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Author(s):

S. A. Hosseini, S. Shahmorad and A. Tari

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EXISTENCE OF AN L^p-SOLUTION FOR TWO DIMENSIONAL INTEGRAL EQUATIONS OF THE HAMMERSTEIN TYPE

S. A. HOSSEINI, S. SHAHMORAD* AND A. TARI

(Communicated by Mohammad Asadzadeh)

Abstract. In this paper, existence of an L^p-solution for 2DIEs (Two Dimensional Integral Equations) of the Hammerstein type is discussed. The main tools in this discussion are Schaefer's and Schauder's fixed point theorems with a general version of Gronwall's inequality.
Keywords: Two dimensional integral equations, Schaefer's and Schauder's fixed point theorems, Gronwall's inequality, Superposition operator.
MSC(2010): Primary: 45G10.

1. Introduction

Most of the integral and integro–differential equations arise from mathematical modeling of scientific problems such as fluid and solid mechanics (for example in modeling piezoelectric materials and utilizing of these materials in nano-tubes), electrical engineering (specially optimal control), mathematical physics, biology, etc. [1, 4, 10, 13].

As we know, much works have been done on analyzing existence and uniqueness of solution and developing numerical algorithms for solving one dimensional nonlinear integral equations [2, 6, 7, 12]. But in two dimensional nonlinear cases a few studies have been done [3, 5, 11, 16].

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*Corresponding author.

In the present work, we study existence of an L^p-solution for the 2DIEs of the Hammerstein type of the form

$$u(x,t) = f(x,t) + \lambda \int_c^d \int_a^b k(x,t,y,z)\phi(y,z,u(y,z))\mathrm{d}y\mathrm{d}z, \tag{1.1}$$

where f and k are given real valued functions, ϕ is a nonlinear function in terms of the unknown function $u(x,t)$, and λ is a real or complex constant.

The fixed point theorems in Banach spaces are powerful tools to prove existence and uniqueness of solution for integral and integro-differential equations. Most of these theorems are based on compactness of operators on a suitable Banach space. In this study, we use Schaefer's and Schauder's fixed point theorems to prove existence of an L^p-solution of Eq. (1.1). Since we will not use the function ϕ satisfying the Lipschitz condition, we claim that our result generalizes literature concerning this equation.

We recall the following well known definitions and theorems from [8, 9, 14, 15, 17] which will be used throughout this paper, where $\Omega = I \times J$, $I = [a,b]$ and $J = [c,d]$ are intervals in $\mathbb{R}$.

Definition 1.1. [9] *The superposition operator N_ϕ associated to $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ assigns to each function $u : \Omega \to \mathbb{R}$, the function $N_\phi u : \Omega \to \mathbb{R}$, defined by*

$$N_\phi u(x,t) = \phi(x,t,u(x,t)). \tag{1.2}$$

Definition 1.2. [17] *Let $p, q \geq 1$. A function $\phi : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ is (p,q)-Carathéodory if the following conditions hold:*

- (i) *If $1 \leq p < \infty$, then $|\phi(\boldsymbol{x},\boldsymbol{y})| \leq C|\boldsymbol{y}|^{\frac{p}{q}} + \psi(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega$, all $\boldsymbol{y} \in \mathbb{R}^m$ and some $\psi \in L^q(\Omega)$, $C \in \mathbb{R}^+$;*
- (ii) *If $p = \infty$, then for every $l > 0$ there is a $\psi_l \in L^q(\Omega)$ with $|\phi(\boldsymbol{x},\boldsymbol{y})| \leq \psi_l(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{y} \in \mathbb{R}^m$ with $|\boldsymbol{y}| \leq l$.*

Theorem 1.3. [15, 17] *Let $p \geq 1$ be a real number and let $q \geq 1$ be the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$). Assume that $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a (p,q)-Carathéodory function such that $u \in L^p(\Omega)$ implies that $\phi(x,t,u(x,t)) \in L^q(\Omega)$. Then, the superposition operator $G : L^p(\Omega) \to L^q(\Omega)$ defined by $Gu(x,t) = \phi(x,t,u(x,t))$ is continuous and bounded. In particular, there exist a constant $C > 0$ and $\psi \in L^q(\Omega)$ such that*

$$|\phi(x,t,u(x,t))| \leq C|u(x,t)|^{p-1} + \psi(x,t),\ \forall x \in I,\ t \in J. \tag{1.3}$$

Theorem 1.4. *(*Arzéla-Ascoli*)* [14] *Let X be a compact space and $B \subseteq C(X)$. Then B is compact if and only if B is uniformly bounded and equicontinuous.*

Theorem 1.5. *(*Schaefer's fixed point theorem*)* [14] *Let T be a continuous and compact operator of a Banach space X into itself such that the set*

$$\{u \in X : u = \lambda T u \ \mathit{for\ some}\ 0 \leq \lambda \leq 1\},$$

is bounded. Then T has a fixed point.

Theorem 1.6. *(*Schauder's fixed point theorem*)* [14] *Let K be a closed convex subset of a Banach space X and $T : K \to K$ a compact, continuous operator. Then T has a fixed point in K.*

Theorem 1.7. *(*General version of Gronwall's inequality*)* [8] *Suppose that ϕ is a positive and measurable function over $I \times J$ and $\rho \in L^1(\Omega)$ with $\rho(x,t) \geq 0$, $\forall x \in I,\ t \in J$ and $\int_c^d \int_a^b \rho(x,t)\phi(x,t)\mathrm{d}x\mathrm{d}t < +\infty$. If for some A we have*

$$\phi(x,t) \leq A + \int_c^d \int_a^b \rho(x,t)\phi(x,t)\mathrm{d}x\mathrm{d}t,\ \forall x \in I,\ t \in J,$$

then

$$\phi(x,t) \leq A \exp\left(\|\rho\|_1\right),\ \forall x \in I,\ t \in J.$$

2. Existence of an L^p-solution

We will use the following notations throughout this section:
For every $f \in L^p(\Omega)$ define

$$\|f\|_p = \left(\int_c^d \int_a^b |f(x,t)|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}},\ \mathit{for}\ 1 \leq p < \infty,$$
$$\|f\|_p = \mathit{ess} \sup_{\substack{x \in I, \\ t \in J}} |f(x,t)|,\ \mathit{for}\ p = \infty.$$

Similarly, for every $k \in L^p(X)$ where $X = \Omega \times \Omega$, define

$$\|k\|_p = \left(\int_c^d \int_a^b \int_c^d \int_a^b |k(x,t,y,z)|^p \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}z\right)^{\frac{1}{p}},\ \mathit{for}\ 1 \leq p < \infty,$$
$$\|k\|_p = \mathit{ess} \sup_{\substack{x,y \in I, \\ t,z \in J}} |k(x,t,y,z)|,\ \mathit{for}\ p = \infty.$$

Theorem 2.1. *Let $1 \leq p, q \leq \infty$ and let q be the conjugate of p. Suppose that the operator $T : L^p(\Omega) \to \mathbb{R}$ is defined by*

$$Tu(x,t) = f(x,t) + \lambda \int_c^d\int_a^b k(x,t,y,z)\phi(y,z,u(y,z))\mathrm{d}y\mathrm{d}z, \tag{2.1}$$

and $f \in L^p(\Omega)$ and there exists a positive constant C and a function $\psi \in L^q(\Omega)$ such that

$$|\phi(x,t,u(x,t))| \leq C|u(x,t)|^{p-1} + \psi(x,t),\ \forall x \in I,\ t \in J,$$

holds. In addition, suppose that $k \in L^p(X)$, where $X = \Omega \times \Omega$. Then for every $u \in L^p(\Omega)$, $Tu \in L^p(\Omega)$ and T is a compact operator.

Proof. In order to show that T is a compact operator on $L^p(\Omega)$, we rewrite T as $T = T_f + T_k$, where

$$T_f u(x,t) = f(x,t),$$

and

$$T_k u(x,t) = \lambda \int_c^d\int_a^b k(x,t,y,z)\phi(y,z,u(y,z))\mathrm{d}y\mathrm{d}z, \tag{2.2}$$

are defined on $L^p(\Omega)$. It is clear that $T_f : L^p(\Omega) \to L^p(\Omega)$ is a rank one operator and so is a compact operator on $L^p(\Omega)$. Thus it suffices to prove that $T_k : L^p(\Omega) \to L^p(\Omega)$ is a compact operator. First, we show that for each $u \in L^p(\Omega)$, we have $T_k u \in L^p(\Omega)$.
Let $u \in L^p(\Omega)$. Then we have

$$\begin{aligned}\|T_k u\|_p^p &= \left\|\lambda \int_c^d\int_a^b k(x,t,y,z)\phi(y,z,u(y,z))\mathrm{d}y\mathrm{d}z\right\|_p^p \\ &\leq |\lambda| \int_c^d\int_a^b \left(\int_c^d\int_a^b |k(x,t,y,z)\phi(y,z,u(y,z))|\mathrm{d}y\mathrm{d}z\right)^p \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.3}$$

By using the Hölder's inequality and the assumptions of the theorem, we rewrite (2.3) as

$$\begin{aligned}\|T_k u\|_p^p &\le |\lambda| \int_c^d\int_a^b \left(\int_c^d\int_a^b |k(x,t,y,z)|^p \mathrm{d}y\mathrm{d}z\right) \\ &\quad \times \left(\int_c^d\int_a^b |\phi(y,z,u(y,z))|^q \mathrm{d}y\mathrm{d}z\right)^{\frac{p}{q}} \mathrm{d}x\mathrm{d}t \\ &\le |\lambda|\|k\|_p^p \left(\int_c^d\int_a^b (C|u(x,t)|^{p-1}+\psi(x,t))^q \mathrm{d}y\mathrm{d}z\right)^{\frac{p}{q}} \\ &\le |\lambda|\|k\|_p^p \left(\|C|u|^{p-1}+\psi\|_q\right)^p \\ &\le |\lambda|\|k\|_p^p \left(C\||u|^{p-1}\|_q+\|\psi\|_q\right)^p \\ &= |\lambda|\|k\|_p^p \left(C\|u\|_p^{p-1}+\|\psi\|_q\right)^p.\end{aligned}$$

Consequently, we have

$$\|T_k u\|_p \le |\lambda|^{\frac{1}{p}}\|k\|_p \left(C\|u\|_p^{p-1}+\|\psi\|_q\right).$$

By the assumptions of the theorem, we have $T_k u \in L^p(\Omega)$.

To prove the compactness of the operator T_k, we consider the following cases:

Case 1: Let $k \in C(X)$. Let $x, x_0 \in I$, $t, t_0 \in J$ and let $u \in L^p(\Omega)$ be arbitrary. Then, we have

$$\begin{aligned}|T_k u(x,t) - T_k u(x_0,t_0)| &\le |\lambda| \int_c^d\int_a^b |k(x,t,y,z)-k(x_0,t_0,y,z)| \\ &\quad \times (C|u(y,z)|^{p-1}+\psi(y,z))\mathrm{d}y\mathrm{d}z \\ &\le |\lambda| \sup_{\substack{y\in I,\\ z\in J}} |k(x,t,y,z)-k(x_0,t_0,y,z)| \\ &\quad \times [(d-c)(b-a)]^{\frac{1}{p}}(C\|u\|_p^{p-1}+\|\psi\|_q). \end{aligned} \tag{2.4}$$

Since X is compact, k is uniformly continuous on X, thus the inequality in (2.4) implies that $T_k u \in C(\Omega)$. Hence T_k is an operator from $L^p(\Omega)$ into $C(\Omega)$.

Suppose that $S = \{u_n\}_{n\in\mathbb{N}}$ is a bounded subset of $L^p(\Omega)$ (i.e., there exists a positive constant M such that $\|u_n\|_p \le M$ for each $n \in \mathbb{N}$). We show that T_k is uniformly bounded operator on $C(\Omega)$. Let $u_n \in S$.

Then

$$\begin{aligned}|T_k u_n(x,t)| &\leq |\lambda| \int_c^d\int_a^b |k(x,t,y,z)| \left(C|u_n(y,z)|^{p-1} + \psi(y,z)\right) \mathrm{d}y\mathrm{d}z \\ &\leq |\lambda| ess \sup_{\substack{x\in I,\\ t\in J}} |k(x,t,y,z)[(d-c)(b-a)]^{\frac{1}{p}}(C\|u_n\|_p^{p-1} + \|\psi\|_q) \\ &\leq |\lambda|\|k\|_\infty[(d-c)(b-a)]^{\frac{1}{p}}(CM^{p-1} + \|\psi\|_q), \\ &\quad \forall n \in \mathbb{N},\ x \in I,\ t \in J.\end{aligned}$$

To prove the compactness of T_k, it suffices to show $\{T_k(u_n) : u_n \in S\}$ is equicontinuous. Indeed, for every $x, x_0, y \in I$ and $t, t_0, z \in J$ by using the Hölder's inequality we have

$$\begin{aligned}|T_k u_n(x,t) - T_k u_n(x_0,t_0)| \leq |\lambda| \sup_{\substack{y\in I,\\ z\in J}} &|k(x,t,y,z) - k(x_0,t_0,y,z)| \\ &\times[(d-c)(b-a)]^{\frac{1}{p}}(CM^{p-1} + \|\psi\|_q).\end{aligned}$$

This implies that T_k is equicontinuous on Ω, since k is uniformly continuous on X. By the Arzéla-Ascoli theorem (Theorem 1.4), T_k is compact on $C(\Omega)$. Therefore T_k is compact on $L^p(\Omega)$, since $C(\Omega)$ is dense in $L^p(\Omega)$.

Case 2: Suppose that $k \in L^p(X)$. Then there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ of continuous kernels such that $\|k_n - k\|_p \to 0$ as $n \to \infty$. Suppose that S is a bounded subset of $L^p(\Omega)$ in previous case. Since $k_1(x,t,y,z)$ is a continuous function on X, by using the result of case 1, T_{k_1} is a compact operator. Thus there exists a subsequence $\{u_n^{(1)}\}_{n\in\mathbb{N}}$ of $\{u_n\}$ such that $\{T_{k_1}(u_n^{(1)})\}$ is convergent. Similarly, there exists a subsequence $\{u_n^{(2)}\}_{n\in\mathbb{N}}$ of $\{u_n^{(1)}\}$ such that $\{T_{k_2}(u_n^{(2)})\}_{n\in\mathbb{N}}$ is convergent. Generally, for each $m \in \mathbb{N}$ there exists a subsequence $\{u_n^{(m)}\}_{n\in\mathbb{N}}$ of $\{u_n^{(m-1)}\}_{n\in\mathbb{N}}$ such that $\{T_{k_m}(u_n^{(m)})\}$ is convergent. Considering the diagonal subsequence $\{u_n^{(n)}\}_{n\in\mathbb{N}}$, we show that $\{T_k(u_n^{(n)})\}$ is a Cauchy sequence on $L^p(\Omega)$. For every $m, n, l \in \mathbb{N}$, we have

$$\begin{aligned}\|T_k(u_m^{(m)}) - T_k(u_l^{(l)})\|_p \leq\ &\|T_k(u_m^{(m)}) - T_{k_n}(u_m^{(m)})\|_p \\ &+\|T_{k_n}(u_m^{(m)}) - T_{k_n}(u_l^{(l)})\|_p \\ &+\|T_{k_n}(u_l^{(l)}) - T_k(u_l^{(l)})\|_p.\end{aligned} \tag{2.5}$$

Since $\{T_{k_n}(u_l^{(l)})\}$ is convergent, for $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\|T_{k_n}(u_m^{(m)}) - T_{k_n}(u_l^{(l)})\|_p < \frac{\epsilon}{3}, \ \forall \ m, l \geq N_1. \tag{2.6}$$

Also, we have

$$\|T_k(u_m^{(m)}) - T_{k_n}(u_m^{(m)})\|_p^p \leq |\lambda| \|k - k_n\|_p^p (CM^{p-1} + \|\psi\|_q)^p. \tag{2.7}$$

Since $\|k_n - k\|_p \to 0$ as $n \to \infty$, for $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$\|T_k(u_m^{(m)}) - T_{k_n}(u_m^{(m)})\|_p < \frac{\epsilon}{3}, \ \forall n \geq N_2. \tag{2.8}$$

By (2.5)-(2.8), we have

$$\|T_k(u_m^{(m)}) - T_k(u_l^{(l)})\|_p < \epsilon.$$

Hence, $\{T_k(u_n^{(n)})\}$ is a Cauchy sequence on $L^p(\Omega)$ and so it is convergent. Thus T_k is a compact operator on $L^p(\Omega)$. $\square$

Theorem 2.2. *Consider the 2DIE* (1.1)*. Suppose that $1 \leq p \leq 2$ and $q \geq 1$ is the conjugate of p, and $k \in L^p(X)$, $f \in L^p(\Omega)$ and ϕ is a (p, q)-Carathéodory function. Then*

(i) *If $1 \leq p < 2$, then Eq.* (1.1) *has an L^p-solution.*
(ii) *If $p = 2$ and the following conditions hold for the kernel k:*
  (a) *$C|\lambda|^{\frac{1}{2}}\|k\|_2 < 1$ where C is the constant given in Theorem 1.3;*
  (b) *$k(x, t, y, z) = 0, \ \forall y \geq x, \ \forall z \geq t$ and*

$$|k(x, t, y, z)| \leq |k_1(x, t)||k_2(y, z)|,$$

where k_1 is bounded and measurable on Ω and $k_2 \in L^p(\Omega)$, then Eq. (1.1) *has an L^2-solution.*

Proof. Since ϕ is a (p, q)-Carathéodory function, it satisfies Theorem 1.3. Thus by Theorem 2.1 the operator T in (2.1) is compact on $L^p(\Omega)$. To prove existence of an L^p-solution of Eq. (1.1), we use the Schaefer's fixed point theorem (Theorem 1.5). First, we show that T is continuous. Let $u \in L^p(\Omega)$ and let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in $L^p(\Omega)$ that converges to u. Also, let G be the superposition operator defined in (1.2). Then by using the Hölder's inequality, we obtain

$$\begin{aligned}\|Tu_n - Tu\|_p^p &\leq |\lambda| \int_c^d\int_a^b \Big(\int_c^d\int_a^b |k(x,t,y,z)| \\ &\qquad \times |Gu_n(y,z) - Gu(y,z))| \mathrm{d}y\mathrm{d}z\Big)^p \mathrm{d}x\mathrm{d}t \\ &\leq |\lambda| \|k\|_p^p \|G(u_n) - G(u)\|_p^p. \qquad (2.9)\end{aligned}$$

Since G is continuous, inequality (2.9) implies that T is continuous.

To prove the existence results, it suffices to show the set $\mathcal{F} = \{u \in L^p(\Omega) : u = \lambda Tu \ for\ some\ 0 \leq \lambda \leq 1\}$ is a bounded set. Let $u \in \mathcal{F}$. Then by reusing (1.3) and the Hölder's inequality, we have

$$\|u\|_p = |\lambda| \|Tu\|_p \leq \|Tu\|_p \leq \|f\|_p + |\lambda|^{\frac{1}{p}} \|k\|_p (C\|u\|_p^{p-1} + \|\psi\|_q), \qquad (2.10)$$

or equivalently,

$$\|u\|_p^{p-1} (\|u\|_p^{2-p} - |\lambda|^{\frac{1}{p}} C\|k\|_p) \leq \|f\|_p + |\lambda|^{\frac{1}{p}} \|k\|_p \|\psi\|_q.$$

Since $1 \leq p < 2$, $p-1$ and $2-p$ are nonnegative constants, then there exists a positive constant M such that $\|u\|_p \leq M$ and so $\mathcal{F}$ is a bounded set. By using the Schaefer's fixed point theorem, we conclude that the operator T has a fixed point in $L^p(\Omega)$. Hence Eq. (1.1) has an L^p-solution.

Let $p = 2$ and $u \in \mathcal{F}$ and the condition (a) holds. Then from (2.10), we have

$$\|u\|_2 \leq \|Tu\|_2 \leq \|f\|_2 + |\lambda|^{\frac{1}{2}} \|k\|_2 (C\|u\|_2 + \|\psi\|_2),$$

or equivalently,

$$\|u\|_2 \leq \frac{\|f\|_2 + |\lambda|^{\frac{1}{2}} \|k\|_2 \|\psi\|_2}{1 - C|\lambda|^{\frac{1}{2}} \|k\|_2},$$

which implies $\|u\|_2 \leq M$, since $C|\lambda|^{\frac{1}{2}} \|k\|_2 < 1$. Hence $\mathcal{F}$ is bounded. It is evident again from the Schaefer's fixed point theorem that Eq. (1.1) has an L^2-solution.

Now, suppose that the condition (b) holds. Then

$$\begin{aligned}
|u(x,t)| &\leq |f(x,t)| + |\lambda| \int_c^t\int_a^x |k(x,t,y,z)||\phi(y,z,u(y,z))|\mathrm{d}y\mathrm{d}z \\
&\leq \|f\|_\infty + |\lambda| \int_c^t\int_a^x |k_1(x,t)||k_2(y,z)|(C|u(y,z)| + \psi(y,z))\mathrm{d}y\mathrm{d}z \\
&\leq \|f\|_\infty + |\lambda|\|k_1\|_\infty\|k_2\|_2\|\psi\|_2 \\
&\quad + |\lambda|C\|k_1\|_\infty \int_c^t\int_a^x |k_2(y,z)||u(y,z)|\mathrm{d}y\mathrm{d}z \\
&\leq A + \int_c^t\int_a^x \rho(y,z)|u(y,z)|\mathrm{d}y\mathrm{d}z,
\end{aligned}$$

where $A = \|f\|_\infty + |\lambda|\|k_1\|_\infty\|k_2\|_2\|\psi\|_2$ and $\rho(y,z) = C|\lambda|\|k_1\|_\infty|k_2(y,z)| \in L^2(\Omega)$. Since $L^2(\Omega) \subset L^1(\Omega)$, by Gronwall's inequality we conclude

$$\|u\|_2 \leq \sqrt{(d-c)(b-a)}A\exp(\|\rho\|_1).$$

Then the Schaefer's fixed point theorem implies that Eq. (1.1) has an L^2-solution. □

Theorem 2.3. *Let $p, q \geq 1$ and let q be the conjugate of p. Suppose that $f \in L^p(\Omega)$ and $k \in L^p(X)$ and there exist a positive constant C' and $\psi \in L^p(\Omega)$ such that*

$$|\phi(y,z,u(y,z))| \leq C'|u(y,z)| + \psi(y,z) \text{ for a.e. } y \in I,\ z \in J.$$

Suppose that there exists a weight function ω on Ω (i.e., ω is nonnegative, measurable and a bounded function on Ω) such that the function

$$\Phi(x,t) = \begin{cases} \int_c^d\int_a^b |k(x,t,y,z)|^q(\omega(y,z))^{\frac{-q}{p}}\mathrm{d}y\mathrm{d}z)^{\frac{1}{q}}, & p > 1, \\ \sup\limits_{\substack{y\in I,\\ z\in J}} \dfrac{|k(x,t,y,z)|}{\omega(y,z)}, & p = 1, \end{cases}$$

belongs to $L^p(\Omega, d\omega)$. If $\|\lambda C'\Phi\|_{p,\omega} < 1$, then Eq. (1.1) has an L^p-solution.

Proof. The weighted space $L^p(\Omega, d\omega)$ is the space of real functions produced by the norm

$$\|f\|_{p,\omega} = \left(\int_c^d\int_a^b |f(x,t)|^p\omega(x,t)\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}.$$

It is easy to show that the norms $\|\cdot\|_p$ and $\|\cdot\|_{p,\omega}$ are equivalent. Hence, a bounded set of $L^p(\Omega, d\omega)$ is also bounded on $L^p(\Omega)$. Consider the ball

$$S_\alpha = \{f \in L^p(\Omega, d\omega) : \|f\|_{p,\omega} \le \alpha\},$$

in $L^p(\Omega, d\omega)$. Since the operator T_k given by (2.2) is compact on $L^p(\Omega)$ and S_α is a bounded subset of $L^p(\Omega)$, $T(S_\alpha)$ is relatively compact on $L^p(\Omega) \subset L^p(\Omega, d\omega)$.

Let $u \in S_\alpha$. Then

$$\begin{aligned}
|T_k u\|_{p,\omega}^p &\le |\lambda| \int_c^d\int_a^b \omega(x,t)\Big(\int_c^d\int_a^b \frac{|k(x,t,y,z)|}{(\omega(y,z))^{\frac{1}{p}}} \\
&\qquad \times(\omega(y,z))^{\frac{1}{p}}|\phi(y,z,u(y,z))|\mathrm{d}y\mathrm{d}z\Big)^p \mathrm{d}x\mathrm{d}t \\
&\le |\lambda| \int_c^d\int_a^b \omega(x,t)\left(\int_c^d\int_a^b |k(x,t,y,z)|^q(\omega(y,z))^{\frac{-q}{p}}\mathrm{d}y\mathrm{d}z\right)^{\frac{p}{q}} \\
&\qquad \times\Big(\int_c^d\int_a^b \omega(y,z)|\phi(y,z,u(y,z))|^p\mathrm{d}y\mathrm{d}z\Big)\mathrm{d}x\mathrm{d}t \\
&\le |\lambda| \int_c^d\int_a^b \omega(x,t)\left(\int_c^d\int_a^b |k(x,t,y,z)|^q(\omega(y,z))^{\frac{-q}{p}}\mathrm{d}y\mathrm{d}z\right)^{\frac{p}{q}} \\
&\qquad \times\Big(\int_c^d\int_a^b \omega(y,z)(C'|u(y,z)|+\psi(y,z))^p\mathrm{d}y\mathrm{d}z\Big)\mathrm{d}x\mathrm{d}t \\
&\le |\lambda|\|C'u+\psi\|_{p,\omega}^p \int_c^d\int_a^b \omega(x,t) \\
&\qquad \times\left(\int_c^d\int_a^b |k(x,t,y,z)|^q(\omega(y,z))^{\frac{-q}{p}}\mathrm{d}y\mathrm{d}z\right)^{\frac{p}{q}}\mathrm{d}x\mathrm{d}t \\
&\le |\lambda|(C'\|u\|_{p,\omega}+\|\psi\|_{p,\omega})^p\|\Phi\|_{p,\omega}^p. \qquad (2.11)
\end{aligned}$$

Since $Tu = f + T_k u$ and $\|u\|_{p,\omega} \le \alpha$, (2.11) implies

$$\|T_u\|_{p,\omega} \le \|f\|_{p,\omega} + |\lambda|\|\psi\|_{p,\omega}\|\Phi\|_{p,\omega} + |\lambda|C'\|u\|_{p,\omega}\|\Phi\|_{p,\omega}.$$

Thus, if $\|\lambda C'\Phi\|_{p,\omega} < 1$, then there exists $\varepsilon > 0$ such that for each $\alpha \ge \varepsilon$, we have

$$\alpha \ge \frac{|f\|_{p,\omega} + |\lambda|\|\psi\|_{p,\omega}\|\Phi\|_{p,\omega}}{1 - |\lambda|C'\|\Phi\|_{p,\omega}},$$

and so $T(S_\alpha) \subseteq S_\alpha$. Therefore T maps the closed convex subset S_α into itself. Since $T(S_\alpha)$ is compact, by the Schauder's fixed point theorem (Theorem 1.6) T has a fixed point. This completes the proof. $\square$

3. Conclusion

In this paper, we proved a major existence theorem for 2DIEs which is important in the numerical solution of these types of equations. It may be extended to the higher dimensional integral equations by authors as a future work.

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(Seyyed Ahmad Hosseini) Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

E-mail address: `a-hosseini@tabrizu.ac.ir`

(Sedaghat Shahmorad) Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

E-mail address: `shahmorad@tabrizu.ac.ir`

(Abolfazl Tari) Department of Mathematics, Faculty of Sciences, Shahed University, Tehran, Iran

E-mail address: `tari@shahed.ac.ir`