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**On quasi-Einstein Finsler spaces**

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## ON QUASI-EINSTEIN FINSLER SPACES

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**ABSTRACT.** The notion of quasi-Einstein metric in physics is equivalent to the notion of Ricci soliton in Riemannian spaces. Quasi-Einstein metrics serve also as solution to the Ricci flow equation. Here, the Riemannian metric is replaced by a Hessian matrix derived from a Finsler structure and a quasi-Einstein Finsler metric is defined. In compact case, it is proved that the quasi-Einstein metrics are solutions to the Finslerian Ricci flow and conversely, certain form of solutions to the Finslerian Ricci flow are quasi-Einstein Finsler metrics.

**Keywords:** Finsler space; quasi-Einstein; Ricci flow; Ricci soliton.  
**MSC(2010):** Primary: 53C60; Secondary: 53C44, 35C08.

### 1. Introduction

The Ricci flow in Riemannian geometry was introduced by R. S. Hamilton in 1982, cf. [5], and since then has been extensively studied thanks to its applications in geometry, physics and different branches of real world problems. Theoretical physicists have also been looking into the equation of quasi-Einstein metrics in relation with string theory. Ricci flow is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat. G. Perelman used Ricci flow to prove the Poincaré conjecture. Quasi-Einstein metrics or Ricci solitons are considered as a solution to the Ricci flow equation and are subject of great interest in geometry and physics. Let  $(M, g)$  be

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a Riemannian manifold, a triple  $(M, g, X)$  is said to be a *quasi-Einstein metric* or *Ricci soliton* if  $g$  satisfies the equation

$$(1.1) \quad 2Rc + \mathcal{L}_X g = 2\lambda g,$$

where  $Rc$  is the Ricci tensor,  $X$  a smooth vector field on  $M$ ,  $\mathcal{L}_X$  the Lie derivative along  $X$  and  $\lambda$  a real constant. If the vector field  $X$  is gradient of a potential function  $f$ , then  $(M, g, X)$  is said to be *gradient* and (1.1) takes the familiar form

$$(1.2) \quad Rc + \nabla \nabla f = \lambda g.$$

Perelman has proved that on a compact Riemannian manifold every Ricci soliton is gradient, cf. [7]. Moreover, on a compact Riemannian manifold a quasi-Einstein metric is a special solution to the Ricci flow equation defined by

$$(1.3) \quad \frac{\partial}{\partial t} g(t) = -2Rc, \quad g(t=0) := g_0.$$

A quasi-Einstein metric is considered as special solution to the Ricci flow in Riemannian geometry.

Some recent work has focused on the natural question of extending this notion to the Finsler geometry as a natural generalization of Riemannian geometry.

In this work, the Akbar-Zadeh's Ricci tensor is used to determine the notion of quasi-Einstein Finsler metric and it is shown that in the case of compact manifolds, it is a solution to the Ricci flow equation used by D. Bao cf. [3] in Finsler geometry and vice versa. More precisely, it is proved that if there is a quasi-Einstein Finsler metric on a compact manifold then there exists a solution to the Ricci flow equation. Conversely, certain form of solutions to the Ricci flow are quasi-Einstein Finsler metrics.

## 2. Preliminaries and notations

Let  $M$  be a real  $n$ -dimensional differentiable manifold. We denote by  $TM$  its tangent bundle and by  $\pi : TM_0 \rightarrow M$ , the fiber bundle of non zero tangent vectors. A *Finsler structure* on  $M$  is a function  $F : TM \rightarrow [0, \infty)$ , with the following properties:

- I. Regularity:  $F$  is  $C^\infty$  on the entire slit tangent bundle  $TM_0 = TM \setminus 0$ .
- II. Positive homogeneity:  $F(x, \lambda y) = \lambda F(x, y)$  for all  $\lambda > 0$ .
- III. Strong convexity: The  $n \times n$  Hessian matrix  $(g_{ij})_F = ([\frac{1}{2} F^2]_{y^i y^j})$  is positive definite at every point of  $TM_0$ .

A *Finsler manifold*  $(M, F)$  is a pair consisting of a differentiable manifold  $M$  and a Finsler structure  $F$ . The formal Christoffel symbols of second kind and spray coefficients are respectively denoted here by

$$(2.1) \quad (\gamma_{jk}^i)_F := g^{is} \frac{1}{2} \left( \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right),$$

where  $g_{ij}(x, y) = [\frac{1}{2}F^2]_{y^i y^j}$ , and

$$(2.2) \quad G_F^i := \frac{1}{2} (\gamma_{jk}^i)_F y^j y^k.$$

We consider also the *reduced curvature tensor*  $R_k^i$  which is expressed entirely in terms of the  $x$  and  $y$  derivatives of spray coefficients  $G_F^i$

$$(2.3) \quad (R_k^i)_F := \frac{1}{F^2} \left( 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k} \right).$$

In the general Finslerian setting, one of the remarkable definitions of Ricci tensors is introduced by H. Akbar-Zadeh [1] as follows.

$$Ric_{jk} := [\frac{1}{2}F^2 Ric]_{y^j y^k},$$

where  $Ric = R_i^i$  and  $R_k^i$  is defined by (2.3). One of the advantages of the Ricci quantity defined here is its independence to the choice of the Cartan, Berwald or Chern(Rund) connections. Based on the Akbar-Zadeh's Ricci tensor, in analogy with the equation (1.3), D. Bao has considered, the following natural extension of *Ricci flow* in Finsler geometry, cf. [3],

$$\frac{\partial}{\partial t} g_{jk} = -2Ric_{jk}, \quad g(t=0) := g_0.$$

This equation is equivalent to the following differential equation

$$(2.4) \quad \frac{\partial}{\partial t} (\log F(t)) = -Ric, \quad F(t=0) := F_0,$$

where  $F_0$  is the initial Finsler structure. Let  $V = v^i(x) \frac{\partial}{\partial x^i}$  be a vector field on  $M$ . If  $\{\varphi_t\}$  is the local one-parameter group of  $M$  generated by  $V$ , then it induces an infinitesimal point transformation on  $M$  defined by  $\varphi_t^*(x^i) := \bar{x}^i$ , where  $\bar{x}^i = x^i + v^i(x)t$ . This is naturally extended to the point transformation  $\tilde{\varphi}_t$  on the tangent bundle  $TM$  defined by  $\tilde{\varphi}_t^* := (\bar{x}^i, \bar{y}^i)$ , where

$$(2.5) \quad \bar{x}^i = x^i + v^i(x)t, \quad \bar{y}^i = y^i + \frac{\partial v^i}{\partial x^m} y^m t.$$

It can be shown that,  $\{\tilde{\varphi}_t\}$  induces a vector field  $\hat{V} = v^i(x) \frac{\partial}{\partial x^i} + y^j \frac{\partial v^i}{\partial x^j} \frac{\partial}{\partial y^i}$  on  $TM$  called the complete lift of  $V$ . The one-parameter group associated to the complete lift  $\hat{V}$  is given by  $\tilde{\varphi}_t(x, y) = (\varphi_t(x), y^i \frac{\partial \varphi_t}{\partial x^i})$ .

The *Lie derivative* of an arbitrary geometric object  $\Upsilon^I(x, y)$  on  $TM$ , where  $I$  is a mixed multi index, with respect to the complete lift  $\hat{V}$  of a vector field  $V$  on  $M$ , is defined by

$$(2.6) \quad \mathcal{L}_{\hat{V}} \Upsilon^I = \lim_{t \rightarrow 0} \frac{\tilde{\varphi}_t^*(\Upsilon^I) - \Upsilon^I}{t} = \frac{d}{dt} \tilde{\varphi}_t^*(\Upsilon^I),$$

where  $\tilde{\varphi}_t^*(\Upsilon^I)$  is the deformation of  $\Upsilon^I(x, y)$  under the extended point transformation (2.5). Whenever the geometric object  $\Upsilon^I$  is a tensor field,  $\tilde{\varphi}_t^*(\Upsilon^I)$  coincides with the classical notation of pullback of  $\Upsilon^I(x, y)$ , cf. [6].

### 3. Quasi-Einstein Finsler metrics

Let  $(M, F_0)$  be a Finsler manifold and  $V = v^i(x) \frac{\partial}{\partial x^i}$  a vector field on  $M$ . We call the triple  $(M, F_0, V)$  a Finslerian *quasi-Einstein* or a *Ricci soliton* if  $g_{jk}$ , the Hessian related to the Finsler structure  $F_0$ , satisfies

$$(3.1) \quad 2Ric_{jk} + \mathcal{L}_{\hat{V}} g_{jk} = 2\lambda g_{jk},$$

where  $\hat{V}$  is complete lift of  $V$  and  $\lambda \in \mathbb{R}$ . A moment's thought shows that this equation leads

$$(3.2) \quad 2F_0^2 Ric_{F_0} + \mathcal{L}_{\hat{V}} F_0^2 = 2\lambda F_0^2.$$

**Lemma 3.1.** *Let  $M$  be a differentiable manifold,  $F_0$  a Finsler structure and  $\varphi_t$  a family of diffeomorphisms on  $M$ . Then the pull back of extended point transformation  $\tilde{\varphi}_t^*(F_0) : TM \rightarrow [0, \infty)$  is also a Finsler structure on  $M$ .*

*Proof.* We put  $\tilde{\varphi}_t^* x^i = \tilde{x}^i$  and  $\tilde{\varphi}_t^* y^i = \tilde{y}^i$ . We should show that  $\tilde{\varphi}_t^*(F_0) := F_0 \circ \tilde{\varphi}_t : TM_0 \rightarrow [0, \infty)$  satisfies the three conditions I, II and III in the definition of Finsler structure. Clearly we have the regularity condition since  $F_0$  and  $\tilde{\varphi}_t$  are  $C^\infty$  on  $TM_0$ , and so is  $F_0 \circ \tilde{\varphi}_t$ . We have

$$\begin{aligned} \tilde{\varphi}_t^* F_0(x, \lambda y) &= F_0(\tilde{\varphi}_t(x, \lambda y)) = F_0(\varphi_t(x), \lambda y^j \frac{\partial \varphi_t}{\partial x^j}) \\ &= \lambda F_0(\varphi_t(x), y^j \frac{\partial \varphi_t}{\partial x^j}) = \lambda F_0(\tilde{\varphi}_t(x, y)) = \lambda \tilde{\varphi}_t^*(F_0)(x, y), \end{aligned}$$

thus the positive homogeneity is satisfied. For strong convexity we have

$$\begin{aligned} \left[\frac{1}{2}(\tilde{\varphi}_t^* F_0)^2\right]_{\tilde{y}^i \tilde{y}^j} &= \frac{1}{2} \frac{\partial^2((\tilde{\varphi}_t^* F_0)^2)}{\partial \tilde{y}^i \partial \tilde{y}^j} = \frac{1}{2} \frac{\partial^2((F_0 \circ \tilde{\varphi}_t)(F_0 \circ \tilde{\varphi}_t))}{\partial \tilde{y}^i \partial \tilde{y}^j} \\ &= \frac{1}{2} \frac{\partial^2(F_0^2 \circ \tilde{\varphi}_t)}{\partial \tilde{y}^i \partial \tilde{y}^j} = \frac{1}{2} \frac{\partial^2(\tilde{\varphi}_t^* F_0^2)}{\partial \tilde{y}^i \partial \tilde{y}^j}. \end{aligned}$$

One can easily check that

$$\frac{\partial(\tilde{\varphi}_t^* F_0^2)}{\partial \tilde{y}^i} = \tilde{\varphi}_t^* \left( \frac{\partial F_0^2}{\partial y^i} \right),$$

from which we get

$$\left[\frac{1}{2}(\tilde{\varphi}_t^* F_0)^2\right]_{\tilde{y}^i \tilde{y}^j} = \frac{1}{2} \frac{\partial^2(\tilde{\varphi}_t^* F_0^2)}{\partial \tilde{y}^i \partial \tilde{y}^j} = \tilde{\varphi}_t^* \left( \frac{1}{2} \frac{\partial^2 F_0^2}{\partial y^i \partial y^j} \right) = \tilde{\varphi}_t^* \left( \left[\frac{1}{2} F_0^2\right]_{y^i y^j} \right).$$

Using the facts that  $[\frac{1}{2} F_0^2]_{y^i y^j}$  is positive definite and  $\tilde{\varphi}_t^*$  is a diffeomorphism,  $\tilde{\varphi}_t^*([\frac{1}{2} F_0^2]_{y^i y^j})$  is also positive definite and hence  $\tilde{\varphi}_t^*(F_0)$  satisfies III. This completes the proof of Lemma 1.  $\square$

**Lemma 3.2.** *Let  $\varphi_t$  be a family of diffeomorphisms on  $M$  and  $(\gamma_{jk}^i)_{F_0}$  and  $G_{F_0}^i$  the Christoffel and spray coefficients related to the Finsler structure  $F_0$ , respectively. Then we have*

$$(3.3) \quad \tilde{\varphi}_t^* ((\gamma_{jk}^i)_{F_0}) = (\gamma_{jk}^i)_{\tilde{\varphi}_t^*(F_0)},$$

$$(3.4) \quad \tilde{\varphi}_t^* (G_{F_0}^i) = G_{\tilde{\varphi}_t^*(F_0)}^i.$$

*Proof.* By definition, we have

$$\begin{aligned} \tilde{\varphi}_t^* (\gamma_{jk}^i)_{F_0} &= \tilde{\varphi}_t^* \left( g^{is} \frac{1}{2} \left( \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right) \right) \\ &= \tilde{\varphi}_t^* (g^{is}) \tilde{\varphi}_t^* \left( \frac{1}{2} \left( \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right) \right) \\ &= \tilde{\varphi}_t^* (g^{is}) \frac{1}{2} \left( \frac{\partial \tilde{\varphi}_t^*(g_{sj})}{\partial \tilde{x}^k} - \frac{\partial \tilde{\varphi}_t^*(g_{jk})}{\partial \tilde{x}^s} + \frac{\partial \tilde{\varphi}_t^*(g_{ks})}{\partial \tilde{x}^j} \right) \\ &= (\gamma_{jk}^i)_{\tilde{\varphi}_t^*(F_0)}. \end{aligned}$$

Next, we have

$$\begin{aligned} \tilde{\varphi}_t^* (G_{F_0}^i) &= \tilde{\varphi}_t^* \left( \frac{1}{2} (\gamma_{jk}^i)_{F_0} y^j y^k \right) = \frac{1}{2} \tilde{\varphi}_t^* ((\gamma_{jk}^i)_{F_0}) \tilde{\varphi}_t^* y^j \tilde{\varphi}_t^* y^k \\ &= \frac{1}{2} (\gamma_{jk}^i)_{\tilde{\varphi}_t^*(F_0)} \tilde{y}^j \tilde{y}^k = G_{\tilde{\varphi}_t^*(F_0)}^i, \end{aligned}$$

which completes the proof of Lemma 2.  $\square$

**Lemma 3.3.** *Let  $\varphi_t$  be a family of diffeomorphisms on  $M$  and  $Ric_{F_0}$  the Ricci scalar related to the Finsler structure  $F_0$ , then we have*

$$(3.5) \quad Ric_{\mu F_0} = \frac{1}{\mu^2} Ric_{F_0},$$

for all  $\mu > 0$ , and

$$(3.6) \quad \tilde{\varphi}_t^*(Ric_{F_0}) = Ric_{\tilde{\varphi}_t^*(F_0)}.$$

*Proof.* The spray coefficient  $G_{F_0}^i$  are two-homogeneous, that is, for all  $\lambda > 0$  we have  $G_{F_0}^i(x, \lambda y) = \lambda^2 G_{F_0}^i(x, y)$ . On the other hand for all  $\mu > 0$ ,  $G_{\mu F_0}^i = G_{F_0}^i$ . In fact

$$(g_{ij})_{\mu F_0} = \frac{1}{2} \frac{\partial^2((\mu F_0)^2)}{\partial y^i \partial y^j} = \frac{1}{2} \frac{\partial^2(\mu^2 F_0^2)}{\partial y^i \partial y^j} = \frac{1}{2} \mu^2 \frac{\partial^2(F_0^2)}{\partial y^i \partial y^j} = \mu^2 (g_{ij})_{F_0},$$

and hence

$$\begin{aligned} (\gamma_{jk}^i)_{\mu F_0} &= \frac{1}{2} (g^{is})_{\mu F_0} \left( \frac{\partial((g_{sj})_{\mu F_0})}{\partial x^k} - \frac{\partial((g_{jk})_{\mu F_0})}{\partial x^s} + \frac{\partial((g_{ks})_{\mu F_0})}{\partial x^j} \right) \\ &= \frac{1}{2} \frac{1}{\mu^2} (g^{is})_{F_0} \left( \frac{\partial(\mu^2 (g_{sj})_{F_0})}{\partial x^k} - \frac{\partial(\mu^2 (g_{jk})_{F_0})}{\partial x^s} + \frac{\partial(\mu^2 (g_{ks})_{F_0})}{\partial x^j} \right) \\ &= \frac{1}{2} \frac{1}{\mu^2} (g^{is})_{F_0} \mu^2 \left( \frac{\partial((g_{sj})_{F_0})}{\partial x^k} - \frac{\partial((g_{jk})_{F_0})}{\partial x^s} + \frac{\partial((g_{ks})_{F_0})}{\partial x^j} \right) \\ &= \frac{1}{2} (g^{is})_{F_0} \left( \frac{\partial((g_{sj})_{F_0})}{\partial x^k} - \frac{\partial((g_{jk})_{F_0})}{\partial x^s} + \frac{\partial((g_{ks})_{F_0})}{\partial x^j} \right) = (\gamma_{jk}^i)_{F_0}, \end{aligned}$$

which implies that

$$(3.7) \quad G_{\mu F_0}^i = \frac{1}{2} (\gamma_{jk}^i)_{\mu F_0} y^i y^j = \frac{1}{2} (\gamma_{jk}^i)_{F_0} y^i y^j = G_{F_0}^i.$$

By means of the definition of Ricci scalar we obtain

$$(3.8) \quad (R_k^i)_{\mu F_0} = \frac{1}{\mu^2 F_0^2} \left( 2 \frac{\partial G_{\mu F_0}^i}{\partial x^k} - \frac{\partial^2 G_{\mu F_0}^i}{\partial x^j \partial y^k} y^j + 2 G^j \frac{\partial^2 G_{\mu F_0}^i}{\partial y^j \partial y^k} - \frac{\partial G_{\mu F_0}^i}{\partial y^j} \frac{\partial G_{\mu F_0}^j}{\partial y^k} \right).$$

Using (3.7) and (3.8) we get

$$\begin{aligned} (R_k^i)_{\mu F_0} &= \frac{1}{\mu^2 F_0^2} \left( 2 \frac{\partial G_{F_0}^i}{\partial x^k} - \frac{\partial^2 G_{F_0}^i}{\partial x^j \partial y^k} y^j + 2 G^j \frac{\partial^2 G_{F_0}^i}{\partial y^j \partial y^k} - \frac{\partial G_{F_0}^i}{\partial y^j} \frac{\partial G_{F_0}^j}{\partial y^k} \right) \\ &= \frac{1}{\mu^2} (R_k^i)_{F_0}. \end{aligned}$$

Putting  $i = k$  in this equation, we get  $Ric_{\mu F_0} = \frac{1}{\mu^2} Ric_{F_0}$ . Therefore we have (3.5). Next, in order to prove (3.6) we use (3.5) as follows:

$$\begin{aligned} \tilde{\varphi}_t^*((R_k^i)_{F_0}) &= \tilde{\varphi}_t^*\left(\frac{1}{F_0^2}\left(2\frac{\partial(G_{F_0}^i)}{\partial x^k} - \frac{\partial^2(G_{F_0}^i)}{\partial x^j \partial y^k} y^j\right.\right. \\ &\quad \left.\left.+ 2G_{F_0}^j \frac{\partial^2(G_{F_0}^i)}{\partial y^j \partial y^k} - \frac{\partial(G_{F_0}^i)}{\partial y^j} \frac{\partial(G_{F_0}^i)}{\partial y^k}\right)\right) \\ &= \tilde{\varphi}_t^*\left(\frac{1}{F_0^2}\right) \tilde{\varphi}_t^*\left(2\frac{\partial(G_{F_0}^i)}{\partial x^k} - \frac{\partial^2(G_{F_0}^i)}{\partial x^j \partial y^k} y^j\right. \\ &\quad \left.+ 2G_{F_0}^j \frac{\partial^2(G_{F_0}^i)}{\partial y^j \partial y^k} - \frac{\partial(G_{F_0}^i)}{\partial y^j} \frac{\partial(G_{F_0}^i)}{\partial y^k}\right). \end{aligned}$$

Thus we get

$$\begin{aligned} \tilde{\varphi}_t^*((R_k^i)_{F_0}) &= \frac{1}{\tilde{\varphi}_t^*(F_0^2)} \left(2\frac{\partial(\tilde{\varphi}_t^*(G_{F_0}^i))}{\partial \tilde{x}^k} - \frac{\partial^2(\tilde{\varphi}_t^*(G_{F_0}^i))}{\partial \tilde{x}^j \partial \tilde{y}^k} \tilde{y}^j\right. \\ &\quad \left.+ 2\tilde{\varphi}_t^*(G_{F_0}^j) \frac{\partial^2(\tilde{\varphi}_t^*(G_{F_0}^i))}{\partial \tilde{y}^j \partial \tilde{y}^k} - \frac{\partial(\tilde{\varphi}_t^*(G_{F_0}^i))}{\partial \tilde{y}^j} \frac{\partial(\tilde{\varphi}_t^*(G_{F_0}^i))}{\partial \tilde{y}^k}\right). \end{aligned}$$

Putting  $i = k$  in this equation together with (3.4) implies

$$\tilde{\varphi}_t^*(Ric_{F_0}) = Ric_{\tilde{\varphi}_t^*(F_0)}.$$

This completes the proof of Lemma 3. □

Now, we are in a position to prove the main result of this work. Let  $M$  be a compact differentiable manifold and  $F_0$  an initial Finsler structure on  $M$ . If  $(M, F_0, V)$  is a solution to (3.2) then there is a one-parameter family of scalars  $\tau(t)$  and a one-parameter family of diffeomorphisms  $\varphi_t$  on  $M$  such that  $(M, F(t))$  is a solution of the Ricci flow (2.4), where  $F(t)$  is defined by

$$(3.9) \quad F^2(t) = \tau(t)\tilde{\varphi}_t^*(F_0^2).$$

The converse of this assertion is also true, that is, if  $(M, F(t))$  is a solution to the Finsler Ricci flow having the special form (3.9), then there is a vector field  $V$  on  $M$  such that  $(M, F_0, V)$  is quasi-Einstein. In short we have the following theorem.

**Theorem 3.4.** *Let  $(M, F_0)$  be a compact Finsler manifold and  $(M, F_0, V)$  a quasi-Einstein space. Then there exists a solution  $(M, F(t))$  in the form (3.9) to the Finslerian Ricci flow. Conversely, if  $(M, F(t))$  is a*



solution to the Finslerian Ricci flow having the form (3.9), then there is a vector field  $V$  on  $M$  such that  $(M, F_0, V)$  is quasi-Einstein.

*Proof.* Suppose that  $(M, F_0, V)$  satisfies (3.2) and consider a family of scalars  $\tau(t)$  defined by

$$\tau(t) := 1 - 2\lambda t > 0,$$

where  $\lambda$  is a constant. Next define a one-parameter family of vector fields  $X_t$  on  $M$  by

$$X_t(x) := \frac{1}{\tau(t)} V(x).$$

Let  $\varphi_t$  denote the diffeomorphisms generated by the family  $X_t$ , where  $\varphi_0 = Id_M$ , and define a smooth one-parameter family of Finsler structures on  $M$  by

$$F^2(t) = \tau(t)\tilde{\varphi}_t^*(F_0^2).$$

Thus we have

$$\log(F(t)) = \frac{1}{2} \log(\tau(t)\tilde{\varphi}_t^*(F_0^2)).$$

Using (2.6) we have  $\frac{d}{dt}\tilde{\varphi}_t^*(F_0^2) = \tilde{\varphi}_t^*(\mathcal{L}_{X_t} F_0^2)$  from which we get

$$\begin{aligned} \frac{\partial}{\partial t} \log(F(t)) &= \frac{1}{2} \frac{\tau'(t)\tilde{\varphi}_t^*(F_0^2) + \tau(t)\tilde{\varphi}_t^*(\mathcal{L}_{X_t} F_0^2)}{\tau(t)\tilde{\varphi}_t^*(F_0^2)} \\ &= \frac{1}{2} \frac{\tilde{\varphi}_t^*(-2\lambda F_0^2 + \mathcal{L}_V F_0^2)}{\tau(t)\tilde{\varphi}_t^*(F_0^2)} \\ &= \frac{1}{2} \frac{\tilde{\varphi}_t^*(-2F_0^2 Ric_{F_0})}{\tau(t)\tilde{\varphi}_t^*(F_0^2)} \\ &= \frac{1}{2} \frac{(-2)\tilde{\varphi}_t^*(F_0^2)\tilde{\varphi}_t^*(Ric_{F_0})}{\tau(t)\tilde{\varphi}_t^*(F_0^2)} \\ &= \frac{-Ric_{\tilde{\varphi}_t^*(F_0)}}{\tau(t)} = -Ric_{(\tau(t))^{\frac{1}{2}}\tilde{\varphi}_t^*(F_0)} \\ &= -Ric_{F(t)}. \end{aligned}$$

Therefore,  $F(t)$  is a solution to the Ricci flow (2.4). Conversely, suppose that  $(M, F(t))$  is a solution to the Ricci flow equation (2.4) having the form (3.9). We may assume without loss of generality that  $\tau(0) = 1$  and

$\varphi_0 = Id_M$ . Then we have

$$\begin{aligned}
 -Ric_{F_0} &= \frac{\partial}{\partial t}(\log F(t))|_{t=0} \\
 &= \frac{\partial}{\partial t}\left(\frac{1}{2} \log(\tau(t)\tilde{\varphi}_t^*F_0^2)\right)|_{t=0} \\
 &= \frac{1}{2} \frac{\tau'(t)\tilde{\varphi}_t^*F_0^2 + \tau(t)\tilde{\varphi}_t^*(\mathcal{L}_{\hat{X}(t)}F_0^2)}{\tau(t)\tilde{\varphi}_t^*(F_0^2)}|_{t=0} \\
 (3.10) \quad &= \frac{1}{2} \frac{\tau'(0)F_0^2 + \mathcal{L}_{\hat{X}(0)}F_0^2}{F_0^2},
 \end{aligned}$$

where  $X(t)$  is the family of vector fields generating the diffeomorphisms  $\varphi_t$ . Thus, (3.10) implies

$$-2F_0^2 Ric_{F_0} = \tau'(0)F_0^2 + \mathcal{L}_{\hat{X}(0)}F_0^2.$$

Replacing  $\lambda = \frac{-1}{2}\tau'(0)$  and  $V = X(0)$  we have the equation (3.2). This completes the proof of the Theorem.  $\square$

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