Title:
Higher order close-to-convex functions associated with Attiya-Srivastava operator

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HIGHER ORDER CLOSE-TO-CONVEX FUNCTIONS ASSOCIATED WITH ATTIYA-SRIVASTAVA OPERATOR

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ABSTRACT. In this paper, we introduce a new class $T_{k}^{s,a}[A, B, \alpha, \beta]$ of analytic functions by using a newly defined convolution operator. This class contains many known classes of analytic and univalent functions as special cases. We derived some interesting results including inclusion relationships, a radius problem and sharp coefficient bound for this class.

Keywords: Close-to-convex functions, bounded boundary rotation, Attiya-Srivastava operator.

1. Introduction

Let $\mathcal{A}$ contain the the functions of the form

\begin{equation}
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\end{equation}

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ containing starlike, convex and close-to-convex univalent functions defined in $E$.

A function $f$ analytic in $E$, is subordinate to a function $F$ if there exists a Schwarz function $h(z)$, analytic in $E$ with $h(0) = 0$ and $|h(z)| < |z|$ in $E$, such that $f(z) = F(h(z))$.

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In [12], the class \( P[A, B, \alpha] \) of generalized Janowski functions was introduced. For arbitrary fixed numbers \( A, B, \alpha, -1 \leq B < A \leq 1, \) \( 0 \leq \alpha < 1, \) a function \( p, \) analytic in \( E \) with \( p(0) = 1 \) is in the class \( P[A, B, \alpha], \) if and only if
\[
p(z) < \frac{1 + [(1 - \alpha)A + \alpha B]z}{1 + Bz} \Leftrightarrow p(z) = \frac{1 + [(1 - \alpha)A + \alpha B]h(z)}{1 + Bh(z)}.
\]
If we take \( \alpha = 0, \) then the class \( P[A, B, \alpha] \) reduces to the class \( P[A, B] \) defined by Janowski in [5]. We note that \( p \in P[A, B, \alpha] \) if and only if there exists \( p_1 \in P[A, B] \) such that, for \( z \in E, \)
\[
(1.2) \quad p(z) = (1 - \alpha)p_1(z) + \alpha.
\]
One can easily verify that \( P[A, B] \subset P(\beta), \beta = \frac{1-A}{1-B}. \) Quite recently, in [8], Noor introduced the class \( P_k[A, B, \alpha], \) and studied some new classes of analytic functions connected with the class \( P_k[A, B, \alpha]. \) A function \( q(z) \) analytic in \( E \) with \( q(0) = 1 \) and
\[
q(z) = \left( \frac{k}{4} + \frac{1}{2} \right) q_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) q_2(z),
\]
is in class \( P_k[A, B, \alpha], \) if and only if, \( q_1, q_2 \in P[A, B, \alpha], -1 \leq B < A \leq 1, 0 \leq \alpha < 1, k \geq 2. \) It was given in [8] that
\[
P_k[A, B, \alpha] \subset P_k(\beta), \beta = \frac{1-A_1}{1-B}, \quad A_1 = (1 - \alpha)A + \alpha B.
\]
Also note that \( P_k[1, -1, 0] \equiv P_k, \) as defined by Pinchuk in [11].

The convolution or Hadamard product is defined as
\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,
\]
where
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n.
\]
We consider the function
\[
(1.4) \quad \Psi(s, a; z) = \sum_{n=0}^{\infty} \frac{z^n}{(a + n)^s},
\]
where \( a \in \mathbb{C}\setminus Z_0, \) \( s \in \mathbb{C}. \) The function \( \Psi(s, a; z) \) contain many well-known functions as its special cases, such as the Riemann and Hurwitz Zeta functions; for more details, see [13, 14, 17] and references therein.
Using the technique of convolution and the function $\Psi(s, a; z)$, Srivastava and Attiya [16] consider the convolution operator $J_{s,a} : A \to A$ as

\begin{equation}
J_{s,a}f(z) = \Phi(s, a, z) \ast f(z), \quad z \in E, \quad f \in A,
\end{equation}

where $\ast$ denotes the convolution and

\begin{equation}
\Phi(s, a, z) = (1 + a)^s[\Psi(s, a, z) - a^{-s}] = z + \sum_{n=2}^{\infty} \left( \frac{a + 1}{a + n} \right)^s z^n.
\end{equation}

Therefore, using (1.5) and (1.6), we obtain

\begin{equation}
J_{s,a}f(z) = z + \sum_{n=2}^{\infty} \left( \frac{a + 1}{a + n} \right)^s a_n z^n.
\end{equation}

For special values of $a$, $s$, the operator $J_{s,a}$ contain many known operators, see [1, 2]. From (1.7), it is clear that the operators $J_{s,a}$ satisfies the following recursive relations

\begin{equation}
z(J_{s+1,a}f(z))' = (a + 1)J_{s,a}f(z) - aJ_{s+1,a}f(z).
\end{equation}

For $f \in A$, $a > -1$ and $s$ real, in [8] Noor introduced and studied the following class of analytic functions

$$
R_{k}^{s,a}[A, B, \alpha] = \left\{ f \in A : \frac{z(J_{s,a}f(z))'}{J_{s,a}f(z)} \in P_{k}[A, B, \alpha] \right\},
$$

and a related class $V_{k}^{s,a}[A, B, \alpha]$ by using the Alexander type relation as

$$
f \in V_{k}^{s,a}[A, B, \alpha] \Leftrightarrow zf' \in R_{k}^{s,a}[A, B, \alpha].
$$

In this paper we study the mapping properties of a new class of analytic function using the methods from convolution theory and Attiya-Srivastava operator. Using the operator $J_{s,a}$, we introduce the following new class of analytic functions.

**Definition 1.1.** Let $f \in A$, $a > -1$ and let $s$ be real. Then $f \in T_{k}^{s,a}[A, B, \alpha, \beta]$, if and only if, there exists $g \in R_{2}^{s,a}[1 - 2\beta, -1, 0]$ such that

$$
\frac{z(J_{s,a}f(z))'}{J_{s,a}g(z)} \in P_{k}[A, B, \alpha],
$$

where $-1 \leq B < A \leq 1$, $0 \leq \alpha, \beta < 1$ and $k \geq 2$.

We note that for special values of $s, \alpha$ and $k$ we obtain several known classes of analytic and univalent functions, see [8, 14, 16] and the references therein, for example $T_{k}^{0,0}[1, -1, 0, \beta] = T_{k}(\beta)$ and $T_{2}^{0,0}[1, -1, 0, \beta] = K(\beta)$, the class of close-to-convex functions of order $\beta$. 


2. Preliminary results

**Lemma 2.1** ([6]). Let $h$ be convex in the open unit disk $E$ and let $F : E \to \mathbb{C}$ with $\text{Re}F(z) > 0, z \in E$. If $p$ is analytic in $E$, then

$$p(z) + F(z)zp'(z) \prec h(z)$$

implies that

$$p(z) \prec h(z).$$

**Lemma 2.2** ([10]). Let $p \in P[A, B]$. Then

$$\frac{1 - Ar}{1 - Br} \leq \text{Re}p(z) \leq \frac{1 + Ar}{1 + Br}.$$

**Proof.** (1.2), (1.3) and Lemma 2.2, the following Lemma can easily be proved. □

**Lemma 2.3.** Let $p \in P[A, B, \alpha]$. Then

$$\frac{1 - \{(1 - \alpha)A - \alpha B\}r}{1 - Br} \leq \text{Re}p(z) \leq \frac{1 + \{(1 - \alpha)A + \alpha B\}r}{1 + Br}.$$

**Lemma 2.4** ([15]). Let $\psi$ be convex and let $g$ be starlike in $E$. Then for $F$ analytic in $E$ with $F(0) = 1$, $\frac{\psi F g}{\psi g}$ is contained in the convex hull of $F(E)$.

**Lemma 2.5.** Let $p_1, p_2 \in P_k[A, B, \alpha]$. Then for any positive real numbers, $\delta_1, \delta_2$

$$\frac{1}{\delta_1 + \delta_2} [\delta_1 p_1 + \delta_2 p_2] \in P_k[A, B, \alpha].$$

**Proof.** Let

(2.1) $p_1(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z),$

and

(2.2) $p_2(z) = \left(\frac{k}{4} + \frac{1}{2}\right) q_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) q_2(z),$

where $h_i, q_i \in P[A, B, \alpha]$ for $i = 1, 2$. From (2.1) and (2.2), we have

$$\frac{1}{\delta_1 + \delta_2} [\delta_1 p_1 + \delta_2 p_2] = \left(\frac{k}{4} + \frac{1}{2}\right) \left\{ \frac{\delta_1}{\delta_1 + \delta_2} h_1(z) + \frac{\delta_2}{\delta_1 + \delta_2} q_1(z) \right\}$$

$$- \left(\frac{k}{4} - \frac{1}{2}\right) \left\{ \frac{\delta_1}{\delta_1 + \delta_2} h_2(z) + \frac{\delta_2}{\delta_1 + \delta_2} q_2(z) \right\}.$$
Since $P[A, B, \alpha]$ is a convex set, so \[ \left\{ \frac{\delta_1}{\delta_{1+\delta_2}} h_1(z) + \frac{\delta_2}{\delta_{1+\delta_2}} g_1(z) \right\} \in P[A, B, \alpha], \]
Hence \[ \frac{1}{\delta_{1^2+\delta_2^2}} [\delta_1 p_1 + \delta_2 p_2] \in P_k[A, B, \alpha] \text{ for } i = 1, 2. \]

Note. For $\alpha = 0, k = 2$, this result was proved in [7].

**Lemma 2.6.** Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P_k[A, B, \alpha]$. Then for $n \geq 1$,
\[
|c_n| \leq \frac{k}{2} (1 - \alpha) (A - B).
\]
This result is sharp.

**Proof.** The proof is straightforward by using (1.3), (1.4) and coefficient bound for $P[A, B]$, see [5]. \qed

### 3. Main results

**Theorem 3.1.** Let $f \in A, a > 0$ and let $s$ be real. Then
\[
T^{s,a}_k[A, B, \alpha, \beta] \subset T^{s+1,a}_k[A, B, \alpha, \beta].
\]

**Proof.** Let $f \in T^{s,a}_k[A, B, \alpha, \beta]$. Then there exists $g \in R^{s,a}_2[1 - 2\beta, -1, 0]$ such that
\[
\frac{z(J_{s,a} f)' - J_{s,a} g'}{J_{s,a} g} \in P_k[A, B, \alpha].
\]
We set
\[
\frac{z(J_{s+1,a} f)' - J_{s+1,a} g'}{J_{s+1,a} g} = H(z)
\]
\[
= \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z),
\]
where $p$ is analytic in $E$ and $p(0) = 1$. Observe that
\[
\frac{z(J_{s,a} f)'}{J_{s,a} g} = \frac{J_{s,a} (z f')}{J_{s,a} g}.
\]
By using (1.8), and making some simplification, we obtain
\[
\frac{z(J_{s,a} f)'}{J_{s,a} g} = \frac{z(J_{s+1,a} (z f'))'}{J_{s+1,a} g} + a \frac{z(J_{s+1,a} f)'}{J_{s+1,a} g} + a
\]

By logarithmic differentiation of (3.3), and some simplifications, we have

\[
\frac{z(J_{s+1,a}(zf'))'}{J_{s+1,a}g} = H(z)h(z) + zH'(z),
\]

where

\[
h(z) = \frac{z(J_{s,a}g(z))'}{J_{s,a}g(z)}.
\]

Since \( g \in R_2^{s,a}[1-2\beta,-1,0] \), by using a result given in [8], \( g \in R_2^{s+1,a}[1-2\beta,-1,0] \subset S^*(\beta) \) which implies \( \text{Re}h(z) > \beta \). From (3.6) – (3.10), we obtain

\[
\frac{z(J_{s,a}f')'}{J_{s,a}g} = H(z) + \frac{zH'(z)}{h(z) + a}.
\]

Thus (3.4) and (3.9) together imply

\[
\frac{z(J_{s,a}f')'}{J_{s,a}g} = \left( \frac{k}{4} + \frac{1}{2} \right) \left\{ h_1(z) + \frac{zh_1'(z)}{h(z) + a} \right\}
- \left( \frac{k}{4} - \frac{1}{2} \right) \left\{ h_2(z) + \frac{zh_2'(z)}{h(z) + a} \right\}.
\]

From (3.2) and (3.10), we have

\[
\left( \frac{k}{4} + \frac{1}{2} \right) \left\{ h_1(z) + \frac{zh_1'(z)}{h(z) + a} \right\} - \left( \frac{k}{4} - \frac{1}{2} \right) \left\{ h_2(z) + \frac{zh_2'(z)}{h(z) + a} \right\} \in P_k[A,B,\alpha],
\]

which gives that

\[
h_i(z) + \frac{zh_i'(z)}{h(z) + a} < \frac{1 + A_1z}{1 + Bz}, \quad i = 1, 2,
\]

where \( A_1 = (1-\alpha)A + \alpha B \). Since \( \text{Re} \left\{ \frac{1}{h(z) + a} \right\} > 0 \), \( z \in E \), so using Lemma 2.1, we have

\[
h_i(z) < \frac{1 + A_1z}{1 + Bz}, \quad i = 1, 2,
\]

which implies that the function \( H \) defined by (3.3) belongs to \( P_k[A,B,\alpha] \) and hence \( f \in T_k^{s+1,a}[A,B,\alpha,\beta] \). \( \square \)
Theorem 3.2. Let $f \in V_k^{s,a}[A, B, \alpha]$ and $g \in R_k^{a,s}[A, B, \alpha]$. Let $H(z)$ be defined as

\begin{equation}
J_{s,a}H(z) = \int_0^z \left[ (J_{s,a}f(t))' \right] \delta_1 \left[ \frac{J_{s,a}g(t)}{t} \right] \delta_2 dt,
\end{equation}

where $\delta_1$ and $\delta_2$ are positive reals with $\delta_1 + \delta_2 = 1$. Then $H \in V_k^{s,a}[A, B, \alpha]$.

Proof. From (3.11), we have

\[(J_{s,a}H(z))' = \left[ (J_{s,a}f(z))' \right] \delta_1 \left[ \frac{J_{s,a}g(z)}{z} \right] \delta_2.
\]

Logarithmic differentiation implies

\[
\frac{z (J_{s,a}H(z))'}{J_{s,a}H(z)} = \delta_1 \left( \frac{z (J_{s,a}f(z))'}{J_{s,a}f(z)} \right) + \delta_2 \frac{z (J_{s,a}g(z))'}{J_{s,a}g(z)}.
\]

(3.6)

Then by using Lemma 2.5, \(\{\delta_1 p_1(z) + \delta_2 p_2(z)\} \in P_k[A, B, \alpha]\). Hence from (3.15), \(\frac{z(J_{s,a}H(z))'}{J_{s,a}H(z)} \in P_k[A, B, \alpha]\) and consequently $H \in V_k^{s,a}[A, B, \alpha]$. \(\square\)

Theorem 3.3. Let $\psi \in C$ and $f \in R_2^{a,s}[A, B, \alpha]$. Then $\psi \ast f \in R_2^{a,s}[A, B, \alpha]$.

Proof. Let $F(z) = \psi \ast f$. Then by using some properties of convolution we have

\begin{equation}
\frac{(J_{s,a}F(z))'}{J_{s,a}F(z)} = \frac{\psi \ast z (J_{s,a}f(z))'}{\psi \ast J_{s,a}f(z)}
\end{equation}

\[
= \frac{\psi \ast \frac{z(J_{s,a}f(z))'}{J_{s,a}f(z)} J_{s,a}f(z)}{\psi \ast J_{s,a}f(z)}
\]

\[
= \frac{\psi \ast p(z) J_{s,a}f(z)}{\psi \ast J_{s,a}f(z)},
\]

where $p(z) = \frac{z(J_{s,a}f(z))'}{J_{s,a}f(z)}$. Since $f \in R_2^{a,s}[A, B, \alpha]$, therefore $\frac{z(J_{s,a}f(z))'}{J_{s,a}f(z)} \in P[A, B, \alpha] \subset P[A_1, B, \alpha] \subset P$ and hence $J_{s,a}f(z) \in S^*$. Then by Lemma 2.4, $F(z)$ lies in the convex hull of $p(z)$ and consequently, $\psi \ast f \in R_2^{a,s}[A, B, \alpha]$. By $S^*(\alpha), (0 \leq \alpha < 1)$, we mean the starlike functions of order $\alpha$ and $f \in S_{s,a}^*(\alpha)$; if and only if, $J_{s,a}f \in S^*(\alpha)$. \(\square\)
Theorem 3.4. Let \( f \in R^{s, a}_{k}[A, B, \alpha] \) and \( A_1 = (1 - \alpha)A + \alpha B \) then \( f \in S^{s, a}_{k}(\frac{1-A_1}{1-B}) \) for \( |z| < r \), with \( r \) given by

\[
(3.14) \quad r = \frac{2N}{M + \sqrt{M^2 - 4LN}},
\]

where

\[
(3.8) \quad L = \left\{ \frac{k}{2} \alpha B^2 - (1 - \alpha)AB \right\} (1 - B) + (1 - A_1)B^2, \\
M = (1 - B) \left\{ \frac{k}{2} B - (1 - \alpha)A \right\} + \alpha B, \\
N = A_1 - B = (1 - \alpha)(A - B).
\]

Proof. Let \( f \in R^{s, a}_{k}[A, B, \alpha] \). Then \( \frac{z((J_{s,a}f)' - 1 - A_1}{J_{s,a}f} \in P_k[A, B, \alpha] \). Let

\[
\frac{z((J_{s,a}f)'}{J_{s,a}f} = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z).
\]

Then \( p_i \in P[A, B, \alpha] \) for \( i = 1, 2 \). Now consider

\[
\text{Re} \left\{ \frac{z((J_{s,a}f)'}{J_{s,a}f} - \frac{1 - A_1}{1 - B} \right\} = \left( \frac{k}{4} + \frac{1}{2} \right) \text{Re} p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) \text{Re} p_2(z) - \frac{1 - A_1}{1 - B}.
\]

Using Lemma 2.3, and making some simplifications, we have

\[
(3.16) \quad \text{Re} \left\{ \frac{z((J_{s,a}f)'}{J_{s,a}f} - \frac{1 - A_1}{1 - B} \right\} \geq \frac{Lr^2 + Mr + N}{(1 - B)(1 - B^2 r^2)},
\]

where \( L, M, N \) are given by (3.15). Clearly the right hand side of (3.16) is positive for \( |z| < r \). Hence \( f \in S^{s, a}_{k}(\frac{1-A_1}{1-B}) \) for \( |z| < r \), where \( r \) is given by (3.17).

For \( \alpha = 0, A = 1, B = -1, s = 0, a = s \), we obtain the radius of starlike-ness for the class of bounded radius rotations. \( \square \)

Theorem 3.5. Let \( f \in T^{s, a}_{k}[A, B, \alpha, \beta] \) be of the form (1.1). Then for \( n \geq 2 \),

\[
(3.9) \quad |a_n| \leq \left( \frac{a + n}{a + 1} \right)^s \frac{k(1 - \beta)(A - B)}{2(n - 1)!} \prod_{j=2}^{\infty} (j - 2\beta).
\]

This bound is sharp.

Proof. For \( f \in T^{s, a}_{k}[A, B, \alpha, \beta] \), we have by definition

\[
(3.18) \quad \frac{z((J_{s,a}f(z))'}{J_{s,a}g(z)} = p(z),
\]
where $J_{s,a}g(z) \in R_2[1 - 2\beta, -1, 0] \equiv S^*(\beta)$ and $p(z) \in P_k[A, B, \alpha]$. If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $J_{s,a}g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then from (3.18), we have

$$z \left( z + \sum_{n=2}^{\infty} \left( \frac{a + 1}{a + n} \right)^n a_n z^n \right) = \left( z + \sum_{n=2}^{\infty} b_n z^n \right) \left( 1 + \sum_{n=1}^{\infty} c_n z^n \right).$$

This implies that

$$z + \sum_{n=2}^{\infty} \frac{a + 1}{a + n} z^n n a_n z^n = z + \sum_{n=2}^{\infty} \left( \sum_{k=2}^{n} b_{n-k} c_k \right) z^n, \quad b_0 = 0, \quad b_1 = c_0 = 1.$$

Equating coefficients of $z^n$, we have

$$\left( \frac{a + 1}{a + n} \right)^n a_n = \sum_{k=0}^{n} b_n c_{n-k} ,$$

which implies that

$$\left| \left( \frac{a + 1}{a + n} \right)^n \right| a_n \leq \sum_{k=0}^{n} |b_n| |c_{n-k}| .$$

Now using the coefficient bounds $|b_n| \leq \frac{1}{(n-1)!} \prod_{j=2}^{\infty} (j - 2\beta)$, (see [4]) and $|c_{n-k}| \leq \frac{k}{2} (1 - \beta) (A - B)$ by Lemma 2.6, we can have the required result. Sharpness follows from the function $f_0$ for which

$$\frac{z(J_{s,a}f_0(z))'}{J_{s,a}g_0(z)} = \left( \frac{k}{4} + \frac{1}{2} \right) \frac{1 - A_1 z}{1 + B z} - \left( \frac{k}{4} - \frac{1}{2} \right) \frac{1 - A_1 z}{1 - B z},$$

where $g_0(z) = \frac{1}{1 - \beta} \left\{ \frac{z}{(1 - z)^2} - \beta \right\}$ and $A_1 = (1 - \alpha) A + \alpha B$.

Note that for $s = 0$, $a = s$, $\alpha = 0$, $A = 1$, $B = -1$, we obtain the coefficient estimate for the class $T_k$ studied by Noor in [9].

**References**


Higher order close-to-convex


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