \in B\} < \infty, \text{ for any bounded subset } B \text{ of } C.$$

Let  $S$  be a mapping from  $C$  into itself defined by  $Sx := \lim_{n \rightarrow \infty} S_nx$  for all  $x \in C$  and suppose that  $Fix(S) := \bigcap_{n=1}^{\infty} Fix(S_n)$ . Then  $\{x_n\}$  converges strongly to an element  $\omega \in Fix(S) \cap VI(C, A)$ , where  $\omega = P_{Fix(S) \cap VI(C, A)}0$ .

*Proof.* First of all, we prove that  $\{x_n\}$  is bounded. Let  $x \in Fix(S) \cap VI(C, A)$ . It follows from (1.3) and (1.4) that  $x = P_C(I - \lambda A)x$  and  $I - \lambda A$  is a nonexpansive mapping. Therefore,

$$\begin{aligned} \|y_n - x\| &= \|P_C[(1 - \lambda_n)(I - \lambda A)x_n] - P_C(I - \lambda A)x\| \\ &\leq \|(1 - \lambda_n)(I - \lambda A)x_n - (I - \lambda A)x\| \\ &\leq (1 - \lambda_n)\|(I - \lambda A)x_n - (I - \lambda A)x\| + \lambda_n\|(I - \lambda A)x\| \\ &\leq (1 - \lambda_n)\|x_n - x\| + \lambda_n\|(I - \lambda A)x\|. \end{aligned}$$

It implies that

$$\begin{aligned} \|x_{n+1} - x\| &= \|(1 - \alpha_n)x_n + \alpha_n S_n y_n - x\| \\ &\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n \|S_n y_n - x\| \\ &\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n \|y_n - x\| \\ &\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n (1 - \lambda_n)\|x_n - x\| \\ &\quad + \alpha_n \lambda_n \|(I - \lambda A)x\| \\ &= (1 - \alpha_n \lambda_n)\|x_n - x\| + \alpha_n \lambda_n \|(I - \lambda A)x\| \\ &\leq \max\{\|x_n - x\|, \|(I - \lambda A)x\|\} \\ &\quad \vdots \\ &\leq \max\{\|x_1 - x\|, \|(I - \lambda A)x\|\}. \end{aligned}$$

Hence,  $\{x_n\}$  is bounded and so are  $\{y_n\}$  and  $\{(I - \lambda A)x_n\}$ . Also, we note that

$$\begin{aligned} \|S_{n+1}y_{n+1} - S_{n+1}y_n\| &\leq \|y_{n+1} - y_n\| \\ &\leq \|(1 - \lambda_{n+1})(I - \lambda A)x_{n+1} - (1 - \lambda_n)(I - \lambda A)x_n\| \\ &\leq (1 - \lambda_{n+1})\|(I - \lambda A)x_{n+1} - (I - \lambda A)x_n\| + |\lambda_{n+1} - \lambda_n|\|(I - \lambda A)x_n\| \\ &\leq (1 - \lambda_{n+1})\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|(I - \lambda A)x_n\| \\ &\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|(I - \lambda A)x_n\|. \end{aligned}$$

Then

$$\begin{aligned} \|S_{n+1}y_{n+1} - S_n y_n\| &\leq \|S_{n+1}y_{n+1} - S_{n+1}y_n\| + \|S_{n+1}y_n - S_n y_n\| \\ &\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|(I - \lambda A)x_n\| \\ &\quad + \sup\{\|S_{n+1}x - S_n x\| : x \in \{y_n\}\}. \end{aligned}$$

Now, as  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_n x\| : x \in \{y_n\}\} < \infty$ , we obtain

$$\limsup_{n \rightarrow \infty} \|S_{n+1}y_{n+1} - S_n y_n\| - \|x_{n+1} - x_n\| \leq 0.$$

Using Lemma 1.1, we get that  $\lim_{n \rightarrow \infty} \|S_n y_n - x_n\| = 0$ . Thus

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \alpha_n \|S_n y_n - x_n\| = 0.$$

By convexity of  $\|\cdot\|^2$  and (1.3), we have

$$\begin{aligned} \|x_{n+1} - x\|^2 &\leq (1 - \alpha_n) \|x_n - x\|^2 + \alpha_n \|S_n y_n - x\|^2 \\ &\leq (1 - \alpha_n) \|x_n - x\|^2 + \alpha_n \|y_n - x\|^2 \\ &\leq (1 - \alpha_n) \|x_n - x\|^2 \\ &\quad + \alpha_n \|P_C[(1 - \lambda_n)(I - \lambda A)x_n] - P_C(I - \lambda A)x\|^2 \\ &\leq (1 - \alpha_n) \|x_n - x\|^2 \\ &\quad + \alpha_n \|(1 - \lambda_n)(I - \lambda A)x_n - (I - \lambda A)x\|^2 \\ &\leq (1 - \alpha_n) \|x_n - x\|^2 + \alpha_n \lambda_n \|(I - \lambda A)x\|^2 \\ &\quad + \alpha_n (1 - \lambda_n) \|(I - \lambda A)x_n - (I - \lambda A)x\|^2 \\ &\leq (1 - \alpha_n) \|x_n - x\|^2 + \alpha_n (1 - \lambda_n) \|x_n - x\|^2 \\ &\quad + \alpha_n (1 - \lambda_n) \lambda (\lambda - 2\alpha) \|Ax_n - Ax\|^2 \\ &\quad + \alpha_n \lambda_n \|(I - \lambda A)x\|^2 \\ &\leq \|x_n - x\|^2 + \alpha_n (1 - \lambda_n) \lambda (\lambda - 2\alpha) \|Ax_n - Ax\|^2 \\ &\quad + \alpha_n \lambda_n \|(I - \lambda A)x\|^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\alpha_n (1 - \lambda_n) \lambda (2\alpha - \lambda) \|Ax_n - Ax\|^2 \\ &\leq \|x_n - x\|^2 - \|x_{n+1} - x\|^2 + \alpha_n \lambda_n \|(I - \lambda A)x\|^2 \\ &\leq \|x_n - x_{n+1}\| [\|x_n - x\| + \|x_{n+1} - x\|] + \alpha_n \lambda_n \|(I - \lambda A)x\|^2. \end{aligned}$$



Since  $0 < \lambda \leq 2\alpha$ ,  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ , we obtain  $\lim_{n \rightarrow \infty} \|Ax_n - Ax\| = 0$ . From (1.1), we have

$$\begin{aligned} & \|y_n - x\|^2 = \|P_C[(1 - \lambda_n)(I - \lambda A)x_n] - P_C(I - \lambda A)x\|^2 \\ & \leq \langle (1 - \lambda_n)(I - \lambda A)x_n - (I - \lambda A)x, y_n - x \rangle \\ & = \frac{1}{2} \{ \|(1 - \lambda_n)(I - \lambda A)x_n - (I - \lambda A)x\|^2 + \|y_n - x\|^2 \\ & \quad - \|(1 - \lambda_n)(I - \lambda A)x_n - (I - \lambda A)x - y_n + x\|^2 \} \\ & = \frac{1}{2} \{ \|(I - \lambda A)x_n - (I - \lambda A)x - \lambda_n(I - \lambda A)x_n\|^2 \\ & \quad + \|y_n - x\|^2 - \|x_n - y_n - \lambda(Ax_n - Ax) - \lambda_n(I - \lambda A)x_n\|^2 \} \\ & = \frac{1}{2} \{ \|(I - \lambda A)x_n - (I - \lambda A)x\|^2 + \lambda_n^2 \|(I - \lambda A)x_n\|^2 \\ & \quad - 2\lambda_n \langle (I - \lambda A)x_n - (I - \lambda A)x, (I - \lambda A)x_n \rangle + \|y_n - x\|^2 \\ & \quad - \|x_n - y_n\|^2 - \|\lambda(Ax_n - Ax) - \lambda_n(I - \lambda A)x_n\|^2 \\ & \quad + 2\langle x_n - y_n, \lambda(Ax_n - Ax) + \lambda_n(I - \lambda A)x_n \rangle \} \\ & \leq \frac{1}{2} \{ \|x_n - x\|^2 + \lambda_n^2 \|(I - \lambda A)x_n\|^2 - 2\lambda_n \|(I - \lambda A)x_n\|^2 \\ & \quad + 2\lambda_n \|(I - \lambda A)x_n\| \|(I - \lambda A)x\| + \|y_n - x\|^2 - \|x_n - y_n\|^2 \\ & \quad + 2\lambda \|x_n - y_n\| \|Ax_n - Ax\| + 2\lambda_n \|x_n - y_n\| \|(I - \lambda A)x_n\| \} \\ & \leq \frac{1}{2} \{ \|x_n - x\|^2 + 2\lambda_n \|(I - \lambda A)x_n\| \|(I - \lambda A)x\| + \|y_n - x\|^2 \\ & \quad - \|x_n - y_n\|^2 + 2\lambda \|x_n - y_n\| \|Ax_n - Ax\| \\ & \quad + 2\lambda_n \|x_n - y_n\| \|(I - \lambda A)x_n\| \}. \end{aligned}$$

It follows that

$$\begin{aligned} \|y_n - x\|^2 & \leq \|x_n - x\|^2 + 2\lambda_n \|(I - \lambda A)x_n\| \|(I - \lambda A)x\| - \|x_n - y_n\|^2 \\ & \quad + 2\lambda \|x_n - y_n\| \|Ax_n - Ax\| + 2\lambda_n \|x_n - y_n\| \|(I - \lambda A)x_n\|, \end{aligned}$$

and hence

$$\begin{aligned} \|x_{n+1} - x\|^2 & \leq (1 - \alpha_n) \|x_n - x\|^2 + \alpha_n \|S_n y_n - x\|^2 \\ & \leq (1 - \alpha_n) \|x_n - x\|^2 + \alpha_n \|y_n - x\|^2 \\ & \leq \|x_n - x\|^2 + 2\alpha_n \lambda_n \|(I - \lambda A)x_n\| \|(I - \lambda A)x\| \\ & \quad - \alpha_n \|x_n - y_n\|^2 + 2\alpha_n \lambda \|x_n - y_n\| \|Ax_n - Ax\| \\ & \quad + 2\alpha_n \lambda_n \|x_n - y_n\| \|(I - \lambda A)x_n\|, \end{aligned}$$

which implies that

$$\begin{aligned} & \alpha_n \|x_n - y_n\|^2 \leq \|x_n - x\|^2 - \|x_{n+1} - x\|^2 \\ & + 2\alpha_n \lambda_n \|(I - \lambda A)x_n\| \|(I - \lambda A)x\| + 2\alpha_n \lambda \|x_n - y_n\| \|Ax_n - Ax\| \\ & + 2\alpha_n \lambda_n \|x_n - y_n\| \|(I - \lambda A)x_n\| \\ \leq & \|x_n - x_{n+1}\| [\|x_n - x\| + \|x_{n+1} - x_n\|] \\ & + 2\alpha_n \lambda_n \|(I - \lambda A)x_n\| \|(I - \lambda A)x\| + 2\alpha_n \lambda \|x_n - y_n\| \|Ax_n - Ax\| \\ & + 2\alpha_n \lambda_n \|x_n - y_n\| \|(I - \lambda A)x_n\|. \end{aligned}$$

As  $\lim_{n \rightarrow \infty} \lambda_n = 0$ ,  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|Ax_n - Ax\| = 0$ , it implies that  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ . Also, using  $\|S_n y_n - y_n\| \leq \|S_n y_n - x_n\| + \|x_n - y_n\|$ , we get that  $\lim_{n \rightarrow \infty} \|S_n y_n - y_n\| = 0$ .

Now, we prove that  $\limsup_{n \rightarrow \infty} \langle \omega, \omega - y_n \rangle \leq 0$ , where  $\omega = P_{Fix(S) \cap VI(C, A)} 0$ .

To show it, we choose a subsequence  $\{y_{n_i}\}$  of  $\{y_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle \omega, \omega - y_n \rangle = \lim_{i \rightarrow \infty} \langle \omega, \omega - y_{n_i} \rangle.$$

As  $\{y_{n_i}\}$  is bounded, we have a subsequence  $\{y_{n_{i_j}}\}$  of  $\{y_{n_i}\}$  which converges weakly to  $z$ . Without loss of generality, assume  $y_{n_i} \rightharpoonup z$ . It readily follows that  $z \in C$ , because  $\{y_{n_i}\}$  is a subsequence in  $C$  and  $C$  is a closed convex subset. We shall show that  $z \in Fix(S) \cap VI(C, A)$ .

First, we suppose that  $z \notin Fix(S)$ . From opial's condition and Lemma 1.3, we have

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|y_{n_i} - z\| & < \liminf_{i \rightarrow \infty} \|y_{n_i} - Sz\| \\ & = \liminf_{i \rightarrow \infty} \|y_{n_i} - S_{n_i} y_{n_i} + S_{n_i} y_{n_i} - S y_{n_i} + S y_{n_i} - Sz\| \\ & \leq \liminf_{i \rightarrow \infty} \|S y_{n_i} - Sz\| \\ & \leq \liminf_{i \rightarrow \infty} \|y_{n_i} - z\|. \end{aligned}$$

This is a contradiction. Thus  $z \in Fix(S)$ .

Now, let us show that  $z \in VI(C, A)$ . The mapping  $A$  is an  $\alpha$ -inverse strongly monotone mapping and so it is monotone. Let  $T$  be the mapping generated by  $A$  as follows:

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C, \end{cases}$$

where  $N_C v$  is the normal cone to  $C$  at  $v \in C$ . Then  $T$  is a maximal monotone mapping. Set  $(v, w) \in G(T)$ . Since  $w - Av \in N_C v$  and

$y_{n_i} \in C$ , we have  $\langle v - y_{n_i}, w - Av \rangle \geq 0$ . On the other hand,  $y_n = P_C[(1 - \lambda_n)(I - \lambda A)x_n]$ , we have  $\langle v - y_{n_i}, y_{n_i} - (1 - \lambda_{n_i})(I - \lambda A)x_{n_i} \rangle \geq 0$  or  $\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + Ax_{n_i} + \frac{\lambda_{n_i}}{\lambda}(I - \lambda A)x_{n_i} \rangle \geq 0$ . So, we get that

$$\begin{aligned} \langle v - y_{n_i}, w \rangle &\geq \langle v - y_{n_i}, Av \rangle \\ &\geq \langle v - y_{n_i}, Av \rangle \\ &\quad - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + Ax_{n_i} + \frac{\lambda_{n_i}}{\lambda}(I - \lambda A)x_{n_i} \rangle \\ &= \langle v - y_{n_i}, Av - Ax_{n_i} \rangle - \frac{1}{\lambda} \langle v - y_{n_i}, y_{n_i} - x_{n_i} \rangle \\ &\quad - \frac{\lambda_{n_i}}{\lambda} \langle v - y_{n_i}, (I - \lambda A)x_{n_i} \rangle \\ &= \langle v - y_{n_i}, Av - Ay_{n_i} \rangle + \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i} \rangle \\ &\quad - \frac{1}{\lambda} \langle v - y_{n_i}, y_{n_i} - x_{n_i} \rangle - \frac{\lambda_{n_i}}{\lambda} \langle v - y_{n_i}, (I - \lambda A)x_{n_i} \rangle. \end{aligned}$$

As  $\lim_{i \rightarrow \infty} \lambda_{n_i} = 0$ ,  $\lim_{i \rightarrow \infty} \|y_{n_i} - x_{n_i}\| = 0$  and  $A$  is monotone and Lipschitz continuous, we obtain  $\langle v - z, w \rangle \geq 0$ . Also, since  $T$  is maximal monotone, we deduce that  $z \in T^{-1}(0)$  and it implies from (1.5) that  $z \in VI(C, A)$ .

Now, as  $z \in Fix(S) \cap VI(C, A)$ , we deduce

$$\limsup_{n \rightarrow \infty} \langle \omega, \omega - y_n \rangle = \lim_{i \rightarrow \infty} \langle \omega, \omega - y_{n_i} \rangle = \langle \omega, \omega - z \rangle \leq 0.$$

Finally, we prove  $x_n \rightarrow \omega$ . From (1.3), we have  $\omega = P_C(\omega - \lambda A\omega)$  for all  $\lambda > 0$ ; in particular,

$$\omega = P_C[\omega - \lambda(1 - \lambda_n)A\omega] = P_C[\lambda_n\omega + (1 - \lambda_n)(I - \lambda A)\omega].$$

Thus, (1.1) implies that

$$\begin{aligned} &\|y_n - \omega\|^2 \\ &= \|P_C[(1 - \lambda_n)(I - \lambda A)x_n] - P_C[\lambda_n\omega + (1 - \lambda_n)(I - \lambda A)\omega]\|^2 \\ &\leq \langle (1 - \lambda_n)(I - \lambda A)x_n - (1 - \lambda_n)(I - \lambda A)\omega, y_n - \omega \rangle \\ &\quad - \langle \lambda_n\omega, y_n - \omega \rangle \\ &\leq (1 - \lambda_n)\|(I - \lambda A)x_n - (I - \lambda A)\omega\| \|y_n - \omega\| + \lambda_n \langle \omega, \omega - y_n \rangle \\ &\leq (1 - \lambda_n)\|x_n - \omega\| \|y_n - \omega\| + \lambda_n \langle \omega, \omega - y_n \rangle \\ &\leq \frac{1 - \lambda_n}{2} [\|x_n - \omega\|^2 + \|y_n - \omega\|^2] + \lambda_n \langle \omega, \omega - y_n \rangle. \end{aligned}$$

Hence

$$(1 - \frac{1 - \lambda_n}{2}) \|y_n - \omega\|^2 \leq \frac{1 - \lambda_n}{2} \|x_n - \omega\|^2 + \lambda_n \langle \omega, \omega - y_n \rangle,$$

or

$$\|y_n - \omega\|^2 \leq \frac{1 - \lambda_n}{1 + \lambda_n} \|x_n - \omega\|^2 + \frac{2\lambda_n}{1 + \lambda_n} \langle \omega, \omega - y_n \rangle.$$

Therefore,

$$\begin{aligned} \|x_{n+1} - \omega\|^2 &\leq (1 - \alpha_n) \|x_n - \omega\|^2 + \alpha_n \|S_n y_n - \omega\|^2 \\ &\leq (1 - \alpha_n) \|x_n - \omega\|^2 + \alpha_n \|y_n - \omega\|^2 \\ &\leq (1 - \alpha_n) \|x_n - \omega\|^2 + \frac{\alpha_n(1 - \lambda_n)}{1 + \lambda_n} \|x_n - \omega\|^2 \\ &\quad + \frac{2\alpha_n\lambda_n}{1 + \lambda_n} \langle \omega, \omega - y_n \rangle \\ &= \left(1 - \frac{2\alpha_n\lambda_n}{1 + \lambda_n}\right) \|x_n - \omega\|^2 + \frac{2\alpha_n\lambda_n}{1 + \lambda_n} \langle \omega, \omega - y_n \rangle. \end{aligned}$$

Hence, by Lemma 1.2, we get that  $\{x_n\}$  converges strongly to  $\omega \in \text{Fix}(S) \cap \text{VI}(C, A)$ , where  $\omega = P_{\text{Fix}(S) \cap \text{VI}(C, A)} 0$ . This completes the proof of this theorem.  $\square$

### 3. Applications

In this section, the equilibrium problem is stated. Then, the problem of finding the zeroes of a maximal monotone operator, a common fixed point of a family of strictly pseudocontractive mappings and a common fixed point of a countable sequence of  $W$ -mappings are investigated.

**3.1. Equilibrium problems.** A considerable amount of literature is being dedicated to the study of equilibrium problems, both from a theoretical point of view (optimization, game theoretic, complementary and variational inequality techniques), and from an applied point of view (networks, agent-based modelling and social scientific methods), which in itself constitutes wide areas of research interest today.

Let  $\varphi : C \rightarrow \mathbb{R}$  be a real-valued function and  $A : C \rightarrow H$  a nonlinear mapping. Also, suppose  $F : C \times C \rightarrow \mathbb{R}$  is a bifunction. The generalized mixed equilibrium problem is to find  $x \in C$  (see [6, 13, 29]) such that

$$(3.1) \quad F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle \geq 0,$$

for all  $y \in C$ .

We shall denote the set of solutions of this generalized mixed equilibrium problem by GMEP; that is

$$\text{GMEP} := \{x \in C : F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle \geq 0, \forall y \in C\}.$$

We now discuss several special cases of GMEP as follows:

1. If  $\varphi = 0$ , then the problem (3.1) is reduced to generalized equilibrium problem, i.e., finding  $x \in C$  such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0,$$

for all  $y \in C$ .

2. If  $A = 0$ , then the problem (3.1) is reduced to mixed equilibrium problem, that is to find  $x \in C$  such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0,$$

for all  $y \in C$ . We shall write the set of solutions of the mixed equilibrium problem by *MEP*.

3. If  $\varphi = 0$ ,  $A = 0$ , then the problem (3.1) is reduced to equilibrium problem, which is to find  $x \in C$  such that

$$F(x, y) \geq 0,$$

for all  $y \in C$ .

4. If  $\varphi = 0$ ,  $F = 0$ , then the problem (3.1) is reduced to variational inequality problem (1.2).

Now, let  $\varphi : C \rightarrow \mathbb{R}$  be a real-valued function. To solve the generalized mixed equilibrium problem for a bifunction  $F : C \times C \rightarrow \mathbb{R}$ , let us assume that  $F$ ,  $\varphi$  and  $C$  satisfy the following conditions:

- (A<sub>1</sub>)  $F(x, x) = 0$  for all  $x \in C$ ;
- (A<sub>2</sub>)  $F$  is monotone, i.e.,  $F(x, y) + F(y, x) \leq 0$  for all  $x, y \in C$ ;
- (A<sub>3</sub>) for each  $x, y, z \in C$ ,  $\lim_{t \rightarrow 0^+} F(tz + (1-t)x, y) \leq F(x, y)$ ;
- (A<sub>4</sub>) for each  $x \in C$ ,  $y \mapsto F(x, y)$  is convex and lower semicontinuous;
- (B<sub>1</sub>) for each  $x \in H$  and  $r > 0$ , there exists a bounded subset  $D_x \subseteq C$  and  $y_x \in C$  such that for each  $z \in C \setminus D_x$ ,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0;$$

- (B<sub>2</sub>)  $C$  is a bounded set.

In what follows, we state some lemmas which are useful to prove our convergence results.

**Lemma 3.1.** ([12]) *Assume that  $F : C \times C \rightarrow \mathbb{R}$  satisfies (A<sub>1</sub>) – (A<sub>4</sub>), and let  $\varphi : C \rightarrow \mathbb{R}$  be a lower semicontinuous and convex function. Assume that either (B<sub>1</sub>) or (B<sub>2</sub>) holds. For  $r > 0$  and  $x \in H$ , define a*

mapping  $T_r^{(F,\varphi)} : H \rightarrow C$  as follows:

$$T_r^{(F,\varphi)}(x) := \{z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C\},$$

for all  $x \in H$ . Then the following assertions hold:

- (1) For each  $x \in H$ ,  $T_r^{(F,\varphi)} \neq \emptyset$ ;
- (2)  $T_r^{(F,\varphi)}$  is single-valued;
- (3)  $T_r^{(F,\varphi)}$  is firmly nonexpansive, i.e., for any  $x, y \in H$ ,
 
$$\|T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y\|^2 \leq \langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y \rangle;$$
- (4)  $\text{Fix}(T_r^{(F,\varphi)}) = \text{MEP}$ ;
- (5)  $\text{MEP}$  is closed and convex.

**Lemma 3.2.** ([9]) Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Assume that  $S_1$  is a nonexpansive mapping from  $C$  into  $H$  and  $S_2$  a firmly nonexpansive mapping from  $H$  into  $C$  such that  $\text{Fix}(S_1) \cap \text{Fix}(S_2) \neq \emptyset$ . Then  $S_1S_2$  is a nonexpansive mapping from  $H$  into itself and  $\text{Fix}(S_1S_2) = \text{Fix}(S_1) \cap \text{Fix}(S_2)$ .

**Lemma 3.3.** ([3, 10]) Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  into  $\mathbb{R}$  satisfying  $(A_1) - (A_4)$  and  $\varphi : C \rightarrow \mathbb{R}$  a lower semicontinuous and convex function. Assume that either  $(B_1)$  or  $(B_2)$  holds. Let  $\{r_n\}$  be a sequence in  $(0, \infty)$ , such that  $\inf\{r_n : n \in \mathbb{N}\} > 0$ ,  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$  and let  $T_{r_n}^{(F,\varphi)}$  be a mapping defined as in Lemma 3.1. Then

- (1)  $\sum_{n=1}^{\infty} \sup\{\|T_{r_{n+1}}^{(F,\varphi)}x - T_{r_n}^{(F,\varphi)}x\| : x \in B\} < \infty$ , for any bounded subset  $B$  of  $C$ ;
- (2)  $\text{Fix}(T_r^{(F,\varphi)}) = \bigcap_{n=1}^{\infty} \text{Fix}(T_{r_n}^{(F,\varphi)})$ , where  $T_r^{(F,\varphi)}$  is a mapping defined by  $T_r^{(F,\varphi)}x := \lim_{n \rightarrow \infty} T_{r_n}^{(F,\varphi)}x$ , for all  $x \in C$ . Moreover,
 
$$\lim_{n \rightarrow \infty} \|T_{r_n}^{(F,\varphi)}x - T_r^{(F,\varphi)}x\| = 0.$$

Now, we have the following theorem that improves and extends the results announced by Shehu [15]. In fact, this theorem shows that the new conditions are better than what is proved in that paper.

**Theorem 3.4.** Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  into  $\mathbb{R}$  satisfying

$(A_1) - (A_4)$  and  $\varphi : C \rightarrow \mathbb{R}$  a lower semicontinuous and convex function. Assume that either  $(B_1)$  or  $(B_2)$  holds. Let  $A$  be an  $\alpha$ -inverse strongly monotone mapping,  $B$  a  $\beta$ -inverse strongly monotone mapping and  $\Theta$  a  $\mu$ -Lipschitzian, relaxed  $(\gamma, \delta)$ -cocoersive mapping from  $C$  into  $H$ . Suppose that  $S$  is a nonexpansive mapping from  $C$  into itself such that  $Fix(S) \cap VI(C, A) \cap VI(C, \Theta) \cap GMEP \neq \emptyset$ . Let  $\{\alpha_n\}$  and  $\{\lambda_n\}$  two real sequences in  $(0, 1)$ ,  $\{r_n\}$  and  $\{s_n\}$  two real sequences in  $(0, \infty)$  and  $0 < \lambda \leq 2\alpha$ . Let  $\{y_n\}$ ,  $\{u_n\}$  and  $\{x_n\}$  be generated by  $x_1 \in C$ ,

$$\begin{cases} y_n := P_C[(1 - \lambda_n)(I - \lambda A)x_n], \\ u_n := T_{r_n}^{(F, \varphi)}(y_n - r_n B y_n), \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n S P_C(u_n - s_n \Theta u_n), \quad (n \geq 1). \end{cases}$$

Suppose that the following conditions are satisfied:

- (1)  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ ,
- (2)  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\sum_{n=1}^{\infty} \lambda_n = \infty$ ,
- (3)  $0 < a \leq r_n \leq 2\beta$ ,  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ ,
- (4)  $0 < s_n \leq \frac{2(\delta - \gamma\mu^2)}{\mu^2}$ ,  $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$ .

Then  $\{x_n\}$  converges strongly to an element

$$\omega \in Fix(S) \cap VI(C, A) \cap VI(C, \Theta) \cap GMEP,$$

where  $\omega = P_{Fix(S) \cap VI(C, A) \cap VI(C, \Theta) \cap GMEP} 0$ .

*Proof.* For all  $x, y \in C$  and  $s_n \in [0, \frac{2(\delta - \gamma\mu^2)}{\mu^2}]$ , we obtain

$$\begin{aligned} & \|(I - s_n \Theta)x - (I - s_n \Theta)y\|^2 = \|x - y - s_n(\Theta x - \Theta y)\|^2 \\ &= \|x - y\|^2 - 2s_n \langle x - y, \Theta x - \Theta y \rangle + s_n^2 \|\Theta x - \Theta y\|^2 \\ &\leq \|x - y\|^2 - 2s_n[-\gamma \|\Theta x - \Theta y\|^2 + \delta \|x - y\|^2] + s_n^2 \|\Theta x - \Theta y\|^2 \\ &\leq \|x - y\|^2 + 2s_n \mu^2 \gamma \|x - y\|^2 - 2s_n \delta \|x - y\|^2 + \mu^2 s_n^2 \|x - y\|^2 \\ &= (1 + 2s_n \mu^2 \gamma - 2s_n \delta + \mu^2 s_n^2) \|x - y\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

This shows that  $I - s_n \Theta$  is nonexpansive for each  $n \in \mathbb{N}$ . By Lemma 3.3, it implies that

$$\sum_{n=1}^{\infty} \sup\{\|T_{r_{n+1}}^{(F, \varphi)} x - T_{r_n}^{(F, \varphi)} x\| : x \in E\} < \infty,$$

for any bounded subset  $E$  of  $C$ . Moreover, the mapping  $T_r^{(F,\varphi)}$ , defined by  $T_r^{(F,\varphi)}x := \lim_{n \rightarrow \infty} T_{r_n}^{(F,\varphi)}x$  for all  $x \in C$ , satisfies  $Fix(T_r^{(F,\varphi)}) =$

$$\bigcap_{n=1}^{\infty} Fix(T_{r_n}^{(F,\varphi)}) = MEP.$$

Put  $S_n := SP_C(I - s_n \Theta)T_{r_n}^{(F,\varphi)}(I - r_n B) = SU_n$ . Then, by Lemmas 3.2, 3.1 and relation (1.3), we get that  $S_n$  is a nonexpansive mapping from  $C$  into itself and for all  $n \in \mathbb{N}$ ,

$$Fix(S_n) = Fix(S) \cap VI(C, A) \cap Fix(T_{r_n}^{(F,\varphi)}(I - r_n B)) = Fix(S) \cap VI(C, A) \cap GM EP.$$

So,  $\bigcap_{n=1}^{\infty} Fix(S_n) = Fix(S) \cap VI(C, A) \cap GM EP$ . Also, we note that

$$\begin{aligned} & \|S_{n+1}x - S_nx\| = \|SU_{n+1}x - SU_nx\| \leq \|U_{n+1}x - U_nx\| \\ & \leq \|(I - s_{n+1}\Theta)T_{r_{n+1}}^{(F,\varphi)}(I - r_{n+1}B)x - (I - s_n\Theta)T_{r_n}^{(F,\varphi)}(I - r_nB)x\| \\ & \leq \|(I - s_{n+1}\Theta)T_{r_{n+1}}^{(F,\varphi)}(I - r_{n+1}B)x - (I - s_{n+1}\Theta)T_{r_n}^{(F,\varphi)}(I - r_nB)x\| \\ & \quad + \|(I - s_{n+1}\Theta)T_{r_n}^{(F,\varphi)}(I - r_nB)x - (I - s_n\Theta)T_{r_n}^{(F,\varphi)}(I - r_nB)x\| \\ & \leq \|T_{r_{n+1}}^{(F,\varphi)}(I - r_{n+1}B)x - T_{r_n}^{(F,\varphi)}(I - r_nB)x\| \\ & \quad + |s_{n+1} - s_n| \|\Theta T_{r_n}^{(F,\varphi)}(I - r_nB)x\| \\ & \leq \|T_{r_{n+1}}^{(F,\varphi)}(I - r_{n+1}B)x - T_{r_n}^{(F,\varphi)}(I - r_{n+1}B)x\| \\ & \quad + \|T_{r_n}^{(F,\varphi)}(I - r_{n+1}B)x - T_{r_n}^{(F,\varphi)}(I - r_nB)x\| \\ & \quad + |s_{n+1} - s_n| \|\Theta T_{r_n}^{(F,\varphi)}(I - r_nB)x\| \\ & \leq \|T_{r_{n+1}}^{(F,\varphi)}v_n - T_{r_n}^{(F,\varphi)}v_n\| + |r_{n+1} - r_n| \|Bx\| \\ & \quad + |s_{n+1} - s_n| \|\Theta T_{r_n}^{(F,\varphi)}(I - r_nB)x\|. \end{aligned}$$

where  $v_n = (I - r_{n+1}B)x$ . Moreover, for any bounded subset  $E$  of  $C$ , the boundedness of  $D = \{(I - r_{n+1}B)x : x \in E, n \in \mathbb{N}\}$  implies that

$$\begin{aligned} & \sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_nx\| : x \in E\} \\ & \leq \sum_{n=1}^{\infty} \sup\{\|U_{n+1}x - U_nx\| : x \in E\} \\ & \leq \sum_{n=1}^{\infty} \sup\{\|T_{r_{n+1}}^{(F,\varphi)}y - T_{r_n}^{(F,\varphi)}y\| : y \in D\} \end{aligned}$$



$$\begin{aligned}
 & + \sum_{n=1}^{\infty} |r_{n+1} - r_n| \sup\{\|Bx\| : x \in E\} \\
 & + \sum_{n=1}^{\infty} |s_{n+1} - s_n| \sup\{\|\Theta T_{r_n}^{(F,\varphi)}(I - r_n B)x\| : x \in E\} < \infty.
 \end{aligned}$$

Hence, by Theorem 2.1,  $\{x_n\}$  converges strongly to an element  $\omega \in \text{Fix}(S) \cap \text{VI}(C, A) \cap \text{VI}(C, \Theta) \cap \text{GMEP}$ , where  $\omega = P_{\text{Fix}(S) \cap \text{VI}(C, A) \cap \text{VI}(C, \Theta) \cap \text{GMEP}} 0$ . This completes the proof.  $\square$

**Corollary 3.5.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  into  $\mathbb{R}$  satisfying  $(A_1) - (A_4)$  and  $\varphi : C \rightarrow \mathbb{R}$  a lower semicontinuous and convex function. Assume that either  $(B_1)$  or  $(B_2)$  holds. Let  $A$  be an  $\alpha$ -inverse strongly monotone mapping,  $B$  a  $\beta$ -inverse strongly monotone mapping and  $\Theta$  a  $\mu$ -Lipschitzian, relaxed  $(\gamma, \delta)$ -cocoersive mapping from  $C$  into  $H$ . Suppose that  $S_n$  is a sequence of nonexpansive mappings from  $H$  into  $C$  such that  $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \text{VI}(C, A) \cap \text{VI}(C, \Theta) \cap \text{GMEP} \neq \emptyset$ .*

*Let  $\{\alpha_n\}$  and  $\{\lambda_n\}$  two real sequences in  $(0, 1)$ ,  $\{r_n\}$  and  $\{s_n\}$  be two real sequences in  $(0, \infty)$  and  $0 < \lambda \leq 2\alpha$ . Let  $\{y_n\}$ ,  $\{u_n\}$  and  $\{x_n\}$  be generated by  $x_1 \in C$ ,*

$$\begin{cases}
 y_n := P_C[(1 - \lambda_n)(I - \lambda A)x_n], \\
 u_n := T_{r_n}^{(F,\varphi)}(y_n - r_n B y_n), \\
 x_{n+1} := (1 - \alpha_n)x_n + \alpha_n S_n P_C(u_n - s_n \Theta u_n), \quad (n \geq 1).
 \end{cases}$$

*Suppose that the following conditions are satisfied:*

- (1)  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ ,
- (2)  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\sum_{n=1}^{\infty} \lambda_n = \infty$ ,
- (3)  $0 < a \leq r_n \leq 2\beta$ ,  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ ,
- (4)  $0 < s_n \leq \frac{2(\delta - \gamma\mu^2)}{\mu^2}$ ,  $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$ ,
- (5)  $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_nx\| : x \in B\} < \infty$ , for any bounded subset  $B$  of  $C$ .

*Then  $\{x_n\}$  converges strongly to an element  $\omega \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \text{VI}(C, A) \cap \text{VI}(C, \Theta) \cap \text{GMEP}$ , where  $\omega = P_{\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap \text{VI}(C, A) \cap \text{VI}(C, \Theta) \cap \text{GMEP}} 0$ .*

**3.2. Maximal monotone operators.** In this section, applying Theorem 2.1, we prove a strong convergence theorem concerning maximal monotone operator.

Let  $T$  be a set-valued maximal monotone operator. It is known that  $T$  is a set-valued maximal monotone operator if and only if  $R(I+rT) = H$  for all  $r > 0$ . Hence, for every  $r > 0$  and  $x \in H$ , there exists a unique  $x_r$  such that  $x \in x_r + rTx_r$  (see [20]). We define the *resolvent* of  $T$  by  $J_r^T x := x_r$ , for all  $r > 0$ . In other words,  $J_r^T = (I + rT)^{-1}$  for all  $r > 0$ . We now consider the problem of finding  $x \in H$  such that  $0 \in T(x)$ . Suppose that the set of solution to this problem is denoted by

$$\Lambda := \{x \in H : 0 \in T(x)\} = T^{-1}(0).$$

We know that  $\Lambda$  is a nonempty closed convex subset of  $H$ . Also,  $J_r^T$  is a single-valued nonexpansive mapping and  $Fix(J_r^T) = T^{-1}(0)$  for all  $r > 0$  (see [8] for more details).

We now give a lemma which will be used in the proof of the next theorem.

**Lemma 3.6.** ([5]) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $T$  be a set-valued maximal monotone operator such that  $T^{-1}(0) \neq \emptyset$  and  $\overline{D(T)} \subseteq C$ , and let  $\{r_n\}$  be a sequence in  $(0, \infty)$ . If  $\inf\{r_n : n \in \mathbb{N}\} > 0$  and  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ , then the following assertions hold:*

- (1)  $\sum_{n=1}^{\infty} \sup_{B} \{\|J_{r_{n+1}}^T x - J_{r_n}^T x\| : x \in B\} < \infty$ , for any bounded subset  $B$  of  $C$ ;
- (2) If  $r = \lim_{n \rightarrow \infty} r_n$ , then  $J_r^T x := \lim_{n \rightarrow \infty} J_{r_n}^T x$ , for all  $x \in C$  and  $Fix(J_r^T) = \bigcap_{n=1}^{\infty} Fix(J_{r_n}^T)$ .

In the following theorem, we observe an iterative sequence to obtain zeroes of a maximal monotone operator and the set of solutions of variational inequality of an  $\alpha$ -inverse strongly monotone mapping.

**Theorem 3.7.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$  and  $T$  be a set-valued maximal monotone operator such that  $T^{-1}(0) \neq \emptyset$  and  $\overline{D(T)} \subseteq C$ . Suppose  $A$  is an  $\alpha$ -inverse strongly monotone mapping from  $C$  into  $H$  such that  $T^{-1}(0) \cap VI(C, A) \neq \emptyset$ . Let  $\{\alpha_n\}$  and  $\{\lambda_n\}$  be two real sequences in  $(0, 1)$ ,  $\{r_n\}$  a real sequence*

in  $(0, \infty)$  and  $0 < \lambda \leq 2\alpha$ . Set  $J_{r_n}^T$  be the resolvent of  $T$ . Let  $\{y_n\}$  and  $\{x_n\}$  be generated by  $x_1 \in C$ ,

$$\begin{cases} y_n := P_C[(1 - \lambda_n)(I - \lambda A)x_n], \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n J_{r_n}^T y_n, \quad (n \geq 1). \end{cases}$$

Suppose that the following conditions are satisfied:

- (1)  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ ,
- (2)  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\sum_{n=1}^{\infty} \lambda_n = \infty$ ,
- (3)  $\inf\{r_n : n \in \mathbb{N}\} > 0$  and  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ .

Then  $\{x_n\}$  converges strongly to an element  $\omega \in T^{-1}(0) \cap VI(C, A)$ , where  $\omega = P_{T^{-1}(0) \cap VI(C, A)} 0$ .

*Proof.*  $J_{r_n}^T$  is a nonexpansive single-valued mapping and  $Fix(J_{r_n}^T) = T^{-1}(0)$ . From Lemma 3.6, we know that  $\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}}^T x - J_{r_n}^T x\| : x \in B\} < \infty$ , for any bounded subset  $B$  of  $C$ . Therefore, applying Theorem 2.1, the desired conclusion is obtained.  $\square$

In the sequel, we provide an algorithm to obtain the common element of the set of zeroes of a maximal monotone operator, the set of solutions of a variational inequality problem of an  $\alpha$ -inverse strongly monotone mapping, the set of solutions of a variational inequality problem of a  $\mu$ -Lipschitzian, relaxed  $(\alpha, \lambda)$ -cocoercive mapping and the set of solutions of a generalized mixed equilibrium problem.

**Theorem 3.8.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  into  $\mathbb{R}$  satisfying  $(A_1) - (A_4)$  and  $\varphi : C \rightarrow \mathbb{R}$  a lower semicontinuous and convex function with assumption  $(B_1)$  or  $(B_2)$ . Let  $A$  be an  $\alpha$ -inverse strongly monotone mapping,  $B$  a  $\beta$ -inverse strongly monotone mapping and  $\Theta$  a  $\mu$ -Lipschitzian, relaxed  $(\gamma, \delta)$ -cocoersive mapping from  $C$  into  $H$ . Assume that  $T$  is a set-valued maximal monotone operator such that  $T^{-1}(0) \cap VI(C, A) \cap VI(C, \Theta) \cap GMEP \neq \emptyset$ . Let  $J_r^T$  be the resolvent of  $T$  for each  $r > 0$ . Suppose that  $\{\alpha_n\}$  and  $\{\lambda_n\}$  are two real sequences in  $(0, 1)$ ,  $\{r_n\}$  and  $\{s_n\}$  are two real sequences in  $(0, \infty)$  such that  $r = \lim_{n \rightarrow \infty} r_n$  and  $0 < \lambda \leq 2\alpha$ . Let  $\{y_n\}$ ,  $\{u_n\}$  and  $\{x_n\}$  be generated by  $x_1 \in C$ ,*

$$\begin{cases} y_n := P_C[(1 - \lambda_n)(I - \lambda A)x_n], \\ u_n := T_{r_n}^{(F, \varphi)}(y_n - r_n B y_n), \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n J_r^T P_C(u_n - s_n \Theta u_n), \quad (n \geq 1). \end{cases}$$

Suppose that the following conditions are satisfied:

- (1)  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ ,
- (2)  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\sum_{n=1}^{\infty} \lambda_n = \infty$ ,
- (3)  $0 < a \leq r_n \leq 2\beta$ ,  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ ,
- (4)  $0 < s_n \leq \frac{2(\delta - \gamma\mu^2)}{\mu^2}$ ,  $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$ .

Then  $\{x_n\}$  converges strongly to an element  $\omega \in T^{-1}(0) \cap VI(C, A) \cap VI(C, \Theta) \cap GMEP$ , where  $\omega = P_{T^{-1}(0) \cap VI(C, A) \cap VI(C, \Theta) \cap GMEP} 0$ .

*Proof.* We know that  $J_r^T$  is nonexpansive and  $Fix(J_r^T) = \bigcap_{n=1}^{\infty} Fix(J_{r_n}^T) = T^{-1}(0)$ . So, using Theorem 3.4, we are done.  $\square$

**3.3. Strictly pseudocontractive mappings.** A mapping  $S : C \rightarrow C$  is said to be  $k$ -strictly pseudocontractive if there exists a constant  $k \in [0, 1)$  such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2,$$

for all  $x, y \in C$ . It is clear that every nonexpansive mapping is  $k$ -strictly pseudocontractive with  $k = 0$ . Before starting the main theorem of this section, we recall the following lemma.

**Lemma 3.9.** ([30]) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ , and let  $S : C \rightarrow C$  be a  $k$ -strictly pseudocontractive mapping with a fixed point. Then  $Fix(S)$  is a closed convex subset. Define  $S_\lambda : C \rightarrow H$  by  $S_\lambda x = \lambda x + (1 - \lambda)Sx$ , for each  $x \in C$ . Then  $S_\lambda$  is a nonexpansive mapping such that  $Fix(S_\lambda) = Fix(S)$ , provided that  $\lambda \in [k, 1)$ .*

**Theorem 3.10.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse strongly monotone mapping from  $C$  into  $H$  and  $\{S_n\}$  a sequence of  $k_n$ -strictly pseudocontractive mappings from  $C$  into itself such that  $\bigcap_{n=1}^{\infty} Fix(S_n) \cap VI(C, A) \neq \emptyset$ . Suppose that  $\{\alpha_n\}$  and  $\{\lambda_n\}$  are two real sequences in  $(0, 1)$  and  $0 < \lambda \leq 2\alpha$ . Let  $x_1 \in C$  and let  $\{x_n\}$  be the iterative sequence defined by*

$$\begin{cases} y_n := P_C[(1 - \lambda_n)(I - \lambda A)x_n], \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T_n y_n, \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $T_n x = k_n x + (1 - k_n) S_n x$ , for all  $x \in C$ . Suppose that the following conditions are satisfied:

- (1)  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ ,
- (2)  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\sum_{n=1}^{\infty} \lambda_n = \infty$ ,
- (3)  $\sum_{n=1}^{\infty} |k_{n+1} - k_n| < \infty$ ,
- (4)  $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_n x\| : x \in B\} < \infty$ , for any bounded subset  $B$  of  $C$ .

Let  $S$  be a mapping from  $C$  into itself defined by  $Sx := \lim_{n \rightarrow \infty} S_n x$  for all  $x \in C$  and suppose that  $Fix(S) := \bigcap_{n=1}^{\infty} Fix(S_n)$ . Then  $\{x_n\}$  converges strongly to an element  $\omega \in Fix(S) \cap VI(C, A)$ , where  $\omega = P_{Fix(S) \cap VI(C, A)} 0$ .

*Proof.* Using Lemma 3.9, we obtain that  $T_n$  is a nonexpansive mapping such that  $Fix(S_n) = Fix(T_n)$ . Moreover, for any bounded subset  $B$  of  $C$ ,

$$\begin{aligned} & \sup\{\|T_{n+1}x - T_n x\| : x \in B\} \\ &= \sup\{\|k_{n+1}x + (1 - k_{n+1})S_{n+1}x - (k_n x + (1 - k_n)S_n x)\| : x \in B\} \\ &\leq |k_{n+1} - k_n| \sup\{\|x - S_n x\| : x \in B\} \\ &\quad + (1 - k_{n+1}) \sup\{\|S_{n+1}x - S_n x\| : x \in B\} \\ &\leq |k_{n+1} - k_n| (\sup\{\|x\| : x \in B\} + \sup\{\|S_n x\| : x \in B\}) \\ &\quad + \sup\{\|S_{n+1}x - S_n x\| : x \in B\}. \end{aligned}$$

Hence, by conditions (3), (4) and Lemma 1.3, we get that

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_n x\| : x \in B\} < \infty.$$

Also, by Lemma 1.3, we can define  $T : C \rightarrow C$  by  $Tx = \lim_{n \rightarrow \infty} T_n x$ , for all  $x \in C$ .

Since  $\{k_n\}$  is a bounded sequence, there exists a subsequence  $\{k_{n_i}\}$  of  $\{k_n\}$  such that  $\lim_{i \rightarrow \infty} k_{n_i} = k$ . It follows that

$$Tx = \lim_{i \rightarrow \infty} T_{n_i} x = \lim_{i \rightarrow \infty} k_{n_i} x + (1 - k_{n_i}) S_{n_i} x = kx + (1 - k)Sx,$$

for all  $x \in C$ . Thus, by the previous lemma,  $T$  is a nonexpansive mapping such that  $Fix(T) = Fix(S)$ . Now the result is an immediate consequence of Theorem 2.1.  $\square$

By considering Lemma 3.9 and using Corollary 3.5, we have the following theorem.

**Theorem 3.11.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  into  $\mathbb{R}$  satisfying  $(A_1) - (A_4)$  and  $\varphi : C \rightarrow \mathbb{R}$  a lower semicontinuous and convex function with assumption  $(B_1)$  or  $(B_2)$ . Let  $A$  be an  $\alpha$ -inverse strongly monotone mapping,  $B$  a  $\beta$ -inverse strongly monotone mapping and  $\Theta$  a  $\mu$ -Lipschitzian, relaxed  $(\gamma, \delta)$ -cocoersive mapping from  $C$  into  $H$ . Assume that  $\{S_n\}$  is a sequence of  $k_n$ -strictly pseudocontractive mappings from  $C$  into itself such that  $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap VI(C, A) \cap VI(C, \Theta) \cap GMEP \neq \emptyset$ .*

*Let  $\{\alpha_n\}$  and  $\{\lambda_n\}$  be two real sequences in  $(0, 1)$ ,  $\{r_n\}$  and  $\{s_n\}$  two real sequences in  $(0, \infty)$  and  $0 < \lambda \leq 2\alpha$ . Let  $\{y_n\}$ ,  $\{u_n\}$  and  $\{x_n\}$  be generated by  $x_1 \in C$ ,*

$$\begin{cases} y_n := P_C[(1 - \lambda_n)(I - \lambda A)x_n], \\ u_n := T_{r_n}^{(F, \varphi)}(y_n - r_n B y_n), \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T_n P_C(u_n - s_n \Theta u_n), \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $T_n x = k_n x + (1 - k_n)S_n x$ ,  $x \in C$ . Suppose that the following conditions are satisfied:

- (1)  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ ,
- (2)  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\sum_{n=1}^{\infty} \lambda_n = \infty$ ,
- (3)  $\sum_{n=1}^{\infty} |k_{n+1} - k_n| < \infty$ ,
- (4)  $0 < a \leq r_n \leq 2\beta$ ,  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ ,
- (5)  $0 < s_n \leq \frac{2(\delta - \gamma\mu^2)}{\mu^2}$ ,  $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$ ,
- (6)  $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}x - S_n x\| : x \in B\} < \infty$ , for any bounded subset  $B$  of  $C$ .

Then  $\{x_n\}$  converges strongly to an element  $\omega \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap VI(C, A) \cap VI(C, \Theta) \cap GMEP$ , where  $\omega = P_{\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap VI(C, A) \cap VI(C, \Theta) \cap GMEP} 0$ .

**3.4.  $W$ -mappings.** The concept of  $W$ -mappings was introduced in [19, 21]. It is now one of the main tools in studying convergence of iterative methods to approach a common fixed point of nonlinear mapping; more recent progresses can be found in [4, 26] and the references cited therein.

Let  $\{S_n\}$  be a countable family of nonexpansive mappings  $S_n : H \rightarrow H$  and  $\beta_1, \beta_2, \dots$  be real numbers such that  $0 \leq \beta_n \leq 1$  for every  $n \in \mathbb{N}$ . We consider the mapping  $W_n$  defined by

$$\begin{aligned}
 &U_{n,n+1} := I, \\
 &U_{n,n} := \beta_n S_n U_{n,n+1} + (1 - \beta_n)I, \\
 &U_{n,n-1} := \beta_{n-1} S_{n-1} U_{n,n} + (1 - \beta_{n-1})I, \\
 &\quad \vdots \\
 (3.2) \quad &U_{n,k} := \beta_k S_k U_{n,k+1} + (1 - \beta_k)I, \\
 &U_{n,k-1} := \beta_{k-1} S_{k-1} U_{n,k} + (1 - \beta_{k-1})I, \\
 &\quad \vdots \\
 &U_{n,2} := \beta_2 S_2 U_{n,3} + (1 - \beta_2)I, \\
 &W_n := U_{n,1} = \beta_1 S_1 U_{n,2} + (1 - \beta_1)I.
 \end{aligned}$$

One can find the proof of the following lemma in [17].

**Lemma 3.12.** *Let  $H$  be a real Hilbert space. Let  $\{S_n\}$  be a sequence of nonexpansive mappings from  $H$  into itself such that  $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \neq \emptyset$ .*

*Suppose that  $\beta_1, \beta_2, \dots$  be real numbers such that  $0 < \beta_n \leq b < 1$  for all  $n \in \mathbb{N}$ . Then*

- (1)  $W_n$  is nonexpansive and  $\text{Fix}(W_n) = \bigcap_{i=1}^n \text{Fix}(S_i)$ , for all  $n \in \mathbb{N}$ ;
- (2)  $\lim_{n \rightarrow \infty} U_{n,k}x$  exists, for all  $x \in H$  and  $k \in \mathbb{N}$ ;
- (3) the mapping  $W : C \rightarrow C$  defined by  $Wx := \lim_{n \rightarrow \infty} W_n x = \lim_{n \rightarrow \infty} U_{n,1}x$ , for all  $x \in C$ , is a nonexpansive mapping satisfying  $\text{Fix}(W) = \bigcap_{n=1}^{\infty} \text{Fix}(S_n)$ ; and it is called  $W$ -mapping generated by  $S_1, S_2, \dots, S_n$  and  $\beta_1, \beta_2, \dots, \beta_n$ .

**Theorem 3.13.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Assume that  $A$  is an  $\alpha$ -inverse strongly monotone mapping from  $C$  into  $H$  and  $\{S_n\}$  a sequence of nonexpansive mappings from  $C$  into itself such that  $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap VI(C, A) \neq \emptyset$ . Let  $\{\alpha_n\}$  and  $\{\lambda_n\}$  be two real sequences in  $(0, 1)$  and  $0 < \lambda \leq 2\alpha$ . Also, suppose that  $W_n$  is the  $W$ -mapping from  $C$  into itself generated by  $S_1, S_2, \dots, S_n$  and  $\beta_1, \beta_2, \dots, \beta_n$  such that  $0 < \beta_n \leq b < 1$ , for every  $n \in \mathbb{N}$ . Let  $x_1 \in C$*

and let  $\{x_n\}$  be the iterative sequence defined by

$$\begin{cases} y_n := P_C[(1 - \lambda_n)(I - \lambda A)x_n], \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n W_n y_n, \quad (n \geq 1), \end{cases}$$

where the following conditions are satisfied:

- (1)  $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ ,
- (2)  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\sum_{n=1}^{\infty} \lambda_n = \infty$ .

Then  $\{x_n\}$  converges strongly to an element  $\omega \in \bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap VI(C, A)$ ,

where  $\omega = P_{\bigcap_{n=1}^{\infty} \text{Fix}(S_n) \cap VI(C, A)} 0$ .

*Proof.* Since  $S_i$  and  $U_{n,i}$  are nonexpansive, by relation (3.2), we deduce that for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} \|W_{n+1}x - W_nx\| &= \|\beta_1 S_1 U_{n+1,2}x - \beta_1 S_1 U_{n,2}x\| \\ &\leq \beta_1 \|U_{n+1,2}x - U_{n,2}x\| \\ &= \beta_1 \|\beta_2 S_2 U_{n+1,3}x - \beta_2 S_2 U_{n,3}x\| \\ &\leq \beta_1 \beta_2 \|U_{n+1,3}x - U_{n,3}x\| \\ &\leq \dots \\ &\leq \beta_1 \beta_2 \dots \beta_n \|U_{n+1,n+1}x - U_{n,n+1}x\| \leq M \prod_{i=1}^n \beta_i, \end{aligned}$$

where  $M > 0$  is a constant such that  $\sup\{\|U_{n+1,n+1}x - U_{n,n+1}x\| : x \in B\} \leq M$ , for any bounded subset  $B$  of  $C$ . Then

$$\sum_{n=1}^{\infty} \sup\{\|W_{n+1}x - W_nx\| : x \in B\} < \infty.$$

Now, by setting  $S_n := W_n$  in Theorem 2.1 and using Lemma 3.12, we obtain the result.  $\square$

#### 4. Numerical examples

The purpose of this section is to illustrate Theorems 2.1 and 3.4 by numerical examples.

**Example 4.1.** Let  $C = [-1, 1] \subset H = \mathbb{R}$  with  $\alpha_n = \frac{2n-1}{10n-9}$ ,  $\lambda_n = \frac{9}{10n}$  and  $\lambda = 1$ . Set  $A(x) = \frac{x}{10}$  and  $S_n(x) = \frac{x}{n}$ . Then  $A$  is a 5-inverse strongly



monotone mapping and  $S_n$  is a sequence of nonexpansive mappings. It readily follows that the sequence  $\{x_n\}$  generated by

$$\begin{cases} y_n := PC[(1 - \lambda_n)(I - \lambda A)x_n] = \frac{90n-81}{100n}x_n, \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n S_n y_n = \frac{8n-8}{10n-9}x_n + \frac{2n-1}{10n^2-9n}y_n, \end{cases}$$

with initial value  $x_1 = 1$ , converges strongly to an element (zero) of  $Fix(S) \cap VI(C, A)$ , where  $Fix(S) = \bigcap_{n=1}^{\infty} Fix(S_n)$  and  $0 = P_{Fix(S) \cap VI(C, A)}0$ .

The next example shows that the conditions of Theorem 3.4 is better than the conditions of the main theorem presented in [15].

**Example 4.2.** Let  $C = [-1, 1] \subset H = \mathbb{R}$  and define  $F(x, y) = -3x^2 + xy + 2y^2$ . Then, we have  $F(x, x) = 0, F(x, y) + F(y, x) = -(x - y)^2 \leq 0$  and  $\lim_{t \rightarrow 0^+} F(tz + (1-t)x, y) = -3x^2 + xy + 2y^2 \leq F(x, y)$  for all  $x, y, z \in C$ .

Also, for all  $x \in C, f(y) = F(x, y) = -3x^2 + xy + 2y^2$  is a lower semicontinuous and convex function.

From Lemma 3.1,  $T_r^{(F, \varphi)}$  is single-valued. Now, we get  $T_r^{(F, \varphi)}$ . For any  $y \in C, r > 0$  and  $\varphi = 0$ , we have

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \Leftrightarrow 2ry^2 + ((r + 1)z - x)y + xz - (3r + 1)z^2 \geq 0.$$

Set  $G(y) = 2ry^2 + ((r + 1)z - x)y + xz - (3r + 1)z^2$ . Then  $G(y)$  is a quadratic function of  $y$  with coefficients  $a = 2r, b = (r + 1)z - x$  and  $c = xz - (3r + 1)z^2$ . So

$$\begin{aligned} \Delta &= [(r + 1)z - x]^2 - 8r(xz - (3r + 1)z^2) \\ &= (r + 1)^2z^2 - 2(r + 1)xz + x^2 - 8rxz + (24r^2 + 8r)z^2 \\ &= ((5r + 1)z - x)^2. \end{aligned}$$

Since  $G(y) \geq 0$ , for all  $y \in C$ , if and only if  $\Delta \leq 0$ . Therefore,  $[(5r + 1)z - x]^2 \leq 0$ . It implies that  $z = \frac{x}{5r+1}$  and hence  $T_r^{(F, \varphi)}(x) = \frac{x}{5r+1}$ .

Let  $A(x) = 0$  and  $B(x) = \Theta(x) = S(x) = \frac{x}{10}$ . Then  $A$  is an  $\alpha$ -inverse strongly monotone mapping,  $B$  is a 5-inverse strongly monotone mapping,  $\Theta$  is a  $\frac{1}{10}$ -Lipschitzian, relaxed  $(1, \frac{1}{10})$ -cocoersive mapping and  $S$  is a nonexpansive mapping. Assume that  $\alpha_n = \frac{2n-1}{10n-9}, \lambda_n = \frac{9}{10n}, r_n = \frac{n+8}{n}$  and  $s_n = \frac{9}{n}$ . It is clear that all the assumptions of Theorem 3.4 are satisfied. Setting  $x_1 = 1$  and using the algorithm in Theorem 3.4, we

obtain the following sequences:

$$\begin{cases} y_n := P_C[(1 - \lambda_n)(I - \lambda A)x_n] = \frac{10n-9}{10n}x_n, \\ u_n := T_{r_n}^{(F,\varphi)}(y_n - r_n B y_n) = \frac{9n-8}{60n+400}y_n, \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n S P_C(u_n - s_n \Theta u_n) = \frac{8n-8}{10n-9}x_n + \frac{2n-1}{100n}u_n. \end{cases}$$

Then  $\{x_n\}$  converges strongly to  $0 \in \text{Fix}(S) \cap \text{VI}(C, \Theta) \cap \text{GMEP}$ , where  $0 = P_{\text{Fix}(S) \cap \text{VI}(C, \Theta) \cap \text{GMEP}}0$ .

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