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HIGHER ORDER CLOSE-TO-CONVEX FUNCTIONS ASSOCIATED WITH ATTIYA-SRIVASTAVA OPERATOR

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Abstract. In this paper, we introduce a new class$T_k^{s,a}[A, B, \alpha, \beta]$ of analytic functions by using a newly defined convolution operator. This class contains many known classes of analytic and univalent functions as special cases. We derived some interesting results including inclusion relationships, a radius problem and sharp coefficient bound for this class.
Keywords: Close-to-convex functions, bounded boundary rotation, Attiya-Srivastava operator.
MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.

1. Introduction

Let $\mathcal{A}$ contain the the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Let $\mathcal{S}^*$, C and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ containing starlike, convex and close-to-convex univalent functions defined in E.
A function f analytic in E, is subordinate to a function F if there exists a Schwarz function $h(z)$, analytic in E with $h(0) = 0$ and $|h(z)| < |z|$ in E, such that $f(z) = F(h(z))$.

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In [12], the class $P[A, B, \alpha]$ of generalized Janowski functions was introduced. For arbitrary fixed numbers A, B, α, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, a function p, analytic in E with $p(0) = 1$ is in the class $P[A, B, \alpha]$, if and only if

$$p(z) \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz} \Leftrightarrow p(z) = \frac{1 + [(1-\alpha)A + \alpha B]h(z)}{1 + Bh(z)}.$$

If we take $\alpha = 0$, then the class $P[A, B, \alpha]$ reduces to the class $P[A, B]$ defined by Janwoski in [5]. We note that $p \in P[A, B, \alpha]$ if and only if there exists $p_1 \in P[A, B]$ such that, for $z \in E$.

$$p(z) = (1-\alpha)p_1(z) + \alpha. \tag{1.2}$$

One can easily verify that $P[A, B] \subset P(\beta)$, $\beta = \frac{1-A}{1-B}$. Quite recently, in [8], Noor introduced the class $P_k[A, B, \alpha]$, and studied some new classes of analytic functions connected with the class $P_k[A, B, \alpha]$. A function $q(z)$ analytic in E with $q(0) = 1$ and

$$q(z) = \left(\frac{k}{4} + \frac{1}{2}\right) q_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) q_2(z), \tag{1.3}$$

is in class $P_k[A, B, \alpha]$, if and only if, $q_1, q_2 \in P[A, B, \alpha]$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, $k \geq 2$. It was given in [8] that

$$P_k[A, B, \alpha] \subset P_k(\beta),\ \beta = \frac{1 - A_1}{1 - B},\ A_1 = (1-\alpha)A + \alpha B.$$

Also note that $P_k[1, -1, 0] \equiv P_k$, as defined by Pinchuk in [11].

The convolution or Hadamard product is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

We consider the function

$$\Psi(s, a; z) = \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s}, \tag{1.4}$$

where $a \in \mathbb{C} \backslash Z_0^-$, $s \in \mathbb{C}$. The function $\Psi(s, a; z)$ contain many well-known functions as its special cases, such as the Riemann and Hurwitz Zeta functions; for more details, see [13, 14, 17] and references therein.

Using the technique of convolution and the function $\Psi(s,a;z)$, Srivastava and Attiya [16] consider the convolution operator $J_{s,a}:\mathcal{A}\to\mathcal{A}$ as

$$J_{s,a}f(z)=\Phi(s,a,z)*f(z),\ z\in E,\quad f\in\mathcal{A}, \tag{1.5}$$

where $*$ denotes the convolution and

$$\Phi(s,a,z)=(1+a)^s[\Psi(s,a,z)-a^{-s}]=z+\sum_{n=2}^{\infty}\left(\frac{a+1}{a+n}\right)^s z^n. \tag{1.6}$$

Therefore, using (1.5) and (1.6), we obtain

$$J_{s,a}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{a+1}{a+n}\right)^s a_n z^n. \tag{1.7}$$

For special values of a, s, the operator $J_{s,a}$ contain many known operators, see [1, 2]. From (1.7), it is clear that the operators $J_{s,a}$ satisfies the following recursive relations

$$z(J_{s+1,a}f(z))'=(a+1)J_{s,a}f(z)-aJ_{s+1,a}f(z). \tag{1.8}$$

For $f\in\mathcal{A}$, $a>-1$ and s real, in [8] Noor introduced and studied the following class of analytic functions

$$R_k^{s,a}[A,B,\alpha]=\left\{f\in\mathcal{A}:\frac{z(J_{s,a}f(z))'}{J_{s,a}f(z)}\in P_k[A,B,\alpha]\right\},$$

and a related class $V_k^{s,a}[A,B,\alpha]$ by using the Alexander type relation as

$$f\in V_k^{s,a}[A,B,\alpha]\Leftrightarrow zf'\in R_k^{s,a}[A,B,\alpha].$$

In this paper we study the mapping properties of a new class of analytic function using the methods from convolution theory and Attiya-Srivastava operator. Using the operator $J_{s,a}$, we introduce the following new class of analytic functions.

Definition 1.1. *Let $f\in\mathcal{A}$, $a>-1$ and let s be real. Then $f\in T_k^{s,a}[A,B,\alpha,\beta]$, if and only if, there exists $g\in R_2^{s,a}[1-2\beta,-1,0]$ such that*

$$\frac{z(J_{s,a}f(z))'}{J_{s,a}g(z)}\in P_k[A,B,\alpha],$$

where $-1\le B<A\le 1$, $0\le\alpha,\beta<1$ and $k\ge 2$.

We note that for special values of s,α and k we obtain several known classes of analytic and univalent functions, see [8, 14, 16] and the references therein, for example $T_k^{0,0}[1,-1,0,\beta]=T_k(\beta)$ and $T_2^{0,0}[1,-1,0,\beta]=K(\beta)$, the class of close-to-convex functions of order β.

2. **Preliminary results**

Lemma 2.1 ([6])**.** *Let h be convex in the open unit disk E and let $F : E \to \mathbb{C}$ with* $\mathrm{Re}F(z) > 0, z \in E$. *If p is analytic in E, then*

$$p(z) + F(z)\, zp'(z) \prec h(z)$$

implies that

$$p(z) \prec h(z).$$

Lemma 2.2 ([10])**.** *Let $p \in P[A, B]$. Then*

$$\frac{1 - Ar}{1 - Br} \leq \mathrm{Re}p(z) \leq |p(z)| \leq \frac{1 + Ar}{1 + Br}.$$

Proof. (1.2), (1.3) and Lemma 2.2, the following Lemma can easily be proved. □

Lemma 2.3. *Let $p \in P[A, B, \alpha]$. Then*

$$\frac{1 - \{(1-\alpha)A - \alpha B\}\, r}{1 - Br} \leq \mathrm{Re}p(z) \leq |p(z)| \leq \frac{1 + \{(1-\alpha)A + \alpha B\}\, r}{1 + Br}.$$

Lemma 2.4 ([15])**.** *Let ψ be convex and let g be starlike in E. Then for F analytic in E with $F(0) = 1$, $\frac{\psi * Fg}{\psi * g}$ is contained in the convex hull of $F(E)$.*

Lemma 2.5. *Let p_1, $p_2 \in P_k[A, B, \alpha]$. Then for any positive real numbers, δ_1, δ_2*

$$\frac{1}{\delta_1 + \delta_2}\left[\delta_1 p_1 + \delta_2 p_2\right] \in P_k[A, B, \alpha].$$

Proof. Let

$$p_1(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \tag{2.1}$$

and

$$p_2(z) = \left(\frac{k}{4} + \frac{1}{2}\right) q_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) q_2(z), \tag{2.2}$$

where $h_i, q_i \in P[A, B, \alpha]$ for $i = 1, 2$. From (2.1) and (2.2), we have

$$\begin{aligned}\frac{1}{\delta_1 + \delta_2}\left[\delta_1 p_1 + \delta_2 p_2\right] &= \left(\frac{k}{4} + \frac{1}{2}\right)\left\{\frac{\delta_1}{\delta_1 + \delta_2} h_1(z) + \frac{\delta_2}{\delta_1 + \delta_2} q_1(z)\right\} \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right)\left\{\frac{\delta_1}{\delta_1 + \delta_2} h_2(z) + \frac{\delta_2}{\delta_1 + \delta_2} q_2(z)\right\}.\end{aligned}$$

Since $P[A,B,\alpha]$ is a convex set, so $\left\{\frac{\delta_1}{\delta_1+\delta_2}h_i(z)+\frac{\delta_2}{\delta_1+\delta_2}q_i(z)\right\}\in P[A,B,\alpha]$. Hence $\frac{1}{\delta_1+\delta_2}\left[\delta_1 p_1+\delta_2 p_2\right]\in P_k[A,B,\alpha]$ for $i=1,2$. □

Note. For $\alpha=0,\ k=2$, this result was proved in [7].

Lemma 2.6. *Let* $p(z)=1+\sum_{n=1}^{\infty}c_n z^n\in P_k[A,B,\alpha]$. *Then for* $n\geq 1$,

$$|c_n|\leq \frac{k}{2}(1-\alpha)(A-B). \tag{2.3}$$

This result is sharp.

Proof. The proof is straightforward by using (1.3), (1.4) and coefficient bound for $P[A,B]$, see [5]. □

3. Main results

Theorem 3.1. *Let* $f\in\mathcal{A}$, $a>0$ *and let* s *be real. Then*

$$T_k^{s,a}[A,B,\alpha,\beta]\subset T_k^{s+1,a}[A,B,\alpha,\beta]. \tag{3.1}$$

Proof. Let $f\in T_k^{s,a}[A,B,\alpha,\beta]$. Then there exists $g\in R_2^{s,a}[1-2\beta,-1,0]$ such that

$$\frac{z(J_{s,a}f)'}{J_{s,a}g}\in P_k[A,B,\alpha]. \tag{3.2}$$

We set

$$\frac{z(J_{s+1,a}f)'}{J_{s+1,a}g} = H(z) \tag{3.3}$$

$$= \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z), \tag{3.4}$$

where p is analytic in E and $p(0)=1$. Observe that

$$\frac{z(J_{s,a}f)'}{J_{s,a}g}=\frac{J_{s,a}(zf')}{J_{s,a}g}. \tag{3.5}$$

By using (1.8), and making some simplification, we obtain

$$\frac{z(J_{s,a}f)'}{J_{s,a}g}=\frac{\frac{z(J_{s+1,a}(zf'))'}{J_{s+1,a}g}+a\frac{z(J_{a,s+1}f)'}{J_{a,s+1}g}}{\frac{z(J_{s+1,a}g)'}{J_{s+1,a}g}+a}. \tag{3.6}$$

By logarithmic differentiation of (3.3), and some simplifications, we have

$$\frac{z(J_{s+1,a}(zf^{'}))^{'}}{J_{s+1,a}g} = H(z)h(z) + zH^{'}(z), \tag{3.7}$$

where

$$h(z) = \frac{z(J_{s,a}g(z))^{'}}{J_{s,a}g(z)}. \tag{3.8}$$

Since $g \in R_2^{s,a}[1-2\beta, -1, 0]$, by using a result given in [8], $g \in R_2^{s+1,a}[1-2\beta, -1, 0] \subset S^*(\beta)$ which implies $\mathrm{Re}h(z) > \beta$. From (3.6) – (3.10), we obtain

$$\frac{z(J_{s,a}f)^{'}}{J_{s,a}g} = H(z) + \frac{zH^{'}(z)}{h(z)+a}. \tag{3.9}$$

Thus (3.4) and (3.9) together imply

$$\begin{aligned} \frac{z(J_{s,a}f)^{'}}{J_{s,a}g} &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{zh_1^{'}(z)}{h(z)+a}\right\} \\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z) + \frac{zh_2^{'}(z)}{h(z)+a}\right\}. \end{aligned} \tag{3.5}$$

From (3.2) and (3.10), we have

$$\left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{zh_1^{'}(z)}{h(z)+a}\right\} - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z) + \frac{zh_2^{'}(z)}{h(z)+a}\right\} \in P_k[A, B, \alpha],$$

which gives that

$$h_i(z) + \frac{zh_i^{'}(z)}{h(z)+a} \prec \frac{1+A_1z}{1+Bz}, \quad i = 1, 2,$$

where $A_1 = (1-\alpha)A + \alpha B$. Since $\mathrm{Re}\left\{\frac{1}{h(z)+a}\right\} > 0,\ z \in E$, so using Lemma 2.1, we have

$$h_i(z) \prec \frac{1+A_1z}{1+Bz}, \quad i = 1, 2,$$

which implies that the function H defined by (3.3) belongs to $P_k[A, B, \alpha]$ and hence $f \in T_k^{s+1,a}[A, B, \alpha, \beta]$. □

Theorem 3.2. *Let $f \in V_k^{s,a}[A,B,\alpha]$ and $g \in R_k^{a,s}[A,B,\alpha]$. Let $H(z)$ be defined as*

$$J_{s,a}H(z) = \int_0^z \left[\left(J_{s,a}f(t)\right)'\right]^{\delta_1} \left[\frac{J_{s,a}g(t)}{t}\right]^{\delta_2} dt, \tag{3.11}$$

where δ_1 and δ_2 are positive reals with $\delta_1+\delta_2 = 1$. Then $H \in V_k^{s,a}[A,B,\alpha]$.

Proof. From (3.11), we have

$$\left(J_{s,a}H(z)\right)' = \left[\left(J_{s,a}f(z)\right)'\right]^{\delta_1} \left[\frac{J_{s,a}g(z)}{z}\right]^{\delta_2}.$$

Logarithmic differentiation implies

$$\begin{aligned} \frac{z\left(J_{s,a}H(z)\right)'}{J_{s,a}H(z)} &= \delta_1 \frac{\left(z\left(J_{s,a}f(z)\right)'\right)'}{\left(J_{s,a}f(z)\right)'} + \delta_2 \frac{z\left(J_{s,a}g(z)\right)'}{J_{s,a}g(z)}. \\ &= \delta_1 p_1(z) + \delta_2 p_2(z), \text{ for all } \ p_1, p_2 \in P_k[A,B,\alpha]. \end{aligned} \tag{3.6}$$

Then by using Lemma 2.5, $\{\delta_1 p_1(z) + \delta_2 p_2(z)\} \in P_k[A,B,\alpha]$. Hence from (3.15), $\frac{z(J_{s,a}H(z))'}{J_{s,a}H(z)} \in P_k[A,B,\alpha]$ and consequently $H \in V_k^{s,a}[A,B,\alpha]$. □

Theorem 3.3. *Let $\psi \in \mathcal{C}$ and $f \in R_2^{a,s}[A,B,\alpha]$. Then $\psi * f \in R_2^{a,s}[A,B,\alpha]$.*

Proof. Let $F(z) = \psi * f$. Then by using some properties of convolution we have

$$\begin{aligned} \frac{\left(J_{s,a}F(z)\right)'}{J_{s,a}F(z)} &= \frac{\psi * z\left(J_{s,a}f(z)\right)'}{\psi * J_{s,a}f(z)} \\ &= \frac{\psi * \frac{z(J_{s,a}f(z))'}{J_{s,a}f(z)} J_{s,a}f(z)}{\psi * J_{s,a}f(z)} \\ &= \frac{\psi * p(z) J_{s,a}f(z)}{\psi * J_{s,a}f(z)}, \end{aligned} \tag{3.7}$$

where $p(z) = \frac{z(J_{s,a}f(z))'}{J_{s,a}f(z)}$. Since $f \in R_2^{a,s}[A,B,\alpha]$, therefore $\frac{z(J_{s,a}f(z))'}{J_{s,a}f(z)} \in P[A,B,\alpha] \subset P[A_1,B,\alpha] \subset P$ and hence $J_{s,a}f(z) \in \mathcal{S}^*$. Then by Lemma 2.4, $F(z)$ lies in the convex hull of $p(z)$ and consequently, $F \in R_2^{a,s}[A,B,\alpha]$.

By $\mathcal{S}^*(\alpha), (0 \le \alpha < 1)$, we mean the starlike functions of order α and $f \in \mathcal{S}^*_{s,a}(\alpha)$; if and only if, $J_{s,a}f \in \mathcal{S}^*(\alpha)$. □

Theorem 3.4. *Let* $f \in R_k^{s,a}[A,B,\alpha]$ *and* $A_1 = (1-\alpha)A + \alpha B$ *then* $f \in \mathcal{S}^*_{s,a}(\frac{1-A_1}{1-B})$ *for* $|z| < r$, *with* r *given by*

$$r = \frac{2N}{M + \sqrt{M^2 - 4LN}}, \tag{3.14}$$

where

$$\begin{aligned} L &= \left[\left\{\frac{k}{2}\alpha B^2 - (1-\alpha)AB\right\}(1-B) + (1-A_1)B^2\right], \\ M &= (1-B)\left[\frac{k}{2}\{B-(1-\alpha)A\} + \alpha B\right], \\ N &= A_1 - B = (1-\alpha)(A-B). \end{aligned} \tag{3.8}$$

Proof. Let $f \in R_k^{s,a}[A,B,\alpha]$. Then $\frac{z((J_{s,a}f)'}{J_{s,a}f} \in P_k[A,B,\alpha]$. Let

$$\frac{z((J_{s,a}f)'}{J_{s,a}f} = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z).$$

Then $p_i \in P[A,B,\alpha]$ for $i = 1, 2$. Now consider

$$\mathrm{Re}\left\{\frac{z((J_{s,a}f)'}{J_{s,a}f} - \frac{1-A_1}{1-B}\right\} = \left(\frac{k}{4}+\frac{1}{2}\right)\mathrm{Re}p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)\mathrm{Re}p_2(z) - \frac{1-A_1}{1-B}.$$

Using Lemma 2.3, and making some simplifications, we have

$$\mathrm{Re}\left\{\frac{z((J_{s,a}f)'}{J_{s,a}f} - \frac{1-A_1}{1-B}\right\} \geq \frac{Lr^2 + Mr + N}{(1-B)(1-B^2r^2)}, \tag{3.16}$$

where $L,\ M,\ N$ are given by (3.15). Clearly the right hand side of (3.16) is positive for $|z| < r$. Hence $f \in \mathcal{S}^*_{a,s}(\frac{1-A_1}{1-B})$ for $|z| < r$, where r is given by (3.17).

For $\alpha = 0$, $A = 1$, $B = -1$, $s = 0$, $a = s$, we obtain the radius of starlike-ness for the class of bounded radius rotations. □

Theorem 3.5. *Let* $f \in T_k^{s,a}[A,B,\alpha,\beta]$ *be of the form* (1.1). *Then for* $n \geq 2$,

$$|a_n| \leq \left|\left(\frac{a+n}{a+1}\right)^s\right| \frac{k\,(1-\beta)\,(A-B)}{2\,(n-1)!} \prod_{j=2}^{\infty}(j-2\beta)\,. \tag{3.9}$$

This bound is sharp.

Proof. For $f \in T_k^{s,a}[A,B,\alpha,\beta]$, we have by definition

$$\frac{z(J_{s,a}f(z))'}{J_{s,a}g(z)} = p\,(z)\,, \tag{3.18}$$

where $J_{s,a}g(z) \in R_2[1-2\beta,-1,0] \equiv S^*(\beta)$ and $p(z) \in P_k[A,B,\alpha]$. If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $J_{s,a}g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then from (3.18), we have

$$z\left(z + \sum_{n=2}^{\infty}\left(\frac{a+1}{a+n}\right)^s a_n z^n\right)' = \left(z + \sum_{n=2}^{\infty} b_n z^n\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right).$$

This implies that

$$z + \sum_{n=2}^{\infty}\left(\frac{a+1}{a+n}\right)^s n a_n z^n = z + \sum_{n=2}^{\infty}\left(\sum_{k=0}^{n} b_n c_{n-k}\right) z^n, \quad b_0 = 0,\ b_1 = c_0 = 1.$$

Equating coefficients of z^n, we have

$$\left(\frac{a+1}{a+n}\right)^s n a_n = \sum_{k=0}^{n} b_n c_{n-k},$$

which implies that

$$\left|\left(\frac{a+1}{a+n}\right)^s\right| n\left|a_n\right| \leq \sum_{k=0}^{n} \left|b_n\right|\left|c_{n-k}\right|.$$

Now using the coefficient bounds $|b_n| \leq \frac{1}{(n-1)!}\prod_{j=2}^{\infty}(j-2\beta)$, (see [4]) and $|c_{n-k}| \leq \frac{k}{2}(1-\beta)(A-B)$ by Lemma 2.6, we can have the required result. Sharpness follows from the function f_0 for which

$$\frac{z(J_{s,a}f_0(z))'}{J_{s,a}g_0(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+A_1 z}{1+Bz} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-A_1 z}{1-Bz},$$

where $g_0(z) = \frac{1}{1-\beta}\left\{\frac{z}{(1-z)^2} - \beta\right\}$ and $A_1 = (1-\alpha)A + \alpha B$.

Note that for $s = 0$, $a = s$, $\alpha = 0$, $A = 1$, $B = -1$, we obtain the coefficient estimate for the class T_k studied by Noor in [9]. □

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