Title:
Reversibility of a module with respect to the bifunctors Hom and Rej

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REVERSIBILITY OF A MODULE WITH RESPECT TO THE BIFUNCTORS HOM AND REJ

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Abstract. Let $M_R$ be a non-zero module and $F : \sigma[M_R] \times \sigma[M_R] \to \text{Mod-Z}$ a bifunctor. The $F$-reversibility of $M$ is defined by $F(X, Y) = 0 \Rightarrow F(Y, X) = 0$ for all non-zero $X, Y$ in $\sigma[M_R]$. Hom (resp. Rej)-reversibility of $M$ is characterized in different ways. Among other things, it is shown that $R_R (\text{resp. } R_R)$ is Hom-reversible if and only if $R = \bigoplus_{i=1}^{n} R_i$ such that each $R_i$ is a perfect ring with a unique simple module (up to isomorphism). In particular, for a duo ring, the concepts of perfectness and Hom-reversibility coincide.

Keywords: Co-retractable, Kasch module, perfect ring, prime module, cogenerator.


1. Introduction and Preliminaries

Throughout this paper rings will have non-zero identity elements and modules will be unitary. Unless stated otherwise modules will be right modules. Let $R$ be a ring, $M$ a non-zero $R$-module, $F : \sigma[M_R] \times \sigma[M_R] \to \text{Mod-Z}$ a bifunctor and $\emptyset \neq C, D \subseteq \sigma[M_R]$, where $\sigma[M_R]$ is the full subcategory of $R$-modules whose objects are all $R$-modules subgenerated by $M$. We say that $M_R$ is $C_D-F$-reversible if for all $C \in C, D \in D$, $F(C, D) = 0$ implies $F(D, C) = 0$. If for $C = D = \sigma[M_R]$ the above condition holds, we say that $M_R$ is $F$-reversible. There are several concepts that may be stated in terms of the notion of reversibility.
For instance, $M_R$ is prime in the sense of Bican [6] (i.e., $\text{Rej}(M, N) = 0$ for all nonzero submodules $N$ of $M$) if and only if $M_R$ is $CD$-Rej-reversible for $C = \{\text{all non-zero submodules of } M_R\}$, $D = \{M_R\}$. Here $\text{Rej}(C, D) = \cap_{f: C \rightarrow D} \ker f$. Also a module $M_R$ is fully Kasch in the sense of Albu and Wisbauer [2] if and only if $M_R$ is $CD$-Hom-reversible for $C = \{\text{all simple } R\text{-modules in } \sigma[M_R]\}$ and $D = \{\text{cyclic modules in } \sigma[M_R]\}$. Note that if $S$ is a simple module in $\sigma[M_R]$, then $S \simeq N/K$ for some cyclic submodule $N \leq M^{(A)}$; see also [5]. The conditions retractable [9], co-retractable [1, 2.13], [7] and weak generator [11] can also be stated by $CD\cdot F$-reversibility by choosing suitable $C, D$ and $F = \text{Hom}$. These observations motivated us to consider questions, such as: “what are $F$-reversible modules when $F$ is $\text{Hom}$ or $\text{Rej}$?”. In this paper, we answer these questions and show first that Hom-reversible modules with semiprime endomorphism rings are precisely semisimple modules [Theorem 2.4]. Then, Hom (resp. Rej)-reversible modules are characterized in several ways in Theorem 2.8 (resp. Theorem 2.14). In particular, it is proved that a duo ring $R$ is perfect if and only if the module $R_R$ is Hom-reversible [Theorem 2.10]. We use the notations $N \leq M$ and $N \leq_e M$ to denote respectively that $N$ is a fully invariant and essential submodule of $M$. We follow [4] and [12] for the terms not defined here, and for the basic results on module and ring theory that are relevant to this work.

2. Main results

We begin with some definitions and lemmas. A non-zero $R$-module $M$ is called:
- Kasch if any simple module in $\sigma[M_R]$ can be embedded in $M_R$ [2].
- retractable if $\text{Hom}_R(M, N) \neq 0$ for any non-zero submodule $N$ of $M_R$ [9].
- co-retractable if $\text{Hom}_R(M/N, M) \neq 0$ for any proper submodule $N$ of $M_R$ [1].

In [2], a module $M_R$ is called fully Kasch if all modules in $\sigma[M_R]$ are Kasch. We say that $M_R$ is fully retractable (resp. fully co-retractable) if all modules in $\sigma[M_R]$ are retractable (resp. co-retractable). Also we call $M_R$, fully max if any non-zero module in $\sigma[M_R]$ has a maximal submodule. In [3], ring $R$ for which the module $R_R$ is fully co-retractable was studied where the authors used the term completely co-retractable for such rings. It is easy to verify that all of the above conditions on a
module are Morita invariant properties.

**Lemma 2.1.** Let $R$ be a semiprime ring. Then the following statements hold.
(i) $\forall 0 \neq I \leq R_R (R_R)$, $\exists a \in I$ with $a^2 \neq 0$.
(ii) $I \leq_e R_R (I \leq_e R_R) \Rightarrow \text{r.ann}_R I = 0 \text{ (l.ann}_R I = 0)$.

*Proof.* We only prove (i). Let $I \leq R_R$. If $a^2 = 0$ for all $a \in I$ then $(x+y)^2 = 0$ for all $x, y \in I$. Consequently, $xy = -yx$ for all $x, y \in I$. It follows that $(xI)^2 = 0$ for all $x \in I$. Thus for all $x \in I$, $xI = 0$ by the semiprime condition on $R$, hence $I^2 = 0$, a contradiction. Therefore, there exists $a \in I$ such that $a^2 \neq 0$. \qed

**Lemma 2.2.** Let $M$ be a non-zero $R$-module with semiprime endomorphism ring $S$.
(i) If $M_R$ is retractable, $N \leq_e M_R$ and $I = \text{Hom}_R(M, N)$ then $I \leq_e S$.
(ii) If $M = N_1 \oplus N_2$ is a direct sum of submodules then $\text{Hom}_R(N_1, N_2) = 0$ if and only if $\text{Hom}_R(N_2, N_1) = 0$.

*Proof.* Let $0 \neq g \in S$. Since $N \leq_e M_R$, $g(M) \cap N = W \neq 0$, $\text{Hom}_R(M, W) = A$ is a non-zero right ideal of $S$ by the retractable condition on $M$. Thus by Lemma 2.1 there exists $f \in A$ such that $f^2 \neq 0$. Since $f^2(M) \subseteq fg(M) \subseteq N$, we have $0 \neq fg \in I$. It follows that $I \leq_e S$.

(ii) Let $A_i = \text{End}_R(N_i)$ $(i = 1, 2)$, $B = \text{Hom}_R(N_1, N_2)$ and $C = \text{Hom}_R(N_2, N_1)$. It is easy to verify that $S \simeq \begin{pmatrix} A_1 & B \\ C & A_2 \end{pmatrix}$. Now if $B = 0$, then $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ is an ideal of $S$ with zero square. Hence, $C = 0$ by hypothesis on $S$. Similarly, $C = 0$ implies $B = 0$. \qed

**Corollary 2.3.** Let $M$ be a non-zero $R$-module.
(i) If for every non-zero $N \leq M_R$, the ring $\text{End}_R(M \oplus N)$ is semiprime, then $M$ is a retractable $R$-module.
(ii) If for every $N < M_R$, the ring $\text{End}_R(M \oplus M/N)$ is semiprime, then $M$ is a co-retractable $R$-module.

*Proof.* Apply Lemma 2.2(ii) and note that for every non-zero $N < M_R$, $\text{Hom}_R(N, M)$ and $\text{Hom}_R(M, M/N)$ are non-zero. \qed
Theorem 2.4. The following statements are equivalent for $M_R$.
(i) For every non-zero $X \in \sigma[M]$, $\text{End}_R(X)$ is a semiprime ring.
(ii) $M_R$ is Hom-reversible and $\text{End}_R(M)$ is a semiprime ring.
(iii) $M_R$ is semisimple.

Proof. i)⇒(ii). Apply Lemma 2.2(ii) and the fact that the class $\sigma[M_R]$ is closed under direct sums.
(ii)⇒(iii). Let $S = \text{End}_R(M)$ and $\text{Soc}(M) = N$. Assume that $X$ is a non-zero submodule of $M_R$. By [12, 14.9] there exists a submodule $Y \leq X$ such that $\text{Soc}(X/Y) \neq 0$. Let $U$ be a simple $R$-module embedded in $X/Y$. It follows by hypothesis that $\text{Hom}_R(U, X) \neq 0$. Hence $\text{Soc}(X)$ is non-zero. This shows that $N \leq M$. Also the condition (ii) implies that $M_R$ is retractable. Thus $I = \text{Hom}_R(M, N)$ is an ideal of $S$ such that $I \leq SS$ by Lemma 2.2(i). Now if $M/N$ is non-zero, then by hypothesis, there exists a non-zero homomorphism $f : M/N \to M$. Let $g = f\pi$ where $\pi : M \to M/N$ is the canonical homomorphism. Then $0 \neq g \in S$ and $g(N) = 0$. It follows that $gI = 0$ which contradicts Lemma 2.1. Therefore, $M/N = 0$ and $M$ is a semisimple $R$-module.
(iii)⇒(i). This is a known result but we give a proof for completeness; see also [8, Theorem 2.6(b)]. If $M_R$ is semisimple, then so is every $X \in \sigma[M]$. Hence, it is enough to show that $S = \text{End}_R(M)$ is a semiprime ring. Let now $0 \neq g \in S$ with $(gS)^2 = 0$ and set $K = \ker g$, then there exists $N \leq M_R$ such that $K \oplus N = M$. Suppose that $M = g(N) \oplus W$ for some $W \leq M_R$. Since $g$ is one to one on $N$, the map $h : M \to M$ defined by $h(g(n)) = n$ and $h(W) = 0$, is an $R$-homomorphism. Now we have $ghg = g$ and so $e = gh$ is a non-zero idempotent in $gS$, a contradiction. Therefore $S$ is a semiprime ring. □

Corollary 2.5. A ring $R$ is semisimple Artinian if and only if all non-zero $R$-modules have semiprime endomorphism rings.

Proof. This is an immediate corollary of Theorem 2.4. □

We are now going to investigate the Hom-reversible $R$-modules.

Lemma 2.6. If every proper factor of $M_R$ has a maximal submodule and $M_R$ is Kasch then $M_R$ is co-retractable. In particular, finitely generated Kasch modules are co-retractable.

Proof. Let $Y < M_R$. By hypothesis, there exists a maximal submodule $K$ of $M$ such that $Y \leq K$. Since $M_R$ is Kasch, $M/K$ can be embedded...
in $M$. It follows that $\text{Hom}_R(M/Y, M) \neq 0$, proving that $M_R$ is co-retractable. The last statement is now clear.

**Proposition 2.7.** (i) Every fully co-retractable module is a fully Kasch module.
(ii) Every fully retractable module is a fully max module.
(iii) Every fully Kasch fully max module is fully co-retractable and fully retractable.

**Proof.** (i) Let $M_R$ be a fully co-retractable module and $X \in \sigma[M]$. We shall show that $X_R$ is Kasch. Let $U$ be a simple $R$-module in $\sigma[X]$, then there exist a set $\Lambda$, $Y \leq X^{(\Lambda)}$ and a surjective homomorphism $\alpha : Y \to U$. Since $Y_R$ is co-retractable, $\text{Hom}_R(Y/K, Y) \neq 0$ where $K = \ker \alpha$. It follows that $U$ can be embedded in $X$, proving that $X_R$ is Kasch.

(ii) Let $M_R$ be a fully retractable module and $0 \neq X \in \sigma[M]$. As seen in the proof of (ii)$\Rightarrow$(iii) of Theorem 2.4, $X_R$ has a maximal submodule.

(iii) Let $M_R$ be a fully Kasch max module and $Y < X \in \sigma[M]$. By Lemma 2.6, $X_R$ is co-retractable. It remains to show that $X_R$ is retractable. By hypothesis, $Y_R$ is Kasch. Hence $Y$ contains a simple $R$-module $U$. Clearly there exists a non-zero homomorphism $\theta : X \to E(U)$. By hypothesis, $\ker \theta =: K$ is contained in a maximal submodule $N$ of $X_R$. Now since $\text{Im} \theta \simeq (X/K)_R$ is Kasch, the simple $R$-module $X/N \in \sigma[\text{Im} \theta]$ must be embedded in $E(U)$. It follows that $X/N \simeq U$ and so $\text{Hom}_R(X, Y) \neq 0$, as desired.

**Theorem 2.8.** The following statements are equivalent for $M_R$.
(i) $M_R$ is Hom-reversible.
(ii) $M_R$ is fully retractable and fully co-retractable.
(iii) $M_R$ is fully Kasch and fully max.

**Proof.** (i)$\Rightarrow$(ii). This follows by a routine argument.
(ii)$\Rightarrow$(i). Let $X, Y \in \sigma[M]$ and $0 \neq f \in \text{Hom}_R(X, Y)$. We shall show that $\text{Hom}_R(Y, X) \neq 0$. Let $\ker f := K$ and consider the exact sequence $0 \to X/K \xrightarrow{f} Y$. Since $X_R$ is co-retractable, $\text{Hom}_R(X/K, X)$ is non-zero. It follows that there exists non-zero element $h \in \text{Hom}_R(Y, E(X))$. Now since $h(Y)$ is a retractable $R$-module, $\text{Hom}_R(h(Y), h(Y) \cap X)$ is non-zero. Thus $\text{Hom}_R(Y, X)$ is non-zero.

(ii)$\Rightarrow$(iii). By Proposition 2.7.

As an application of Theorem 2.8, we will give a new characterization of certain perfect rings in terms of Hom-reversibility of $R$. A ring $R$ is
called *right fully Kasch* if $R/I$ is a right Kasch ring for any proper ideal $I$ of $R$. A ring $R$ is right fully Kasch if and only if $R_R$ is fully Kasch [2, Proposition 3.15].

**Proposition 2.9.** Let $R$ be a ring. The following statements are equivalent.

(i) Every cyclic $R$-module is co-retractable.
(ii) $R$ is a right fully Kasch ring.
(iii) $R = \bigoplus_{i=1}^{n} R_i$ such that each $R_i$ is a left perfect ring with a unique simple module (up to isomorphism).

*Proof.* (i)$\Rightarrow$(ii). Let $M$ be a non-zero $R$-module. It is easy to see that every simple module in $\sigma[M_R]$ is isomorphic to a factor of a cyclic submodule of $M^{(\Lambda)}$ for some set $\Lambda$. Thus $M$ is a Kasch module by (i).

(ii)$\Rightarrow$(iii). By [5, Proposition 2.8], $R$ is left perfect. Thus $R = \bigoplus_{i=1}^{t} e_i R$ where $e_1, ..., e_t$ are orthogonal idempotents and $e_i R/J(e_i R) \cong S_i^{(e_i R)}$ such that $S_i, S_j$ are non-isomorphic simple $R$-modules for $i \neq j$. It is enough to show that $\text{Hom}_R(e_i R, e_j R) = 0$ for $1 \leq i \neq j \leq t$. For $i \neq j$ let $f : e_i R \rightarrow e_j R$ be a non-zero homomorphism. By hypothesis, $S_i$ can be embedded in $f(e_i R) \leq e_j R$. It follows that $S_i$ can be essentially embedded in $e_j R/K$ for some $K \leq e_j R$. On the other hand, $S_j$ embeds in $e_j R/K$ by hypothesis. Thus $S_j \cong S_i$, a contradiction.

(iii)$\Rightarrow$(i). This is by Lemma 2.6 and the fact that a left perfect ring with a unique simple module is a fully Kasch ring. $\square$

**Theorem 2.10.** The following are equivalent for a ring $R$.

(i) $R_R (R_R)$ is Hom-reversible.
(ii) $R_R (R_R)$ is fully co-retractable.
(iii) $R = \bigoplus_{i=1}^{n} R_i$ such that each $R_i$ is a perfect ring with a unique simple module.

*Proof.* (i)$\Rightarrow$(ii). By Theorem 2.8.

(ii)$\Rightarrow$(iii). By Proposition 2.9 and [3, Theorem 3.10].

(iii)$\Rightarrow$(i). By Theorem 2.8(iii) and the fact that if $R$ is a perfect ring with a unique simple module then $R_R$ is fully Kasch. $\square$

A ring $R$ is said to be duo if every right (left) ideal of $R$ is a two sided ideal.

**Corollary 2.11.** Let $R$ be a ring Morita equivalent to a duo ring. Then the following statements are equivalent for $R$. 

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(i) Every cyclic $R$-module is co-retractable.
(ii) $R$ is a perfect ring.
(iii) $R_R$ is Hom-reversible.

Proof. Suppose that $R$ is Morita equivalent to a duo ring $T$.
(i)⇒(ii). By Proposition 2.9, $R$ and hence $T$ is a left perfect ring. Now since $T$ is a duo ring, it must be a (right) perfect ring.
(ii)⇒(iii). Since $T$ is a duo ring, all idempotent elements in $T$ are central. Now the condition (ii) implies that $T$ satisfies the condition (iii) of Theorem 2.10, proving that $R_R$ is Hom-reversible.
(iii)⇒(i). By Theorem 2.10.

In the following, we show that the converse of parts (i) and (ii) of Proposition 2.7 do not necessarily hold.

Examples 2.12. (i) Let $R$ be a local left perfect ring that is not right perfect; see [10, Example 23.22]. Hence $R_R$ is fully Kasch but it is not fully co-retractable by Proposition 2.9 and Theorem 2.10.
(ii) Let $R = \left(\begin{array}{cc} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{array}\right)$, $e = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right)$ and $I = \left(\begin{array}{cc} 0 & \mathbb{Q} \\ 0 & 0 \end{array}\right)$. Then since $I^2 = 0$ and $R/I \cong \mathbb{Q} \oplus \mathbb{Q}$, it is easily seen that $R_R$ is fully max. But $eR$ is not a retractable $R$-module, because $\text{Hom}_R(eR, I) = 0$.
(iii) Let $M = \mathbb{Q} \oplus A$ and $A = \bigoplus_p \mathbb{Z}_p$ where $p$ runs over the set of all prime numbers. It is easy to verify that $M$ is a Kasch $\mathbb{Z}$-module but it is not co-retractable, because $\text{Hom}_R(M/N, M) = 0$, where $N = \mathbb{Z} \oplus A$.

We are now going to investigate the Rej-reversible modules. We say that $M_R$ is left (resp. right) Rej-reversible whenever for every non-zero $X \in \sigma[M_R]$, $\text{Rej}(X, M) = 0$ (resp. $\text{Rej}(M, X) = 0$) implies $\text{Rej}(M, X) = 0$ (resp. $\text{Rej}(X, M) = 0$). Recall from [12, Proposition 15.1] that $U_e = \bigoplus \{ U \mid U_R \text{ is a finitely generated submodule of } M^{(N)} \}$ is a generator in $\sigma[M_R]$ and $\text{Tr}(U_e, \prod_{\Lambda} N_\lambda)$ (dented by $\prod_{\Lambda}^M N_\lambda$) is a product for any family $\{ N_\lambda \}_\Lambda$ of modules in $\sigma[M_R]$.

Proposition 2.13. Let $M$ be a non-zero $R$-module.
(i) $M$ is right Rej-reversible if and only if $M_R$ is a co-generator in $\sigma[M_R]$.
(ii) $M$ is left Rej-reversible if and only if $\prod_{\Lambda}^M M$ is a prime module for any set $\Lambda$. 
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Proof. (i) Just note that for all $X \in \sigma[M_R]$, $\text{Rej}(M, M \oplus X) = 0$.

(ii) ($\Rightarrow$). Let $N \leq \prod_A^M M$ for some set $\Lambda$. Then $\text{Rej}(N, M) = 0$ and so $\text{Rej}(M, N) = 0$ by hypothesis. Thus $\text{Rej}(\prod_A^M M, N) = 0$, proving that $\prod_A^M M$ is a prime module.

($\Leftarrow$). Let $X \in \sigma[M_R]$ and $\text{Rej}(X, M) = 0$, then $X$ embeds in $\prod_A^M M$ for some $\Lambda$ and so $X = \text{Tr}(U_e, X)$ embeds in $\text{Tr}(U_e, \prod_A^M M) = \prod_A^M M$. Since $\prod_A^M M$ is prime, $\text{Rej}(M, X) = 0$, proving that $M$ is left Rej-reversible.

An $R$-module $M$ is called co-semisimple if any proper submodule of $M$ is an intersection of maximal submodules [12, 23.1].

Theorem 2.14. The following statements are equivalent for a non-zero module $M_R$.

(i) $M$ is left and right Rej-reversible.

(ii) $\text{Rej}(X, Y) = 0$ for all non-zero $X, Y \in \sigma[M_R]$.

(iii) $M_R$ is semi-Artinian co-semisimple and $\sigma[M_R]$ has a unique simple module.

(iv) Any non-zero $X \in \sigma[M_R]$ is a co-generator in $\sigma[M_R]$.

(v) Any non-zero $X \in \sigma[M_R]$ is prime.

(vi) $M_R$ is Rej-reversible.

Proof. (i)$\Rightarrow$(ii). Let $X, Y \in \sigma[M_R]$ be non-zero $R$-modules. Then by (i) and Proposition 2.13 (i), $\text{Rej}(X, M) = 0 = \text{Rej}(Y, M)$ and so $\text{Rej}(M, Y) = 0$, because $M$ is left Rej-reversible. Thus $\text{Rej}(X, Y) = 0$.

(ii)$\Rightarrow$(iii). Let $S$ be a non-zero simple $R$-module in $\sigma[M_R]$ and $0 \neq X \in \sigma[M_R]$. By (ii), $\text{Rej}(X, S) = 0$, therefore $X$ is co-semisimple, $\sigma[M_R]$ has a unique simple module and $X$ is semi-Artinian because by hypothesis $\text{Rej}(S, X) = 0$.

(iii)$\Rightarrow$(iv). Let $0 \neq X \in \sigma[M_R]$. Since $M_R$ is co-semisimple, $X$ is co-semisimple by [12, 23.1] and since $\sigma[M_R]$ has a unique simple module, $\text{Rej}(X, S) = 0$ for any simple $R$-module $S$ in $\sigma[M_R]$. Hence $\text{Rej}(X, Y) = 0$ for all non-zero $Y \in \sigma[M_R]$, because $Y$ has a copy of $S$.

(iv)$\Rightarrow$(v). This is clear.

(v)$\Rightarrow$(i). By (v), $\prod_A^M M$ is prime hence by Proposition 2.13 (ii), $M$ is left Rej-reversible and since $M \bigoplus X$ is prime for all non-zero $X$ in $\sigma[M_R]$, it is easy to see that $M$ is a co-generator in $\sigma[M_R]$. Thus $M_R$ is right Rej-reversible.

(ii)$\Rightarrow$(vi)$\Rightarrow$(i). These are clear by definitions. □
Corollary 2.15. If $M_R$ is Rej-reversible then $M_R$ is Hom-reversible.

Proof. This follows from Theorems 2.8 and 2.14. □

An $R$-module $M$ is called semi-projective if $\text{Hom}_R(M, fM) = f\text{End}_R(M)$ for any $f \in \text{End}_R(M)$ [12, p. 260].

Proposition 2.16. Let $M$ be a semi-projective $R$-module. Then $M$ is Rej-reversible if and only if $M$ is homogeneous semisimple.

Proof. The sufficiency is clear. Conversely, by Corollary 2.15, $M$ is Hom-reversible. Hence by Theorem 2.4 it is enough to show that $\text{End}_R(M)$ is semiprime. Let $I$ be a non-zero principal right ideal in $\text{End}_R(M)$. Since $M$ is Rej-reversible, it can be embedded in $\prod_{\Lambda} IM$ for some set $\Lambda$. Hence there exists $f \in \text{Hom}_R(M, IM)$ such that $f|_{IM} \neq 0$. Thus $0 \neq fI \in I^2 = \text{Hom}_R(M, IM)I$. Therefore $\text{End}_R(M)$ is semisimple and by Theorem 2.14(iii), $M$ is homogeneous semisimple. □

We remark that if $M_R$ is Ext-reversible, then a module in $\sigma[M_R]$ is $M$-projective if and only if it is $M$-injective. The diligent readers may then be interested in the following project.

Project: Let $R$ be a ring and $M_R$ be a non-zero module. Characterize the Ext-reversibility of $M_R$.

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