Title:

Symmetry classes of polynomials associated with the dihedral group

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SYMMETRY CLASSES OF POLYNOMIALS ASSOCIATED WITH THE DIHEDRAL GROUP

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ABSTRACT. In this paper, we obtain the dimensions of symmetry classes of polynomials associated with the irreducible characters of the dihedral group as a subgroup of the full symmetric group. Then we discuss the existence of o-basis of these classes.

Keywords: Relative symmetric polynomials, irreducible characters, dihedral group, linear Diophantine equations, p-adic valuation.

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1. Introduction and preliminaries

One of the most classical areas of algebra, the theory of symmetric polynomials has been known to be connected to combinatorics, representation theory, and other fields of mathematics. For a review of the theory of symmetric polynomials, one can see the book of Macdonald [10]. The relative symmetric polynomials as a generalization of symmetric polynomials are introduced in [12] by M. Shahryari. In [15, 16], the authors studied the space of relative symmetric polynomials (symmetry class of polynomials) with respect to the irreducible characters of the cyclic and dicyclic groups. In this paper we study the symmetry class of polynomials with respect to an irreducible character of the dihedral group as a subgroup of the full symmetric group. We first give a review of this notion (for more details, see [12]).

Let $H_d[x_1, \ldots, x_n]$ be the complex space of homogeneous polynomials of degree d with the independent commuting variables $x_1, \ldots, x_n$. Let $\Gamma_{n,d}^+$ be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers such that $\sum_{i=1}^n \alpha_i = d$. For any $\alpha \in \Gamma_{n,d}^+$, let X^α be the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$. Then the set $\{X^\alpha |\ \alpha \in \Gamma_{n,d}^+\}$ is a basis for $H_d[x_1, \ldots, x_n]$. An inner product on $H_d[x_1, \ldots, x_n]$ is defined by

$$\langle X^\alpha, X^\beta \rangle = \delta_{\alpha,\beta}.$$

Let G be a subgroup of the symmetric group S_n and suppose χ is an irreducible complex character of G. Define the action of G on $H_d[x_1, \ldots, x_n]$ by

$$q^\sigma(x_1, \ldots, x_n) = q(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

So we can consider σ as an operator on $H_d[x_1, \ldots, x_n]$. Define

$$T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma.$$

It is easy to see that $T(G, \chi)$ is an idempotent element in the group algebra $\mathbb{C}G$. By the orthogonality relations of characters, the set

$$\{T(G, \chi) |\ \chi \in \mathrm{Irr}(G)\}$$

is a complete set of orthogonal idempotents, where $\mathrm{Irr}(G)$ is the set of irreducible complex characters of G.

The *symmetry class of polynomials of degree d with respect to G and χ* is the image of $H_d[x_1, \ldots, x_n]$ under the operator $T(G, \chi)$ and is denoted by $H_d(G, \chi)$. So, we have the following orthogonal direct sum decomposition

$$H_d[x_1, \ldots, x_n] = \bigoplus_{\chi \in \mathrm{Irr}(G)} H_d(G, \chi).$$

For any $q \in H_d[x_1, \ldots, x_n]$,

$$q^* = T(G, \chi)(q)$$

is called a *symmetrized polynomial with respect to G and χ*. In particular, if $\alpha \in \Gamma_{n,d}^+$, we denote symmetrized monomial $(X^\alpha)^*$ by $X^{\alpha,*}$. Clearly

$$H_d(G, \chi) = \langle X^{\alpha,*} | \alpha \in \Gamma_{n,d}^+ \rangle.$$

Also, we define the action of G on $\Gamma_{n,d}^+$ by

$$\alpha\sigma = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma_{n,d}^+$ and $\sigma \in G$. Let $O(\alpha) = \{\alpha\sigma | \sigma \in G\}$ be the orbit of α and let G_α be the stabilizer subgroup of α under the action of G, i.e., $G_\alpha = \{\sigma \in G | \alpha\sigma = \alpha\}$. For any $\alpha, \beta \in \Gamma_{n,d}^+$, we have (see [12, Proposition 2.4])

$$\langle X^{\alpha,*}, X^{\beta,*} \rangle = \begin{cases} 0 & \text{if } O(\alpha) \neq O(\beta), \\ \\ \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\beta} \chi(\tau^{-1}\sigma) & \text{if } \alpha = \beta\tau \text{ for some } \tau \in G. \end{cases}$$

(1.1)

In particular

$$\|X^{\alpha,*}\|^2 = \chi(1) \frac{[\chi, 1]_{G_\alpha}}{|G : G_\alpha|},$$

where $[\ ,\ ]_G$ is the inner product of characters. Hence $X^{\alpha,*} \neq 0$ if and only if $[\chi, 1]_{G_\alpha} \neq 0$. Let Δ be a set of representatives of orbits of $\Gamma_{n,d}^+$ under the action of G and define

$$\bar{\Delta} = \{\alpha \in \Delta | \ [\chi, 1]_{G_\alpha} \neq 0\}.$$

For $\alpha \in \bar{\Delta}$, let $H_d^{\alpha,*} = \langle X^{\alpha\sigma,*} | \sigma \in G \rangle$. It follows that

(1.2)
$$H_d(G, \chi) = \bigoplus_{\alpha \in \bar{\Delta}} H_d^{\alpha,*}$$

is an orthogonal direct sum. Also we have [12]

(1.3)
$$\dim H_d^{\alpha,*} = \chi(1)[\chi, 1]_{G_\alpha}.$$

If $\alpha = \gamma g$ and $\beta = \gamma g'$, then $\beta {g'}^{-1} g = \alpha$, and so if we let $\tau = {g'}^{-1}g$ and use (1.1), the equality

(1.4)
$$\langle X^{\alpha,*}, X^{\beta,*} \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in g^{-1}G_\gamma g'} \chi(\sigma)$$

is derived.

Let $\sigma \in G$ be any element with the cyclic structure $[a_1, \ldots, a_m]$, (i.e., σ is equal to a product of m disjoint cycles of lengths $a_1, \ldots, a_m$, respectively). Let $Q(d, \sigma)$ be the number of non-negative integer solutions of the equation

$$a_1 t_1 + a_2 t_2 + \cdots + a_m t_m = d.$$

For a compact formula of $Q(d, \sigma)$, see [13].

By [12, Proposition 2.10], we have

(1.5) $$\dim H_d(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) Q(d, \sigma).$$

Recall that the number of non-negative integer solutions of the equation $t_1 + \cdots + t_m = d$ is (see [1])

$$\binom{d + m - 1}{m - 1} = \frac{(d + m - 1)!}{(m - 1)! \, d!}.$$

An orthogonal basis of $H_d(G, \chi)$ of the form $\{X^{\alpha,*} | \alpha \in S\}$, where S is a subset of $\Gamma_{n,d}^+$, is called an *o-basis* of $H_d(G, \chi)$. If χ is linear, then the set $\{X^{\alpha,*} | \alpha \in \bar{\Delta}\}$ is an o-basis of $H_d(G, \chi)$. Otherwise, the existence of an o-basis for $H_d(G, \chi)$ is not guaranteed. The existence of o-basis for symmetry classes of tensors has been studied in several articles (see e.g., [2, 3, 4, 5, 6, 14]).

In this paper we obtain the dimension of $H_d(G, \chi)$, when G is the dihedral group. Then a necessary and sufficient condition for existence of an o-basis of $H_d(G, \chi)$ is given. A similar result has been obtained for symmetry classes of tensors in [7, 11].

2. The Dimensions

The subgroup D_{2n} of $S_n (n \geq 3)$ generated by the elements

$$\sigma = (1 \; 2 \; \cdots \; n) \;\; \text{and} \;\; \tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & n & n-1 & \cdots & 3 & 2 \end{pmatrix}$$

is the *dihedral group of degree n*. The generators σ and τ satisfy

$$\sigma^n = 1 = \tau^2 \; \text{and} \; \tau^{-1} \sigma \tau = \sigma^{-1}$$

(see [8], p. 50). In particular,

$$D_{2n} = \{\sigma^k, \; \tau\sigma^k | \; 0 \leq k < n\}.$$

For each integer h with $0 < h < n/2$, D_{2n} has an irreducible character χ_h of degree 2 given by

$$\chi_h(\sigma^k) = 2 \cos \frac{2kh\pi}{n}, \;\; \chi_h(\sigma^k \tau) = 0, \;\;\; 0 \leq k < n.$$

The other characters of D_{2n} are of degree 1, namely ψ_j. For the character table of D_{2n}, we refer the reader to [9, p. 182].

In this section, we obtain the dimensions of the symmetry classes of polynomials associated with the irreducible characters of the dihedral group. For any positive integer d, define the multiplicative function $f_d : N \to N$ by

$$f_d(k) = \begin{cases} 1 & \text{if } k \text{ divides } d, \\ 0 & \text{otherwise.} \end{cases}$$

The above function will be used in the following theorems.

Theorem 2.1. *Let* $G = D_{2n}\,(n \geq 3)$*. If* n *is even, then we have*

(a)

$$
\begin{aligned}
dimH_d(G, \psi_1) &= \frac{1}{2n}\left(\sum_{k=0}^{n-1}\binom{\frac{d(n,k)}{n} + (n,k) - 1}{(n,k) - 1}f_d\left(\frac{n}{(n,k)}\right)\right. \\
&\quad + \left.\frac{n}{2}Q(d,\tau) + \frac{n}{2}\binom{\frac{d}{2}+\frac{n}{2}-1}{\frac{n}{2}-1}f_d(2)\right),
\end{aligned}
$$

(b)

$$
\begin{aligned}
dimH_d(G, \psi_2) &= \frac{1}{2n}\left(\sum_{k=0}^{n-1}\binom{\frac{d(n,k)}{n}+(n,k)-1}{(n,k)-1}f_d\left(\frac{n}{(n,k)}\right)\right. \\
&\quad - \left.\frac{n}{2}Q(d,\tau) - \frac{n}{2}\binom{\frac{d}{2}+\frac{n}{2}-1}{\frac{n}{2}-1}f_d(2)\right),
\end{aligned}
$$

(c)

$$
\begin{aligned}
dimH_d(G, \psi_3) &= \frac{1}{2n}\left(\sum_{k=0}^{n-1}\binom{\frac{d(n,k)}{n}+(n,k)-1}{(n,k)-1}f_d\left(\frac{n}{(n,k)}\right)\right. \\
&\quad + \left.\frac{n}{2}Q(d,\tau) - \frac{n}{2}\binom{\frac{d}{2}+\frac{n}{2}-1}{\frac{n}{2}-1}f_d(2)\right),
\end{aligned}
$$

(d)

$$
\begin{aligned}
dimH_d(G, \psi_4) &= \frac{1}{2n}\left(\sum_{k=0}^{n-1}\binom{\frac{d(n,k)}{n}+(n,k)-1}{(n,k)-1}f_d\left(\frac{n}{(n,k)}\right)\right. \\
&\quad - \left.\frac{n}{2}Q(d,\tau) + \frac{n}{2}\binom{\frac{d}{2}+\frac{n}{2}-1}{\frac{n}{2}-1}f_d(2)\right),
\end{aligned}
$$

(e)

$$dim H_d(G, \chi_h) = \frac{2}{n}\left(\sum_{k=0}^{n-1} \cos\frac{2kh\pi}{n}\left(\begin{array}{c}\frac{d(n,k)}{n}+(n,k)-1\\(n,k)-1\end{array}\right)f_d(\frac{n}{(n,k)})\right),$$

where $[u]$ is the integer part of u, (n, k) denotes the greatest common divisor of n and k, and

$$Q(d, \tau) = \begin{cases} \left(\begin{array}{c}\frac{d}{2}+\frac{n}{2}-1\\\frac{n}{2}-1\end{array}\right) + 2\left(\begin{array}{c}\frac{d}{2}+\frac{n}{2}-1\\\frac{n}{2}\end{array}\right) & \textit{if } d \textit{ is even,} \\[3em] 2\left(\begin{array}{c}[\frac{d}{2}]+\frac{n}{2}\\\frac{n}{2}\end{array}\right) & \textit{if } d \textit{ is odd.} \end{cases}$$

Proof. If π is a cycle of length a and $(k, a) = d$, then π^k has d cycles of length a/d and therefore $c(\pi^k) = d = (k, a)$, where $c(\pi)$ denotes the number of cycles, including cycles of length one, in the disjoint cycle factorization of σ. So we have

(i) $c(1) = n$ and then $Q(d, 1)$ is the number of non-negative integer solutions of the equation $t_1 + \cdots + t_n = d$ and it is equal to $\left(\begin{array}{c}d+n-1\\n-1\end{array}\right)$,

(ii) $c(\sigma^{\frac{n}{2}}) = n/2$ and the length of each cycle is 2, thus

$$Q(d, \sigma^{\frac{n}{2}}) = \left(\begin{array}{c}\frac{d}{2}+\frac{n}{2}-1\\\frac{n}{2}-1\end{array}\right)f_d(2),$$

(iii) $c(\sigma^k) = (n, k)$ and the length of each cycle is $n/(n, k)$, so

$$Q(d, \sigma^k) = \left(\begin{array}{c}\frac{d(n,k)}{n}+(n,k)-1\\(n,k)-1\end{array}\right)f_d(\frac{n}{(n, k)}), \ 1 \le k \le \frac{n}{2}-1.$$

(iv) Considering the cycle structure of $\sigma\tau$ given as

$$\sigma\tau = (1\ 2)(2\ n-1)\ \cdots\ (\frac{n}{2}\ \frac{n+2}{2}),$$

we have $c(\sigma\tau) = n/2$ and the length of each cycle is 2, thus $Q(d, \sigma\tau)$ is the number of non-negative integer solutions of the equation

$$2t_1 + \cdots + 2t_{\frac{n}{2}} = d$$

and it is equal to $\begin{pmatrix} \frac{d}{2} + \frac{n}{2} - 1 \\ \frac{n}{2} - 1 \end{pmatrix} f_d(2)$.

(v) Considering the cycle structure of τ given as

$$\tau = (1)(2\ n)(3\ n - 1)\ \cdots\ (\frac{n}{2}\ \frac{n+4}{2})(\frac{n+2}{2}),$$

we have $c(\tau) = (n+2)/2$, τ has two cycles of length 1 and $(n-2)/2$ cycles of length 2. In this case $Q(d, \tau)$ is the number of non-negative integer solutions of the equation

(2.1) $\qquad t_1 + t_2 + 2t_3 + \ \cdots\ + 2t_{\frac{n}{2}} + 2t_{\frac{n}{2}+1} = d.$

For solving (2.1), we can give the possible values of t_1, t_2 and rewrite (2.1) of the form

$$t_3 + \ \cdots\ + t_{\frac{n}{2}} + t_{\frac{n}{2}+1} = \frac{d - t_1 - t_2}{2}.$$

Therefore,

$$Q(d, \tau) = \sum_{t_1=0}^{d} \sum_{t_2=0}^{d-t_1} \begin{pmatrix} \frac{d - t_1 - t_2}{2} + \frac{n}{2} - 2 \\ \frac{n}{2} - 2 \end{pmatrix} f_{d-t_1-t_2}(2).$$

Now, if we set $k = (d - t_1 - t_2)/2$, then by using the following equality

(2.2) $\qquad \sum_{j=0}^{n-m} \begin{pmatrix} m+j \\ m \end{pmatrix} = \begin{pmatrix} n+1 \\ m+1 \end{pmatrix},$

we get

$$
\begin{aligned}
Q(d,\tau) &= \sum_{t_1=0}^{d} \sum_{k=0}^{[\frac{d-t_1}{2}]} \binom{k+\frac{n}{2}-2}{\frac{n}{2}-2} \\
&= \sum_{t_1=0}^{d} \binom{[\frac{d-t_1}{2}]+\frac{n}{2}-1}{\frac{n}{2}-1} \\
&= \begin{cases} \binom{\frac{d}{2}+\frac{n}{2}-1}{\frac{n}{2}-1} + 2\sum_{j=0}^{\frac{d}{2}-1}\binom{j+\frac{n}{2}-1}{\frac{n}{2}-1} & \text{if } d \text{ is even,} \\[2mm] 2\sum_{j=0}^{[\frac{d}{2}]}\binom{j+\frac{n}{2}-1}{\frac{n}{2}-1} & \text{if } d \text{ is odd,} \end{cases} \\
&= \begin{cases} \binom{\frac{d}{2}+\frac{n}{2}-1}{\frac{n}{2}-1} + 2\binom{\frac{d}{2}+\frac{n}{2}-1}{\frac{n}{2}} & \text{if } d \text{ is even,} \\[2mm] 2\binom{[\frac{d}{2}]+\frac{n}{2}}{\frac{n}{2}} & \text{if } d \text{ is odd.} \end{cases}
\end{aligned}
$$

Now by using the character table of D_{2n}, the proof is completed by (1.5). $\square$

Theorem 2.2. *Let* $G = D_{2n}\,(n \geq 3)$. *If* n *is odd, then we have*

(a)
$$
dim H_d(G,\psi_1) = \frac{1}{2n}\left(\sum_{k=0}^{n-1} \binom{\frac{d(n,k)}{n}+(n,k)-1}{(n,k)-1} f_d\!\left(\frac{n}{(n,k)}\right) + n\binom{[\frac{d}{2}]+\frac{n-1}{2}}{\frac{n-1}{2}} \right),
$$

(b)
$$
dim H_d(G,\psi_2) = \frac{1}{2n}\left(\sum_{k=0}^{n-1} \binom{\frac{d(n,k)}{n}+(n,k)-1}{(n,k)-1} f_d\!\left(\frac{n}{(n,k)}\right) - n\binom{[\frac{d}{2}]+\frac{n-1}{2}}{\frac{n-1}{2}} \right),
$$

(c)
$$
dim H_d(G,\chi_h) = \frac{2}{n}\left(\sum_{k=0}^{n-1} \cos\frac{2kh\pi}{n} \binom{\frac{d(n,k)}{n}+(n,k)-1}{(n,k)-1} f_d\!\left(\frac{n}{(n,k)}\right) \right).
$$

Proof. If n is odd, we have

$$\tau = (1)(2\ n)(3\ n-1)\ \cdots\ (\frac{n+1}{2}\ \frac{n+3}{2}),$$

thus $c(\tau) = (n+1)/2$, τ has one cycle of length 1 and $(n-1)/2$ cycles of length 2. In this case $Q(d,\tau)$ is the number of non-negative integer solutions of the equation

(2.3) $t_1 + 2t_2 + \ \cdots\ + 2t_{\frac{n-1}{2}} + 2t_{\frac{n+1}{2}} = d.$

Then, we can give the possible values of t_1 and rewrite (2.3) of the form

$$2t_2 + \ \cdots\ + 2t_{\frac{n-1}{2}} + 2t_{\frac{n+1}{2}} = d - t_1.$$

Therefore,

$$Q(d,\tau) = \sum_{t_1=0}^{d} \binom{\frac{d-t_1}{2}+\frac{n-3}{2}}{\frac{n-3}{2}} f_{d-t_1}(2).$$

Now, if we set $k = (d-t_1)/2$, then by using (2.2) we have

$$Q(d,\tau) = \sum_{k=0}^{[\frac{d}{2}]} \binom{k+\frac{n-3}{2}}{\frac{n-3}{2}} = \binom{[\frac{d}{2}]+\frac{n-1}{2}}{\frac{n-1}{2}}.$$

Similar to the proof of Theorem 2.1, we have

$$Q(d,1) = \binom{d+n-1}{n-1},$$

$$Q(d,\sigma^k) = \binom{\frac{d(n,k)}{n}+(n,k)-1}{(n,k)-1} f_d(\frac{n}{(n,k)}),\ 1 \le k \le \frac{n-1}{2}.$$

Then by using the character table of D_{2n}, the proof is completed by (1.5).   □

3. Orthogonal Basis

Now, we consider $G = D_{2n}(n \ge 3)$ and give a necessary and sufficient condition for existence of an o-basis for $H_d(G,\chi)$.
For any linear character χ, $H_d(G,\chi)$ has an o-basis. Thus we investigate the problem for the irreducible characters of degree 2 of $G = D_{2n}$, i.e., $\chi = \chi_h,\ 0 < h < n/2$.
For a given prime number p, the *p-adic valuation* of a number n is the highest exponent ν such that p^ν divides n; it is denoted by $\nu_p(n)$.

Lemma 3.1. *Let $G = D_{2n}(n \geq 3)$, and let $\chi = \chi_h(0 < h < \frac{n}{2})$. Then for $\gamma \in \bar{\Delta}$, we have $G_\gamma = \langle \sigma^k \rangle$ or $\langle \sigma^k \rangle < G_\gamma$, for some $0 \leq k < n$. In both cases $G_\gamma \cap \langle \sigma \rangle = \langle \sigma^k \rangle$, where $kh \equiv 0 \mod n$.*

Proof. See [11, Lemma 3]. □

Lemma 3.2. *Let $0 < h < \frac{n}{2}$. Then there exist t, t', $0 \leq t, t' < n$ such that $\cos\left(\frac{2(t-t')h\pi}{n}\right) = 0$ if and only if $\nu_2(\frac{2h}{n}) < 0$, where ν_2 is the 2-adic valuation.*

Proof. It is straightforward. □

Lemma 3.3. *Suppose $G = D_{2n}(n \geq 3)$, and $\chi = \chi_h(0 < h < \frac{n}{2})$. Let $\gamma \in \bar{\Delta}$ and suppose that G_γ is defined as $G_\gamma = \langle \sigma^k \rangle$, where $kh \equiv 0 \mod n$. If $\nu_2(\frac{2h}{n}) < 0$, then the subspace $H_d^{\alpha,*}$ has an o-basis.*

Proof. The proof is similar to the *case 1* of the part "only if" of the proof of [11, Theorem 1]. □

Lemma 3.4. *Let $G = D_{2n}(n \geq 3)$, and let $\chi = \chi_h(0 < h < \frac{n}{2})$. Let $\gamma \in \bar{\Delta}$ be such that $\langle \sigma^k \rangle < G_\gamma$ and $G_\gamma \cap \langle \sigma \rangle = \langle \sigma^k \rangle$, where $kh \equiv 0 \mod n$. If $\nu_2(\frac{h}{n}) < 0$, then the subspace $H_d^{\alpha,*}$ has an o-basis.*

Proof. The proof is similar to the *case 2* of the part "only if" of the proof of [11, Theorem 1]. □

Theorem 3.5. *If $G = D_{2n}(n \geq 2)$ and $\chi = \chi_h(0 < h < \frac{n}{2})$, then $H_d(G, \chi)$ has an o-basis if and only if $\nu_2(\frac{2h}{n}) < 0$.*

Proof. Assume that $H_d(G, \chi)$ has an orthogonal o-basis. Then by (1.2) for any $\gamma \in \bar{\Delta}$, the subspace $H_d^{\gamma,*}$ has an o-basis, in particular for $\gamma = (0, 0, \ldots, 0, d)$. In this case $\tau \in G_\gamma$, and $G_\gamma \cap C_n = \{1\}$, where C_n is the cyclic subgroup of G generated by σ. Moreover $G_\gamma = \{1, \tau\}$, because if $\tau\sigma^k \in G_\gamma$ with $0 \leq k < n$, then $\sigma^k = \tau\tau\sigma^k \in G_\gamma \cap C_n$, which implies $k = 0$. Then $[\chi, 1]_{G_\gamma} = 1$, so $\gamma \in \bar{\Delta}$. For all $g, g' \in G$, we have

$$
g^{-1}G_\gamma g' = \begin{cases} \{\sigma^{s-r}, \tau\sigma^{s+r}\} & \text{if } g = \sigma^r, \ g' = \sigma^s, \\ \{\tau\sigma^{s+r}, \sigma^{s-r}\} & \text{if } g = \sigma^r\tau, \ g' = \sigma^s, \\ \{\sigma^{s-r}, \tau\sigma^{s+r}\} & \text{if } g = \sigma^r\tau, \ g' = \sigma^s\tau. \end{cases}
$$

Therefore, by (1.4) we have

$$
(3.1) \ \langle X^{\gamma g,*}, X^{\gamma g',*} \rangle = \begin{cases} \frac{2}{n}\cos\frac{2(s-r)h\pi}{n} & \text{if } g = \sigma^r, \ g' = \sigma^s, \\ \frac{2}{n}\cos\frac{2(s-r)h\pi}{n} & \text{if } g = \sigma^r\tau, \ g' = \sigma^s, \\ \frac{2}{n}\cos\frac{2(s-r)h\pi}{n} & \text{if } g = \sigma^r\tau, \ g' = \sigma^s\tau. \end{cases}
$$

Using (1.3), we have $\dim H_d^{\gamma,*} = 2$. Since $H_d^{\gamma,*}$ has an o-basis, by (3.1), there exist t, t', $0 \leq t, t' < 2n$ such that $\cos\left(\frac{2(t-t')h\pi}{n}\right) = 0$. Therefore, Lemma 3.2 implies that $\nu_2(2h/n) < 0$.

Conversely, if $\nu_2(2h/n) < 0$, then applying Lemmas 3.1, 3.3 and 3.4 we conclude that for any $\gamma \in \bar{\Delta}$, the subspace $H_d^{\gamma,*}$ has orthogonal o-basis, therefore by (1.2) so does $H_d(G, \chi)$. $\hfill\square$

Remark 3.6. *If $0 < h < \frac{n}{2}$ and $h = h_2 h_{2'}$, where h_2 is a power of 2 and $h_{2'}$ is odd, then $\nu_2\left(\frac{2h}{n}\right) < 0$ if and only if $4h_2 | n$.*

Corollary 3.7. *Let $G = D_{2n}(n \geq 3)$. Then $H_d[x_1, \ldots, x_n]$ has an o-basis if and only if n is a power of 2.*

Proof. Suppose $n = n_2 n_{2'}$, where n_2 is a power of 2 and $n_{2'}$ is odd. Then $0 < n_2 < n/2$ and $4n_2 \nmid n$. Therefore if $\chi = \chi_{n_2}$, then Theorem 3.5 implies that $H_d(G, \chi)$ has no o-basis and so $H_d[x_1, \ldots, x_n]$ has no o-basis.

Conversely, assume that n is a power of 2. If $0 < h < n/2$, then $h_2 < n/4$ and so $4h_2 | n$, where $h = h_2 h_{2'}$ with h_2 a power of 2 and $h_{2'}$ odd. Then by Theorem 3.5 we conclude $H_d(G, \chi_h)$ has an o-basis, and so $H_d[x_1, \ldots, x_n]$ has no o-basis. $\hfill\square$

Acknowledgments

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