

ISSN: 1017-060X (Print)



ISSN: 1735-8515 (Online)

**Bulletin of the**  
**Iranian Mathematical Society**

Vol. 40 (2014), No. 4, pp. 851–862

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**Published by Iranian Mathematical Society**  
**<http://bims.ims.ir>**

## EXISTENCE OF AN $L^p$ -SOLUTION FOR TWO DIMENSIONAL INTEGRAL EQUATIONS OF THE HAMMERSTEIN TYPE

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(Communicated by Mohammad Asadzadeh)

**ABSTRACT.** In this paper, existence of an  $L^p$ -solution for 2DIEs (Two Dimensional Integral Equations) of the Hammerstein type is discussed. The main tools in this discussion are Schaefer's and Schauder's fixed point theorems with a general version of Gronwall's inequality.

**Keywords:** Two dimensional integral equations, Schaefer's and Schauder's fixed point theorems, Gronwall's inequality, Superposition operator.

**MSC(2010):** Primary: 45G10.

### 1. Introduction

Most of the integral and integro–differential equations arise from mathematical modeling of scientific problems such as fluid and solid mechanics (for example in modeling piezoelectric materials and utilizing of these materials in nano-tubes), electrical engineering (specially optimal control), mathematical physics, biology, etc. [1, 4, 10, 13].

As we know, much works have been done on analyzing existence and uniqueness of solution and developing numerical algorithms for solving one dimensional nonlinear integral equations [2, 6, 7, 12]. But in two dimensional nonlinear cases a few studies have been done [3, 5, 11, 16].

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Article electronically published on August 23, 2014.

Received: 6 May 2012, Accepted: 20 June 2013.

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In the present work, we study existence of an  $L^p$ -solution for the 2DIEs of the Hammerstein type of the form

$$(1.1) \quad u(x, t) = f(x, t) + \lambda \int_c^d \int_a^b k(x, t, y, z) \phi(y, z, u(y, z)) dy dz,$$

where  $f$  and  $k$  are given real valued functions,  $\phi$  is a nonlinear function in terms of the unknown function  $u(x, t)$ , and  $\lambda$  is a real or complex constant.

The fixed point theorems in Banach spaces are powerful tools to prove existence and uniqueness of solution for integral and integro-differential equations. Most of these theorems are based on compactness of operators on a suitable Banach space. In this study, we use Schaefer's and Schauder's fixed point theorems to prove existence of an  $L^p$ -solution of Eq. (1.1). Since we will not use the function  $\phi$  satisfying the Lipschitz condition, we claim that our result generalizes literature concerning this equation.

We recall the following well known definitions and theorems from [8, 9, 14, 15, 17] which will be used throughout this paper, where  $\Omega = I \times J$ ,  $I = [a, b]$  and  $J = [c, d]$  are intervals in  $\mathbb{R}$ .

**Definition 1.1.** [9] *The superposition operator  $N_\phi$  associated to  $\phi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  assigns to each function  $u : \Omega \rightarrow \mathbb{R}$ , the function  $N_\phi u : \Omega \rightarrow \mathbb{R}$ , defined by*

$$(1.2) \quad N_\phi u(x, t) = \phi(x, t, u(x, t)).$$

**Definition 1.2.** [17] *Let  $p, q \geq 1$ . A function  $\phi : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is  $(p, q)$ -Carathéodory if the following conditions hold:*

- (i) *If  $1 \leq p < \infty$ , then  $|\phi(\mathbf{x}, \mathbf{y})| \leq C|\mathbf{y}|^{\frac{p}{q}} + \psi(\mathbf{x})$  for a.e.  $\mathbf{x} \in \Omega$ , all  $\mathbf{y} \in \mathbb{R}^m$  and some  $\psi \in L^q(\Omega)$ ,  $C \in \mathbb{R}^+$ ;*
- (ii) *If  $p = \infty$ , then for every  $l > 0$  there is a  $\psi_l \in L^q(\Omega)$  with  $|\phi(\mathbf{x}, \mathbf{y})| \leq \psi_l(\mathbf{x})$  for a.e.  $\mathbf{x} \in \Omega$  and all  $\mathbf{y} \in \mathbb{R}^m$  with  $|\mathbf{y}| \leq l$ .*

**Theorem 1.3.** [15, 17] *Let  $p \geq 1$  be a real number and let  $q \geq 1$  be the conjugate of  $p$  ( $\frac{1}{p} + \frac{1}{q} = 1$ ). Assume that  $\phi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a  $(p, q)$ -Carathéodory function such that  $u \in L^p(\Omega)$  implies that  $\phi(x, t, u(x, t)) \in L^q(\Omega)$ . Then, the superposition operator  $G : L^p(\Omega) \rightarrow L^q(\Omega)$  defined by  $Gu(x, t) = \phi(x, t, u(x, t))$  is continuous and bounded. In particular, there exist a constant  $C > 0$  and  $\psi \in L^q(\Omega)$  such that*

$$(1.3) \quad |\phi(x, t, u(x, t))| \leq C|u(x, t)|^{p-1} + \psi(x, t), \quad \forall x \in I, t \in J.$$

**Theorem 1.4.** (Arzela-Ascoli) [14] *Let  $X$  be a compact space and  $B \subseteq C(X)$ . Then  $B$  is compact if and only if  $B$  is uniformly bounded and equicontinuous.*

**Theorem 1.5.** (Schaefer's fixed point theorem) [14] *Let  $T$  be a continuous and compact operator of a Banach space  $X$  into itself such that the set*

$$\{u \in X : u = \lambda Tu \text{ for some } 0 \leq \lambda \leq 1\},$$

*is bounded. Then  $T$  has a fixed point.*

**Theorem 1.6.** (Schauder's fixed point theorem) [14] *Let  $K$  be a closed convex subset of a Banach space  $X$  and  $T : K \rightarrow K$  a compact, continuous operator. Then  $T$  has a fixed point in  $K$ .*

**Theorem 1.7.** (General version of Gronwall's inequality) [8] *Suppose that  $\phi$  is a positive and measurable function over  $I \times J$  and  $\rho \in L^1(\Omega)$  with  $\rho(x, t) \geq 0, \forall x \in I, t \in J$  and  $\int_c^d \int_a^b \rho(x, t) \phi(x, t) dx dt < +\infty$ . If for some  $A$  we have*

$$\phi(x, t) \leq A + \int_c^d \int_a^b \rho(x, t) \phi(x, t) dx dt, \quad \forall x \in I, t \in J,$$

*then*

$$\phi(x, t) \leq A \exp(\|\rho\|_1), \quad \forall x \in I, t \in J.$$

## 2. Existence of an $L^p$ -solution

We will use the following notations throughout this section:  
For every  $f \in L^p(\Omega)$  define

$$\|f\|_p = \left( \int_c^d \int_a^b |f(x, t)|^p dx dt \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty,$$

$$\|f\|_p = \text{ess sup}_{\substack{x \in I, \\ t \in J}} |f(x, t)|, \quad \text{for } p = \infty.$$

Similarly, for every  $k \in L^p(X)$  where  $X = \Omega \times \Omega$ , define

$$\|k\|_p = \left( \int_c^d \int_a^b \int_c^d \int_a^b |k(x, t, y, z)|^p dx dt dy dz \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty,$$

$$\|k\|_p = \text{ess sup}_{\substack{x, y \in I, \\ t, z \in J}} |k(x, t, y, z)|, \quad \text{for } p = \infty.$$

**Theorem 2.1.** *Let  $1 \leq p, q \leq \infty$  and let  $q$  be the conjugate of  $p$ . Suppose that the operator  $T : L^p(\Omega) \rightarrow \mathbb{R}$  is defined by*

$$(2.1) \quad Tu(x, t) = f(x, t) + \lambda \int_c^d \int_a^b k(x, t, y, z) \phi(y, z, u(y, z)) dy dz,$$

and  $f \in L^p(\Omega)$  and there exists a positive constant  $C$  and a function  $\psi \in L^q(\Omega)$  such that

$$|\phi(x, t, u(x, t))| \leq C|u(x, t)|^{p-1} + \psi(x, t), \quad \forall x \in I, t \in J,$$

holds. In addition, suppose that  $k \in L^p(X)$ , where  $X = \Omega \times \Omega$ . Then for every  $u \in L^p(\Omega)$ ,  $Tu \in L^p(\Omega)$  and  $T$  is a compact operator.

*Proof.* In order to show that  $T$  is a compact operator on  $L^p(\Omega)$ , we rewrite  $T$  as  $T = T_f + T_k$ , where

$$T_f u(x, t) = f(x, t),$$

and

$$(2.2) \quad T_k u(x, t) = \lambda \int_c^d \int_a^b k(x, t, y, z) \phi(y, z, u(y, z)) dy dz,$$

are defined on  $L^p(\Omega)$ . It is clear that  $T_f : L^p(\Omega) \rightarrow L^p(\Omega)$  is a rank one operator and so is a compact operator on  $L^p(\Omega)$ . Thus it suffices to prove that  $T_k : L^p(\Omega) \rightarrow L^p(\Omega)$  is a compact operator. First, we show that for each  $u \in L^p(\Omega)$ , we have  $T_k u \in L^p(\Omega)$ .

Let  $u \in L^p(\Omega)$ . Then we have

$$(2.3) \quad \begin{aligned} \|T_k u\|_p^p &= \left\| \lambda \int_c^d \int_a^b k(x, t, y, z) \phi(y, z, u(y, z)) dy dz \right\|_p^p \\ &\leq |\lambda| \int_c^d \int_a^b \left( \int_c^d \int_a^b |k(x, t, y, z) \phi(y, z, u(y, z))| dy dz \right)^p dx dt. \end{aligned}$$

By using the Hölder's inequality and the assumptions of the theorem, we rewrite (2.3) as

$$\begin{aligned}
\|T_k u\|_p^p &\leq |\lambda| \int_c^d \int_a^b \left( \int_c^d \int_a^b |k(x, t, y, z)|^p dy dz \right) \\
&\quad \times \left( \int_c^d \int_a^b |\phi(y, z, u(y, z))|^q dy dz \right)^{\frac{p}{q}} dx dt \\
&\leq |\lambda| \|k\|_p^p \left( \int_c^d \int_a^b (C|u(x, t)|^{p-1} + \psi(x, t))^q dy dz \right)^{\frac{p}{q}} \\
&\leq |\lambda| \|k\|_p^p (\|C|u|^{p-1} + \psi\|_q)^p \\
&\leq |\lambda| \|k\|_p^p (C\|u\|_p^{p-1} + \|\psi\|_q)^p \\
&= |\lambda| \|k\|_p^p (C\|u\|_p^{p-1} + \|\psi\|_q)^p.
\end{aligned}$$

Consequently, we have

$$\|T_k u\|_p \leq |\lambda|^{\frac{1}{p}} \|k\|_p (C\|u\|_p^{p-1} + \|\psi\|_q).$$

By the assumptions of the theorem, we have  $T_k u \in L^p(\Omega)$ .

To prove the compactness of the operator  $T_k$ , we consider the following cases:

**Case 1:** Let  $k \in C(X)$ . Let  $x, x_0 \in I$ ,  $t, t_0 \in J$  and let  $u \in L^p(\Omega)$  be arbitrary. Then, we have

$$\begin{aligned}
|T_k u(x, t) - T_k u(x_0, t_0)| &\leq |\lambda| \int_c^d \int_a^b |k(x, t, y, z) - k(x_0, t_0, y, z)| \\
&\quad \times (C|u(y, z)|^{p-1} + \psi(y, z)) dy dz \\
&\leq |\lambda| \sup_{\substack{y \in I, \\ z \in J}} |k(x, t, y, z) - k(x_0, t_0, y, z)| \\
(2.4) \quad &\quad \times [(d-c)(b-a)]^{\frac{1}{p}} (C\|u\|_p^{p-1} + \|\psi\|_q).
\end{aligned}$$

Since  $X$  is compact,  $k$  is uniformly continuous on  $X$ , thus the inequality in (2.4) implies that  $T_k u \in C(\Omega)$ . Hence  $T_k$  is an operator from  $L^p(\Omega)$  into  $C(\Omega)$ .

Suppose that  $S = \{u_n\}_{n \in \mathbb{N}}$  is a bounded subset of  $L^p(\Omega)$  (i.e., there exists a positive constant  $M$  such that  $\|u_n\|_p \leq M$  for each  $n \in \mathbb{N}$ ). We show that  $T_k$  is uniformly bounded operator on  $C(\Omega)$ . Let  $u_n \in S$ .

Then

$$\begin{aligned}
 |T_k u_n(x, t)| &\leq |\lambda| \int_c^d \int_a^b |k(x, t, y, z)| (C|u_n(y, z)|^{p-1} + \psi(y, z)) \, dydz \\
 &\leq |\lambda| \operatorname{ess\,sup}_{\substack{x \in I, \\ t \in J}} |k(x, t, y, z)| [(d - c)(b - a)]^{\frac{1}{p}} (C\|u_n\|_p^{p-1} + \|\psi\|_q) \\
 &\leq |\lambda| \|k\|_\infty [(d - c)(b - a)]^{\frac{1}{p}} (CM^{p-1} + \|\psi\|_q), \\
 &\quad \forall n \in \mathbb{N}, \quad x \in I, \quad t \in J.
 \end{aligned}$$

To prove the compactness of  $T_k$ , it suffices to show  $\{T_k(u_n) : u_n \in S\}$  is equicontinuous. Indeed, for every  $x, x_0, y \in I$  and  $t, t_0, z \in J$  by using the Hölder’s inequality we have

$$\begin{aligned}
 |T_k u_n(x, t) - T_k u_n(x_0, t_0)| &\leq |\lambda| \sup_{\substack{y \in I, \\ z \in J}} |k(x, t, y, z) - k(x_0, t_0, y, z)| \\
 &\quad \times [(d - c)(b - a)]^{\frac{1}{p}} (CM^{p-1} + \|\psi\|_q).
 \end{aligned}$$

This implies that  $T_k$  is equicontinuous on  $\Omega$ , since  $k$  is uniformly continuous on  $X$ . By the Arzela-Ascoli theorem (Theorem 1.4),  $T_k$  is compact on  $C(\Omega)$ . Therefore  $T_k$  is compact on  $L^p(\Omega)$ , since  $C(\Omega)$  is dense in  $L^p(\Omega)$ .

**Case 2:** Suppose that  $k \in L^p(X)$ . Then there exists a sequence  $\{k_n\}_{n \in \mathbb{N}}$  of continuous kernels such that  $\|k_n - k\|_p \rightarrow 0$  as  $n \rightarrow \infty$ . Suppose that  $S$  is a bounded subset of  $L^p(\Omega)$  in previous case. Since  $k_1(x, t, y, z)$  is a continuous function on  $X$ , by using the result of case 1,  $T_{k_1}$  is a compact operator. Thus there exists a subsequence  $\{u_n^{(1)}\}_{n \in \mathbb{N}}$  of  $\{u_n\}$  such that  $\{T_{k_1}(u_n^{(1)})\}$  is convergent. Similarly, there exists a subsequence  $\{u_n^{(2)}\}_{n \in \mathbb{N}}$  of  $\{u_n^{(1)}\}$  such that  $\{T_{k_2}(u_n^{(2)})\}_{n \in \mathbb{N}}$  is convergent. Generally, for each  $m \in \mathbb{N}$  there exists a subsequence  $\{u_n^{(m)}\}_{n \in \mathbb{N}}$  of  $\{u_n^{(m-1)}\}_{n \in \mathbb{N}}$  such that  $\{T_{k_m}(u_n^{(m)})\}$  is convergent. Considering the diagonal subsequence  $\{u_n^{(n)}\}_{n \in \mathbb{N}}$ , we show that  $\{T_k(u_n^{(n)})\}$  is a Cauchy sequence on  $L^p(\Omega)$ . For every  $m, n, l \in \mathbb{N}$ , we have

$$\begin{aligned}
 \|T_k(u_m^{(m)}) - T_k(u_l^{(l)})\|_p &\leq \|T_k(u_m^{(m)}) - T_{k_n}(u_m^{(m)})\|_p \\
 &\quad + \|T_{k_n}(u_m^{(m)}) - T_{k_n}(u_l^{(l)})\|_p \\
 (2.5) \qquad &\quad + \|T_{k_n}(u_l^{(l)}) - T_k(u_l^{(l)})\|_p.
 \end{aligned}$$

Since  $\{T_{k_n}(u_l^{(l)})\}$  is convergent, for  $\epsilon > 0$ , there exists  $N_1 \in \mathbb{N}$  such that

$$(2.6) \quad \|T_{k_n}(u_m^{(m)}) - T_{k_n}(u_l^{(l)})\|_p < \frac{\epsilon}{3}, \quad \forall m, l \geq N_1.$$

Also, we have

$$(2.7) \quad \|T_k(u_m^{(m)}) - T_{k_n}(u_m^{(m)})\|_p^p \leq |\lambda| \|k - k_n\|_p^p (CM^{p-1} + \|\psi\|_q)^p.$$

Since  $\|k_n - k\|_p \rightarrow 0$  as  $n \rightarrow \infty$ , for  $\epsilon > 0$ , there exists  $N_2 \in \mathbb{N}$  such that

$$(2.8) \quad \|T_k(u_m^{(m)}) - T_{k_n}(u_m^{(m)})\|_p < \frac{\epsilon}{3}, \quad \forall n \geq N_2.$$

By (2.5)-(2.8), we have

$$\|T_k(u_m^{(m)}) - T_k(u_l^{(l)})\|_p < \epsilon.$$

Hence,  $\{T_k(u_n^{(n)})\}$  is a Cauchy sequence on  $L^p(\Omega)$  and so it is convergent. Thus  $T_k$  is a compact operator on  $L^p(\Omega)$ .  $\square$

**Theorem 2.2.** *Consider the 2DIE (1.1). Suppose that  $1 \leq p \leq 2$  and  $q \geq 1$  is the conjugate of  $p$ , and  $k \in L^p(X)$ ,  $f \in L^p(\Omega)$  and  $\phi$  is a  $(p, q)$ -Carathéodory function. Then*

- (i) *If  $1 \leq p < 2$ , then Eq. (1.1) has an  $L^p$ -solution.*
- (ii) *If  $p = 2$  and the following conditions hold for the kernel  $k$ :*
  - (a)  *$C|\lambda|^{\frac{1}{2}}\|k\|_2 < 1$  where  $C$  is the constant given in Theorem 1.3;*
  - (b)  *$k(x, t, y, z) = 0$ ,  $\forall y \geq x$ ,  $\forall z \geq t$  and*

$$|k(x, t, y, z)| \leq |k_1(x, t)||k_2(y, z)|,$$

*where  $k_1$  is bounded and measurable on  $\Omega$  and  $k_2 \in L^p(\Omega)$ , then Eq. (1.1) has an  $L^2$ -solution.*

*Proof.* Since  $\phi$  is a  $(p, q)$ -Carathéodory function, it satisfies Theorem 1.3. Thus by Theorem 2.1 the operator  $T$  in (2.1) is compact on  $L^p(\Omega)$ . To prove existence of an  $L^p$ -solution of Eq. (1.1), we use the Schaefer's fixed point theorem (Theorem 1.5). First, we show that  $T$  is continuous. Let  $u \in L^p(\Omega)$  and let  $\{u_n\}_{n \in \mathbb{N}}$  be a sequence in  $L^p(\Omega)$  that converges to  $u$ . Also, let  $G$  be the superposition operator defined in (1.2). Then



by using the Hölder's inequality, we obtain

$$\begin{aligned}
 \|Tu_n - Tu\|_p^p &\leq |\lambda| \int_c^d \int_a^b \left( \int_c^d \int_a^b |k(x, t, y, z)| \right. \\
 &\quad \left. \times |Gu_n(y, z) - Gu(y, z)| dydz \right)^p dxdt \\
 (2.9) \qquad &\leq |\lambda| \|k\|_p^p \|G(u_n) - G(u)\|_p^p.
 \end{aligned}$$

Since  $G$  is continuous, inequality (2.9) implies that  $T$  is continuous.

To prove the existence results, it suffices to show the set  $F = \{u \in L^p(\Omega) : u = \lambda Tu \text{ for some } 0 \leq \lambda \leq 1\}$  is a bounded set. Let  $u \in F$ . Then by reusing (1.3) and the Hölder's inequality, we have

$$(2.10) \quad \|u\|_p = |\lambda| \|Tu\|_p \leq \|Tu\|_p \leq \|f\|_p + |\lambda|^{\frac{1}{p}} \|k\|_p (C\|u\|_p^{p-1} + \|\psi\|_q),$$

or equivalently,

$$\|u\|_p^{p-1} (\|u\|_p^{2-p} - |\lambda|^{\frac{1}{p}} C \|k\|_p) \leq \|f\|_p + |\lambda|^{\frac{1}{p}} \|k\|_p \|\psi\|_q.$$

Since  $1 \leq p < 2$ ,  $p - 1$  and  $2 - p$  are nonnegative constants, then there exists a positive constant  $M$  such that  $\|u\|_p \leq M$  and so  $F$  is a bounded set. By using the Schaefer's fixed point theorem, we conclude that the operator  $T$  has a fixed point in  $L^p(\Omega)$ . Hence Eq. (1.1) has an  $L^p$ -solution.

Let  $p = 2$  and  $u \in F$  and the condition (a) holds. Then from (2.10), we have

$$\|u\|_2 \leq \|Tu\|_2 \leq \|f\|_2 + |\lambda|^{\frac{1}{2}} \|k\|_2 (C\|u\|_2 + \|\psi\|_2),$$

or equivalently,

$$\|u\|_2 \leq \frac{\|f\|_2 + |\lambda|^{\frac{1}{2}} \|k\|_2 \|\psi\|_2}{1 - C|\lambda|^{\frac{1}{2}} \|k\|_2},$$

which implies  $\|u\|_2 \leq M$ , since  $C|\lambda|^{\frac{1}{2}} \|k\|_2 < 1$ . Hence  $F$  is bounded. It is evident again from the Schaefer's fixed point theorem that Eq. (1.1) has an  $L^2$ -solution.

Now, suppose that the condition (b) holds. Then

$$\begin{aligned}
 |u(x, t)| &\leq |f(x, t)| + |\lambda| \int_c^t \int_a^x |k(x, t, y, z)| |\phi(y, z, u(y, z))| dy dz \\
 &\leq \|f\|_\infty + |\lambda| \int_c^t \int_a^x |k_1(x, t)| |k_2(y, z)| (C|u(y, z)| + \psi(y, z)) dy dz \\
 &\leq \|f\|_\infty + |\lambda| \|k_1\|_\infty \|k_2\|_2 \|\psi\|_2 \\
 &\quad + |\lambda| C \|k_1\|_\infty \int_c^t \int_a^x |k_2(y, z)| |u(y, z)| dy dz \\
 &\leq A + \int_c^t \int_a^x \rho(y, z) |u(y, z)| dy dz,
 \end{aligned}$$

where  $A = \|f\|_\infty + |\lambda| \|k_1\|_\infty \|k_2\|_2 \|\psi\|_2$  and  $\rho(y, z) = C|\lambda| \|k_1\|_\infty |k_2(y, z)| \in L^2(\Omega)$ . Since  $L^2(\Omega) \subset L^1(\Omega)$ , by Gronwall's inequality we conclude

$$\|u\|_2 \leq \sqrt{(d-c)(b-a)} A \exp(\|\rho\|_1).$$

Then the Schaefer's fixed point theorem implies that Eq. (1.1) has an  $L^2$ -solution.  $\square$

**Theorem 2.3.** *Let  $p, q \geq 1$  and let  $q$  be the conjugate of  $p$ . Suppose that  $f \in L^p(\Omega)$  and  $k \in L^p(X)$  and there exist a positive constant  $C'$  and  $\psi \in L^p(\Omega)$  such that*

$$|\phi(y, z, u(y, z))| \leq C'|u(y, z)| + \psi(y, z) \text{ for a.e. } y \in I, z \in J.$$

*Suppose that there exists a weight function  $\omega$  on  $\Omega$  (i.e.,  $\omega$  is nonnegative, measurable and a bounded function on  $\Omega$ ) such that the function*

$$\Phi(x, t) = \begin{cases} \left( \int_c^d \int_a^b |k(x, t, y, z)|^q (\omega(y, z))^{-\frac{q}{p}} dy dz \right)^{\frac{1}{q}}, & p > 1, \\ \sup_{\substack{y \in I, \\ z \in J}} \frac{|k(x, t, y, z)|}{\omega(y, z)}, & p = 1, \end{cases}$$

*belongs to  $L^p(\Omega, d\omega)$ . If  $\|\lambda C' \Phi\|_{p, \omega} < 1$ , then Eq. (1.1) has an  $L^p$ -solution.*

*Proof.* The weighted space  $L^p(\Omega, d\omega)$  is the space of real functions produced by the norm

$$\|f\|_{p, \omega} = \left( \int_c^d \int_a^b |f(x, t)|^p \omega(x, t) dx dt \right)^{\frac{1}{p}}.$$

It is easy to show that the norms  $\|\cdot\|_p$  and  $\|\cdot\|_{p,\omega}$  are equivalent. Hence, a bounded set of  $L^p(\Omega, d\omega)$  is also bounded on  $L^p(\Omega)$ . Consider the ball

$$S_\alpha = \{f \in L^p(\Omega, d\omega) : \|f\|_{p,\omega} \leq \alpha\},$$

in  $L^p(\Omega, d\omega)$ . Since the operator  $T_k$  given by (2.2) is compact on  $L^p(\Omega)$  and  $S_\alpha$  is a bounded subset of  $L^p(\Omega)$ ,  $T(S_\alpha)$  is relatively compact on  $L^p(\Omega) \subset L^p(\Omega, d\omega)$ .

Let  $u \in S_\alpha$ . Then

$$\begin{aligned} \|T_k u\|_{p,\omega}^p &\leq |\lambda| \int_c^d \int_a^b \omega(x, t) \left( \int_c^d \int_a^b \frac{|k(x, t, y, z)|}{(\omega(y, z))^{\frac{1}{p}}} \right. \\ &\quad \left. \times (\omega(y, z))^{\frac{1}{p}} |\phi(y, z, u(y, z))| dy dz \right)^p dx dt \\ &\leq |\lambda| \int_c^d \int_a^b \omega(x, t) \left( \int_c^d \int_a^b |k(x, t, y, z)|^q (\omega(y, z))^{\frac{-q}{p}} dy dz \right)^{\frac{p}{q}} \\ &\quad \times \left( \int_c^d \int_a^b \omega(y, z) |\phi(y, z, u(y, z))|^p dy dz \right) dx dt \\ &\leq |\lambda| \int_c^d \int_a^b \omega(x, t) \left( \int_c^d \int_a^b |k(x, t, y, z)|^q (\omega(y, z))^{\frac{-q}{p}} dy dz \right)^{\frac{p}{q}} \\ &\quad \times \left( \int_c^d \int_a^b \omega(y, z) (C'|u(y, z)| + \psi(y, z))^p dy dz \right) dx dt \\ &\leq |\lambda| \|C'u + \psi\|_{p,\omega}^p \int_c^d \int_a^b \omega(x, t) \\ &\quad \times \left( \int_c^d \int_a^b |k(x, t, y, z)|^q (\omega(y, z))^{\frac{-q}{p}} dy dz \right)^{\frac{p}{q}} dx dt \\ (2.11) \quad &\leq |\lambda| (C'\|u\|_{p,\omega} + \|\psi\|_{p,\omega})^p \|\Phi\|_{p,\omega}^p. \end{aligned}$$

Since  $Tu = f + T_k u$  and  $\|u\|_{p,\omega} \leq \alpha$ , (2.11) implies

$$\|Tu\|_{p,\omega} \leq \|f\|_{p,\omega} + |\lambda| \|\psi\|_{p,\omega} \|\Phi\|_{p,\omega} + |\lambda| C' \|u\|_{p,\omega} \|\Phi\|_{p,\omega}.$$

Thus, if  $\|\lambda C' \Phi\|_{p,\omega} < 1$ , then there exists  $\varepsilon > 0$  such that for each  $\alpha \geq \varepsilon$ , we have

$$\alpha \geq \frac{\|f\|_{p,\omega} + |\lambda| \|\psi\|_{p,\omega} \|\Phi\|_{p,\omega}}{1 - |\lambda| C' \|\Phi\|_{p,\omega}},$$

and so  $T(S_\alpha) \subseteq S_\alpha$ . Therefore  $T$  maps the closed convex subset  $S_\alpha$  into itself. Since  $T(S_\alpha)$  is compact, by the Schauder's fixed point theorem (Theorem 1.6)  $T$  has a fixed point. This completes the proof.  $\square$

### 3. Conclusion

In this paper, we proved a major existence theorem for 2DIEs which is important in the numerical solution of these types of equations. It may be extended to the higher dimensional integral equations by authors as a future work.

### Acknowledgments

The authors would like to thank the associate editor and anonymous reviewers for their useful comments that has improved the structure of this paper. Also, the authors wish to thank Dr. A. Ranjbari and Prof. M. R. Jabbarzadeh for their careful reading of the manuscript and for giving useful comments.

### REFERENCES

- [1] K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, 1997.
- [2] D. Bahuguna and J. Dabas, Existence and uniqueness of a solution to a parabolic integro-differential equation by the method of lines, *Electron. J. Qual. Theory Differ. Equ.* **4** (2008) 1–12.
- [3] H. Brunner and J. P. Kauthen, The numerical solution of two-dimensional Volterra integral equations by collocation and iterated collocation, *IMA J. Numer. Anal.* **9** (1989), no. 1, 47–59.
- [4] S. Chena, G. Wang and M. Chien, Analytical modeling of piezoelectric vibration-induced micro power generator, *Mechatronics* **16** (2006), no. 7, 379–387.
- [5] H. Guoqiang and W. Jiong, Extrapolation of Nystrom solution for two dimensional nonlinear Fredholm integral equations, *J. Comput. Appl. Math.* **134** (2001), no. 1-2, 259–268.
- [6] A. Karoui and A. Jawahdou, Existence and approximate  $L^p$  and continuous solutions of nonlinear integral equations of the Hammerstein and Volterra types, *Appl. Math. Comput.* **216** (2010), no. 7, 2077–2091.
- [7] J. P. Kauthen, The method of lines for parabolic partial integro-differential equations, *J. Integral Equations Appl.* **4** (1992), no. 1, 69–81.
- [8] Y. H. Kim, Gronwall, Bellman and Pachpatte type integral inequalities with applications, *Nonlinear Anal.* **71** (2009), no. 12, 2641–2656.
- [9] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, A Pergamon Press Book The Macmillan Co., New York, 1964.
- [10] A. V. Manzhirov, On a method of solving two-dimensional integral equations of axisymmetric contact problems for bodies with complex rheology, *J. Appl. Math. Mech.* **49** (1985), no. 6, 777–782.

- [11] S. Mckee, T. Tang and T. Diogo, An Euler-type method for two-dimensional Volterra integral equations of the first kind, *IMA J. Numer. Anal.* **20** (2000), no. 3, 423–440.
- [12] M. Meehan and D. O'Regan, Existence theory for nonlinear Volterra integro-differential and integral equations, *Nonlinear Anal.* **31** (1998), no. 3-4, 317–341.
- [13] W. E. Olmstead, A nonlinear integral equation associated with gas absorption in a liquid, *Z. Angew. Math. Phys.* **28** (1977), no. 3, 513–523.
- [14] D. O'Regan and M. Meehan, Existence Theory for Nonlinear Integral and Integrodifferential Equations, Kluwer Academic Publisher, Dordrecht, 1998.
- [15] R. Precup, Methods in Nonlinear Integral Equations, Kluwer Academic Publisher, Dordrecht, 2002.
- [16] A. Tari, M. Y. Rahimi, S. Shahmorad and F. Talati, Solving a class of two dimensional linear and nonlinear Volterra integral equation by the differential transform method, *J. Comput. Appl. Math.* **228** (2009), no. 1, 70–76.
- [17] P. P. Zabrejko, A. I. Koshelev, M. A. Krasnosel'skii, S. G. Mikhlin, L. S. Rakovschik and V. J. Stetsenko, Integral Equations, Nordhoff, 1975.

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