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**Title:**

**Convergence results: A new type iteration scheme for two asymptotically nonexpansive mappings in uniformly convex Banach spaces**

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## CONVERGENCE RESULTS: A NEW TYPE ITERATION SCHEME FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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**ABSTRACT.** In this article, we introduce a new type iterative scheme for approximating common fixed points of two asymptotically nonexpansive mappings is defined, and weak and strong convergence theorem are proved for the new iterative scheme in a uniformly convex Banach space. The results obtained in this article represent an extension as well as refinement of previous known results.

**Keywords:** Two-step iteration process, Asymptotically nonexpansive, Opial's condition, Weak and strong convergence, Common fixed point.

**MSC(2010):** Primary: 47H09; Secondary: 47H10.

### 1. Introduction

Let  $E$  be a uniformly convex Banach space,  $K$  a nonempty closed convex subset of  $E$ . Throughout this paper,  $\mathbb{N}$  denotes the set of all positive integers and  $F(T) := \{x : Tx = x\}$ . A mapping  $T : K \rightarrow K$  is said to be asymptotically nonexpansive if for a sequence  $\{k_n\} \subset [1, \infty)$  with  $\lim_{n \rightarrow \infty} k_n = 1$ , one has  $\|T^n x - T^n y\| \leq k_n \|x - y\|$ , for all  $x, y \in K$  and for all  $n \in \mathbb{N}$ . This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [8] in 1972. They proved that, if  $K$  is a nonempty bounded closed convex subset of a uniformly

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convex Banach space  $E$ , then every asymptotically nonexpansive self-mapping  $T$  of  $K$  has a fixed point. The fixed point iteration process for asymptotically nonexpansive mapping in Banach spaces, including Mann and Ishikawa iterations processes, have been studied extensively by many authors; see ([2]-[22]).

In 1991, Schu ([16, 17]) introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-map defined on nonempty closed convex and bounded subset of Hilbert space  $H$ .

In 2001, Xu and Ori [22] introduced the following implicit iteration scheme for common fixed points of a finite family of nonexpansive mappings  $\{T_i\}_{i=1}^N$  in Hilbert spaces:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, n \in \mathbb{N},$$

where  $T_n = T_{n \bmod N}$ , and they proved weak convergence theorem.

In 2008 Zhao et al. [23] introduced the following iteration scheme for common fixed points of nonexpansive mapping  $T$  in Banach space and proved weak and strong convergence theorems.:

$$x_n = \alpha_n x_{n-1} + \beta_n T x_{n-1} + \gamma_n T x_n, n \in \mathbb{N},$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequence in  $[0, 1]$ , and  $\alpha_n + \beta_n + \gamma_n = 1$ .

The Picard and Mann [12] iteration schemes for a mapping  $T : K \rightarrow K$  are defined by

$$(1.1) \quad \begin{cases} x_1 = x \in K, \\ x_{n+1} = T^n x_n, \end{cases}$$

and

$$(1.2) \quad \begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, n \in \mathbb{N}, \end{cases}$$

where  $\{\alpha_n\}$  is in  $(0, 1)$ . It is well-known that Picard iteration scheme converges for contractions but does not converge for nonexpansive mapping whereas Mann iteration scheme converges for nonexpansive mapping.

Several authors have studied weak and strong convergence problems of iterative sequence (with errors) for asymptotically nonexpansive type mappings in a Hilbert space or a Banach space (see [2, 13, 14, 16]). In

2007, Agrawal et al. [1] introduced the following iteration process:

$$(1.3) \quad \begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, n \in \mathbb{N} \end{cases}$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are in  $(0,1)$ . They showed that this process converges at a rate the same as that the Picard iteration and faster than Mann for contractions.

The above process deals with one mapping only. The case of two mappings in iterative processes has also remained under study since the work of Das and Debat [3] on a two mappings process. Also see, for example [11] and [19]. The problem of approximating common fixed points of finitely many mapping plays an important role in applied mathematics especially in the theory of evaluation equation and the minimization problems. See ([4, 5, 6, 20]), for example.

In 2001, Khan and Takahashi [11] approximated the fixed points of two asymptotically nonexpansive mappings  $S, T : K \rightarrow K$  through the sequence  $\{x_n\}$  given by

$$(1.4) \quad \begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, n \in \mathbb{N}, \end{cases}$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $(0,1)$ .

Recently, Khan et al. [10] modified the iteration process (1.4) to the case of two mappings as follows:

$$(1.5) \quad \begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, n \in \mathbb{N} \end{cases}$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $(0,1)$ .

In this paper, we introduced a new implicit iteration scheme as below:

$$(1.6) \quad \begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n = (1 - \beta_n)S^n x_n + \beta_n T^n x_n, n \in \mathbb{N}, \end{cases}$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $[0,1]$  for fixed points of asymptotically nonexpansive mapping  $T$  in a uniformly convex Banach space. Observe that if in (1.6) we set  $S = I, \beta_n = 0$ , then the scheme will

reduce to:

$$(1.7) \quad \begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n x_n, n \in \mathbb{N}, \end{cases}$$

where  $\alpha_n$  is a sequence in  $[0, 1]$ . This iteration process is referred to as Mann iteration process [12] and has been studied extensively by many authors to approximate fixed points of various mappings including nonexpansive mappings. Obviously if  $K$  is a nonempty compact convex subset of a real Banach space and  $T : K \rightarrow K$  is nonexpansive mapping then the Mann iteration process converges strongly to a fixed point of  $T$ . Also, the results of Guo et.al. [9] are special cases of our main results.

## 2. Preliminaries

Let us now gather some pre-requisites. Let  $X = \{x \in E : \|x\| = 1\}$  and let  $E^*$  be the dual of  $E$ . The space  $E$  has :

(i) Gâteaux differentiable norm if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each  $x, y \in S$ ;

(ii) Frèchet differentiable norm (see e.g. [18]) if for each  $x$  in  $S$ , the above limit exists and is attained uniformly for  $y$  in  $S$  and in this case, it is also well-known that

$$(2.1) \quad \begin{aligned} \langle h, J(x) \rangle + \frac{1}{2}\|x\|^2 &\leq \frac{1}{2}\|x + h\|^2 \\ &\leq \langle h, J(x) \rangle + \frac{1}{2}\|x\|^2 + b(\|h\|) \end{aligned}$$

for all  $x, h \in E$ , where  $J$  is the Frèchet derivative of the function  $\frac{1}{2}\|\cdot\|^2$  at  $x \in E$ ,  $\langle \cdot, \cdot \rangle$  is the dual pairing between  $E$  and  $E^*$ , and  $b$  is an increasing function defined on  $[0, \infty)$  such that  $\lim_{t \rightarrow 0} \frac{b(t)}{t} = 0$ ;

(iii) Opial's condition [15] if for any sequence  $\{x_n\}$  in  $E$ ,  $x_n \rightharpoonup x$  implies that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

for all  $y \in E$  with  $y \neq x$ .

The following are the definitions and lemma used to prove the results in the next section.

**Definition 2.1.** Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ . A mapping  $T : K \rightarrow K$  is said to be asymptotically nonexpansive on  $K$  if there exists a sequence  $k_n, k_n \geq 1$  with  $\lim_{n \rightarrow \infty} k_n = 1$ , such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for each  $x, y \in K$  and each  $n \geq 1$ . If  $k_n = 1$ , then  $T$  is known as a nonexpansive mapping.

**Definition 2.2.** A mapping  $T : K \rightarrow K$  is uniformly  $k$ -Lipschitzian if for some  $k > 0, \|T^n x - T^n y\| \leq k \|x - y\|, \forall x, y \in K$  and for all  $n \in \mathbb{N}$ .

**Definition 2.3.** Let  $E$  be a uniformly convex Banach space,  $K$  a nonempty closed convex subset of  $E$ , and  $T : K \rightarrow K$  an asymptotically nonexpansive mapping. Then  $I - T$  is said to be demi-closed at 0, if  $x_n \rightarrow x$  converges weakly and  $x_n - Tx_n \rightarrow 0$  converges strongly, then it implies that  $x \in K$  and  $Tx = x$ .

**Definition 2.4.** [7] Two mappings  $S, T : K \rightarrow K$ , where  $K$  is a subset of a normed space  $E$ , are said to satisfy condition  $(A')$  if there exists a nondecreasing function  $F : [0, \infty) \rightarrow [0, \infty)$  with  $F(0) = 0$  and  $f(r) > 0$  for all  $r \in (0, \infty)$  such that either  $\|x - Sx\| \geq f(d(x, F))$  or  $\|x - Tx\| \geq f(d(x, F))$  for all  $x \in K$  where  $d(x, F) = \inf\{\|x - p\| : p \in F = F(S) \cap F(T)\}$ .

**Lemma 2.5.** ([21], Lemma 1) : Let  $\{a_n\}, \{b_n\}$  and  $\{\delta_n\}$  be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n.$$

If  $\sum_{n=1}^{\infty} \delta_n < \infty$  and  $\sum_{n=1}^{\infty} b_n < \infty$ , then  $\lim_{n \rightarrow \infty} a_n$  exists. In particular, if  $\{a_n\}$  has a subsequence converging to 0, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 2.6.** [16] : Suppose that  $E$  is a uniformly convex Banach space and  $0 < p \leq t_n \leq q < 1$  for all  $n \in \mathbb{N}$ . Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences of  $E$  such that  $\limsup_{n \rightarrow \infty} \|x_n\| \leq r, \limsup_{n \rightarrow \infty} \|y_n\| \leq r$  and  $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$  hold for some  $r \geq 0$ . Then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .

### 3. Main results

In this section, we prove the approximate common fixed points of two asymptotically nonexpansive mappings for weak and strong sequel

results. In the consequence,  $F$  denotes the set of common fixed point of the mappings  $S$  and  $T$ .

**Theorem 3.1.** *Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ . Suppose  $S, T : K \rightarrow K$  are asymptotically nonexpansive mappings with  $\sum_{n=1}^{\infty} (k_n - 1) < \infty$  and  $\lim_{n \rightarrow \infty} k_n = 1$ . Consider a sequence  $\{x_n\}$  defined by the iteration process (1.6), where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequence in  $[0, 1]$ . If  $F(S) \cap F(T) \neq \phi$ , and*

$$(3.1) \quad \|x - S^n y\| \leq \lambda \|T^n x - S^n y\|,$$

for all  $x, y \in K$ , where  $\lambda > 1$ , then

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$$

for all  $p \in F(S) \cap F(T)$ .

*Proof.* Let  $p \in F(T)$ , using (1.6), we get

$$(3.2) \quad \begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)S^n x_n + \beta_n T^n x_n - p\| \\ &\leq (1 - \beta_n)\|S^n x_n - p\| + \beta_n \|T^n x_n - p\| \\ &\leq (1 - \beta_n)k_n \|x_n - p\| + \beta_n k_n \|x_n - p\| \\ &\leq k_n \|x_n - p\|. \end{aligned}$$

Now, from (1.6) and (3.2), we get

$$(3.3) \quad \begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)T^n x_n + \alpha_n S^n y_n - p\| \\ &\leq (1 - \alpha_n)\|T^n x_n - p\| + \alpha_n \|S^n y_n - p\| \\ &\leq (1 - \alpha_n)k_n \|x_n - p\| + \alpha_n k_n \|y_n - p\| \\ &\leq (1 - \alpha_n)k_n \|x_n - p\| + \alpha_n k_n^2 \|x_n - p\| \\ &\leq (\alpha_n k_n^2 - \alpha_n k_n + k_n) \|x_n - p\| \\ &\leq [1 + (\alpha_n k_n + 1)(k_n - 1)] \|x_n - p\|. \end{aligned}$$

Since  $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ , then by Lemma 2.5,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists. Suppose  $\lim_{n \rightarrow \infty} \|x_n - p\| = c$ , where  $c \geq 0$  is a real number. Suppose  $c > 0$ .

Now

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \|x_{n+1} - p\| \\ &= \lim_{n \rightarrow \infty} \|(1 - \alpha_n)T^n x_n + \alpha_n S^n y_n - p\| \\ &= \lim_{n \rightarrow \infty} (1 - \alpha_n)\|T^n x_n - p\| + \alpha_n \|S^n y_n - p\|, \end{aligned}$$

thus by Lemma 2.6, we get

$$(3.4) \quad \lim_{n \rightarrow \infty} \|T^n x_n - S^n y_n\| = 0.$$

Again, by (3.2), we have

$$(3.5) \quad \limsup_{n \rightarrow \infty} \|S y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\|,$$

also

$$(3.6) \quad \limsup_{n \rightarrow \infty} \|T x_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = c.$$

It follows then that from (3.1) and (3.4), we have

$$\begin{aligned} \|T^n x_n - x_n\| &= \|T^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\ &\leq \|T^n x_n - S^n y_n\| + \|x_n - S^n y_n\| \\ &\leq \|T^n x_n - S^n y_n\| + \lambda \|T^n x_n - S^n y_n\| \\ &\leq (1 + \lambda) \|T^n x_n - S^n y_n\| \\ (3.7) \quad &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Taking limsup on both sides of the above inequality, we get

$$(3.8) \quad \lim_{n \rightarrow \infty} \|T^n x_n - x_n\| = 0,$$

Notice that

$$\|y_n - x_n\| = \beta_n \|T^n x_n - x_n\|.$$

Hence by (3.8)

$$(3.9) \quad \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0.$$

Now

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1 - \alpha_n)T^n x_n + \alpha_n S^n y_n - x_n\| \\ &\leq \|T^n x_n - x_n\| + \alpha_n \|T^n x_n - S^n y_n\|. \end{aligned}$$

This gives

$$(3.10) \quad \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

so that

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \|y_n - x_n\|, \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

and we find that

$$(3.11) \quad \lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0.$$



Moreover, from

$$\begin{aligned} \|x_{n+1} - S^n y_n\| &\leq \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\quad + \|T^n x_n - S^n y_n\| \end{aligned}$$

which gives that

$$(3.12) \quad \lim_{n \rightarrow \infty} \|x_{n+1} - S^n y_n\| = 0.$$

Using (3.4), (3.8) and (3.9) we obtain

$$\begin{aligned} \|x_n - S^n x_n\| &\leq \|x_n - T^n x_n\| + \|T^n x_n - S^n y_n\| \\ &\quad + \|S^n y_n - S^n x_n\| \\ &\leq \|x_n - T^n x_n\| + \|T^n x_n - S^n y_n\| \\ &\quad + k \|y_n - x_n\| \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \|x_n - S^n x_n\| = 0.$$

And

$$\begin{aligned} \|x_{n+1} - Sx_{n+1}\| &\leq \|x_{n+1} - S^{n+1}x_{n+1}\| + \|S^{n+1}x_{n+1} - Sx_{n+1}\| \\ &\leq \|x_{n+1} - S^{n+1}x_{n+1}\| + k \|S^n x_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - S^{n+1}x_{n+1}\| + k (\|S^n x_{n+1} - S^n y_n\| \\ &\quad + \|S^n y_n - x_{n+1}\|) \\ &\leq \|x_{n+1} - S^{n+1}x_{n+1}\| + k^2 \|x_{n+1} - y_n\| \\ &\quad + k \|S^n y_n - x_{n+1}\|. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0.$$

Now

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| \\ &\quad + \|T^{n+1}x_n - Tx_{n+1}\| \\ &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + k \|x_{n+1} - x_n\| \\ &\quad + k \|T^n x_n - x_{n+1}\| \\ &= \|x_{n+1} - T^{n+1}x_{n+1}\| + k \|x_{n+1} - x_n\| \\ &\quad + k \alpha_n \|T^n x_n - S^n y_n\|, \end{aligned}$$

which yields

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

**Example 3.2.** 3.1 : Let  $E$  be the real line with the usual norm  $|\cdot|$  and suppose  $K = [0, 1]$ . Define  $S, T : K \rightarrow K$  by

$$Tx = \frac{2-x}{2}$$

and

$$Sy = \frac{2y+1}{4}$$

for all  $x, y \in K$ . Obviously both  $S$  and  $T$  are asymptotically nonexpansive with the common fixed point  $\frac{2}{3}$  for all  $x, y \in K$ . Now we check that our condition  $\|x - Sy\| \leq \lambda \|Tx - Sx\|$  for all  $x, y \in K$  is true. If  $x, y \in [0, 1]$  and  $\lambda > 1$ , then

$$\begin{aligned} |x - Sy| &= \left| x - \frac{(2y+1)}{4} \right| \\ &= \left| \frac{4x - 2y - 1}{4} \right|, \end{aligned}$$

and

$$\begin{aligned} |Tx - Sy| &= \left| \frac{2-x}{2} - \frac{2y+1}{4} \right| \\ &= \left| \frac{4-2x-2y-1}{4} \right|. \end{aligned}$$

Clearly,  $\left| \frac{4x-2y-1}{4} \right| \leq \lambda \left| \frac{4-2x-2y-1}{4} \right|$ , where  $\lambda > 1$ , so that  $|x - Sy| \leq \lambda |Tx - Sx|$  exists, for all  $x, y \in K$ . Now, we check that  $S$  and  $T$  are quasi-nonexpansive type mappings. In fact, if  $x \in [0, 1]$  and  $p = 0 \in [0, 1]$ , then

$$\begin{aligned} |Tx - p| &= \left| \frac{2-x}{2} - 0 \right| = \left| \frac{2-x}{2} \right| \\ &= \left| \frac{2-x}{2} \right| \leq |x| = |x - 0| = |x - p|, \end{aligned}$$

that is

$$|Tx - p| \leq |x - p|.$$

Similarly, we prove that

$$|Sx - p| \leq |x - p|.$$

Therefore,  $S$  and  $T$  are quasi-nonexpansive type mappings.

**Lemma 3.3.** : *Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ . Suppose  $\{x_n\}$  is the sequence defined in Theorem 3.1 with  $F \neq \phi$ . Then, for any  $p_1, p_2 \in F$ ,  $\lim_{n \rightarrow \infty} \langle x_n, J(p_1 - p_2) \rangle$  exists. In particular,  $\langle p - q, J(p_1 - p_2) \rangle = 0$  for all  $p, q \in \omega_\omega(x_n)$ .*

*Proof.* Take  $x = p_1 - p_2$ , with  $p_1 \neq p_2$  and  $h = t(x_n - p_1)$  in the inequality (2.1) to get:

$$\begin{aligned} \frac{1}{2} \|p_1 - p_2\|^2 + t \langle x_n - p_1, J(p_1 - p_2) \rangle \\ \leq \frac{1}{2} \|tx_n + (1-t)p_1 - p_2\|^2 \\ \leq \frac{1}{2} \|p_1 - p_2\|^2 + t \langle x_n - p_1, J(p_1 - p_2) \rangle + b(t \|x_n - p_1\|). \end{aligned}$$

As  $\sup_{n \geq 1} \|x_n - p_1\| \leq M'$  for some  $M' > 0$ , it follows that

$$\begin{aligned} \frac{1}{2} \|p_1 - p_2\|^2 + t \limsup_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle \\ \leq \frac{1}{2} \lim_{n \rightarrow \infty} \|tx_n + (1-t)p_1 - p_2\|^2 \\ \leq \frac{1}{2} \|p_1 - p_2\|^2 + b(tM') + t \liminf_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle. \end{aligned}$$

That is,

$$\limsup_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle \leq \liminf_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle + \frac{b(tM')}{tM'} M'.$$

If  $t \rightarrow 0$ , then  $\lim_{n \rightarrow \infty} \langle x_n - p_1, J(p_1 - p_2) \rangle$  exists for all  $p_1, p_2 \in F$ . In particular, we get

$$\langle p - q, J(p_1 - p_2) \rangle = 0$$

for all  $p, q \in \omega_\omega(x_n)$ .  $\square$

**Theorem 3.4.** *Let  $E$  be a uniformly convex Banach space satisfying Opial condition and let  $K, T, S$  and  $\{x_n\}$  be taken as Theorem 3.1. If  $F(S) \cap F(T) \neq \phi$ ,  $I - T$  and  $I - S$  are demiclosed at zero, then  $\{x_n\}$  converges weakly to a common fixed point of  $S$  and  $T$ .*

*Proof.* Let  $p \in F(S) \cap F(T)$ . Then as proved in Theorem 3.1,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists. Since  $E$  is uniformly convex. Thus there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\{x_{n_k}\}$  converges weakly to  $z_1 \in K$ . From Theorem 3.1, we have

$$\lim_{n \rightarrow \infty} \|x_{n_k} - Tx_{n_k}\| = 0,$$

$$\lim_{n \rightarrow \infty} \|x_{n_k} - Sx_{n_k}\| = 0.$$

Since  $I - T$  and  $I - S$  are demiclosed at zero, therefore,  $Sz_1 = z_1$ . Similarly  $Tz_1 = z_1$ . Again by the same way, we can prove that  $z_2 \in F(S) \cap F(T)$ . Next, we prove the uniqueness. From Theorem 3.1  $\lim_{n \rightarrow \infty} \|x_n - z_2\|$  exists. For this suppose that  $z_1 \neq z_2$ . Then by the Opial's condition

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - z_1\| &= \lim_{n_i \rightarrow \infty} \|x_{n_i} - z_1\| \\ &< \lim_{n_i \rightarrow \infty} \|x_{n_i} - z_2\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_2\| \\ &= \lim_{n_j \rightarrow \infty} \|x_{n_j} - z_2\| \\ &< \lim_{n_j \rightarrow \infty} \|x_{n_j} - z_1\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_1\|. \end{aligned}$$

This is a contradiction so  $z_1 = z_2$ . Hence  $\{x_n\}$  converges weakly to a common fixed point of  $T$  and  $S$ .  $\square$

**Theorem 3.5.** *Let  $E$  be a real uniformly convex Banach space and  $K, S, T, F, \{x_n\}$  as in Theorem 3.1. Then  $\{x_n\}$  converges strongly to a point of  $F$  if and only if*

$$\liminf_{n \rightarrow \infty} d(x_n, F) = 0,$$

where  $d(x, F) = \inf\{\|x - q\| : q \in F\}$ .

*Proof.* Necessity is evident. Conversely, let  $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ . From Theorem 3.1,  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exists for all  $q \in F$ , so that  $\lim_{n \rightarrow \infty} d(x_n, F)$  exists. Since by hypothesis,  $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ , we get

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0.$$

Next, we show that  $\{x_n\}$  is a Cauchy sequence in  $K$ . Suppose  $\epsilon > 0$  is arbitrarily chosen. Since  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ , there exists a positive integer  $n_0$  such that

$$d(x_n, F) < \frac{\epsilon}{4}, \forall n \geq n_0.$$

In particular,  $\inf\{\|x_{n_0} - q\| : p \in F\} < \epsilon/4$ . Thus there exists  $q \in F$  such that  $\|x_{n_0} - q\| < \epsilon/2$ . Now, for all  $m, n, \geq n_0$ , we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq 2\|x_{n_0} - q\| \\ &< 2 \times \left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned}$$

Hence  $\{x_n\}$  is a Cauchy sequence in closed subset  $K$  of a Banach space  $E$  and therefore it converges to a point  $q$  in  $K$ . But  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ , which implies that  $d(q, F) = 0$ . Therefore we have  $q \in F$ .  $\square$

Using theorem 3.5, we obtain a strong convergence theorem of the iteration scheme (1.6) under the condition  $(A')$  as below:

**Theorem 3.6.** *Let  $E$  be a uniformly convex Banach space and  $K, S, T, F, \{x_n\}$  be as in Theorem 3.1. Let  $S, T$  satisfy the condition  $(A')$  and  $F \neq \phi$ . Then  $\{x_n\}$  converges strongly to a point of  $F$ .*

*Proof.* We proved in Theorem 3.1, that

$$(3.13) \quad \lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0 = \lim_{n \rightarrow \infty} \|x_n - Tx_n\|.$$

Then from the definition of condition  $(A')$  and (3.13), we obtain

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0,$$

or

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0.$$

In the above cases, we get

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0.$$

But  $f : [0, \infty) \rightarrow [0, \infty)$  is a nondecreasing function satisfying  $f(0) = 0$ ,  $f(r) > 0$  for all  $r \in (0, \infty)$ , thus we get

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0.$$

All the condition of Theorem 3.5 are satisfied, therefore by its conclusion,  $\{x_n\}$  converges strongly to a fixed point of  $F$ .  $\square$

**Corollary 3.7.** *Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ . Suppose  $T$  is an asymptotically nonexpansive mapping of  $K$ . Let  $\{x_n\}$  be defined by the iteration (1.3), where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $[0, 1]$  for all  $n \in \mathbb{N}$ . Then  $\{x_n\}$  converges strongly to a fixed point of  $T$ .*

*Proof.* Take  $S = T$  in the above theorem.  $\square$

**Corollary 3.8.** *Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$ . Suppose  $T$  is a asymptotically nonexpansive mapping of  $K$ . Let  $\{x_n\}$  be defined by the iteration (1.2), where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  for all  $n \in \mathbb{N}$ . Then  $\{x_n\}$  converges strongly to a fixed point of  $T$ .*

*Proof.* Suppose  $T = I$  in the above theorem.  $\square$

**Corollary 3.9.** *Suppose  $E$  is a Banach space satisfying Opial condition and let  $K$  and  $T$  be taken as in Theorem 3.1. Let  $F(T) \neq \phi$ . Now if the mapping  $I - T$  is demiclosed at zero, then  $\{x_n\}$  defined by (1.3) converges weakly to a fixed point of  $T$ .*

**Corollary 3.10.** *Let  $E$  be a uniformly convex Banach space which has a Frechet differentiable norm and let  $K$  and  $T$  be taken as theorem 3.1. Let  $F(T) \neq \phi$ . Then  $\{x_n\}$  defined by (1.2) converges weakly to a fixed point of  $T$ .*

**Corollary 3.11.** *Let  $E$  be a uniformly convex Banach space satisfying Opial condition and let  $K$  and  $T$  be taken as in Theorem 3.1. Let  $F(T) \neq \phi$ . If the mapping  $I - T$  is demiclosed at zero, then  $\{x_n\}$  defined by (1.2) converges weakly to a fixed point of  $T$ .*

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