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FROBENIUS KERNEL AND WEDDERBURN'S LITTLE THEOREM

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Dedicated to Professor Saieed Akbari

Abstract. We give a new proof of the well known Wedderburn's little theorem (1905) that a finite division ring is commutative. We apply the concept of Frobenius kernel in Frobenius representation theorem in finite group theory to build a proof.
Keywords: Division ring, maximal subfield, Frobenius representation theorem.
MSC(2010): Primary: 12E15; Secondary: 17A35, 17C60.

1. Introduction

In 1905, Wedderburn proved that [8] "Any division algebra with finitely many elements is commutative". All over the past century, new proofs of this theorem have been given by a number of mathematicians. One can find a good survey of these proofs in [1]. Another approach to this theorem applying Frobenius groups can be found in [2]. Two recent new proofs can be found in [4, 7]. Two of the most famous proofs of this theorem are due to Witt [9, 6] and B.L. van der Waerden ([3], p. 97). Here we give a new proof of this theorem based on a famous theorem of Frobenius. First we recall this latter theorem and one of its corollaries ([5], p. 196]).

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Theorem A. (Frobenius). *Let G be a finite group and let H be a subgroup of G. Also suppose that the condition $H \cap (gHg^{-1}) = \{e\}$ holds for all $g \in G \setminus H$ (Hence $N_G(H) = H$). Then*

$$F = G \setminus \bigcup_{g\in G} g(H^*)g^{-1}, \text{ where } H^* = H \setminus \{e\}$$

is a normal subgroup of G and we have $G = FH$, where $F \cap H = \{e\}$. We call G a Frobenius group with respect to H, and F is called the Frobenius kernel of G.

Proposition B. *a) Let G be a Frobenius group for H with Frobenius kernel F. If $N \trianglelefteq G$, then either $N \leq F$ or $F \leq N$.*

b) If G is a Frobenius group, then its Frobenius kernel is the maximal nilpotent normal subgroup (known as Fitting subgroup of G and denoted by Fit(G)).

c) If G is a Frobenius group for H_1 and H_2, then H_1 and H_2 are conjugate in G.

2. **Results**

We begin with the following lemma which has a main role in this study.

Lemma 2.1. *Let D be a finite division ring with center F such that any proper subdivision ring of D is commutative. Also let the intersection of any two maximal subfield of D be equal to F. If T is a maximal subfield of D, then $N(T^*) = T^*$.*

Proof. Let $y \in N(T^*)\backslash T^*$. Since T is a finite field, there exists an element $x \in T$ such that $T^* = < x >$ and for an integer $j \neq 1$ we have $yxy^{-1} = x^j$. Consequently for any integer n the equality $y^n x = x^{j^n} y^n$ holds. As T/F is a Galois extension, for any prime divisor s of $[T : F]$ there exists a subfield K of T with dimension $[K : F] = s$. Since K is finite, for some element $b \in K$ we have $K^* = < b >$ and $K = F(b)$. Clearly there exists an integer u such that $b = x^u$. Let L be a maximal subfield of D containing y. This implies that $L \neq T$. Consider $yF^* \in D^*/F^*$ and let $w = o(yF^*)$ be the order of $\overline{y} = yF^*$ in the quotient group D^*/F^*. Clearly $w \neq 1$. Let p be a prime divisor of w and put $m = p^{-1}w$. Let $z = y^m$. Hence $o(zF^*) = p$. Now note that $D = F(x, y)$. Since $z, b \in D\backslash F$, the centralizers of z and b are maximal subfields of D. Since $x \in C_D(b)$ and $y \in C_D(z)$ we have $C_D(b) \neq C_D(z)$. By assumption

$C_D(b) \cap C_D(z) = F$, hence $zb \neq bz$ and we have $D = F(b, z)$. Since $zb = b^{j^m} z$, every element of D has a representation as $\sum c_i z^i$, where $c_i \in F(b)$. Note that for all t in $1 \leq t \leq p-1$ we have $b^{j^{tm}} \neq b$, otherwise, $z^t b = b^{j^{tm}} z^t = bz^t$, which implies $z^t \in F$. This contradicts the fact that the order of zF^* is p. Now the set $\{1, z, z^2, \cdots, z^{p-1}\}$ is a basis of D over $F(b)$. Consequently $[D : F] = [D : F(b)][F(b) : F] = ps$. This implies that $[T : F] = s$ and $[L : F] = p$. This also implies that every subfield of D has dimension p or s. Now let $C_D(v)$ be the centralizer of an element $v \in D \backslash F$. Clearly $C_D(v)$ is a maximal subfield of D. Let q be the cardinality of F, then for some positive integers f and e, the class equation of the finite group D^* shows that:

$$q^{ps} - 1 = q - 1 + f(\frac{q^{ps} - 1}{q^p - 1}) + e(\frac{q^{ps} - 1}{q^s - 1}).$$

Equivalently:

$$q^{ps} - 1 - (q - 1) = \frac{q^{ps} - 1}{(q^p - 1)(q^s - 1)}(f(q^s - 1) + e(q^p - 1)).$$

Since $q^s - 1$ and $q^p - 1$ divide $q^{ps} - 1$ (the cardinality of D^*), D^* has subgroups with these cardinalities. If we let d to be the greatest common divisor of $q^p - 1$ and $q^s - 1$, then we have

$$q^{ps} - q = \frac{(q^{ps} - 1)d}{(q^p - 1)(q^s - 1)}(\frac{f(q^s - 1)}{d} + \frac{e(q^p - 1)}{d}).$$

Note that $\frac{(q^{ps}-1)d}{(q^p-1)(q^s-1)} \geq q$ and $\frac{(q^p-1)(q^s-1)}{d}$ is the least common multiple of $(q^p - 1)$ and $(q^s - 1)$. These facts imply that $q^{ps} - 1$ has a divisor h, which is greater than or equal to q. Now since h divides $q^{ps} - q$, it should also divides $q^{ps} - 1 - q^{ps} + q = q - 1$, which is a clear contradiction. This contradiction shows that $N(T^*) = T^*$ and the claim is fulfilled. □

We apply the above Lemma to present a new proof of the Wedderburn's little theorem based on the Frobenius kernel in Frobenius representation theorem.

Theorem 2.2 (Wedderburn)**.** *Any finite division ring is commutative.*

Proof. The proof is by induction on the size of D. The first step in size 2 is clear. Now suppose the claim holds for all division rings with elements fewer than $|D|$. Let $Z(D) = F$ be the center of D. If $D \neq F$, then we have at least two maximal subfields containing two non-commutative elements. Let T be a proper maximal subfield of D. We have two separate cases: either all pairs of maximal subfields of D have F as their intersection or there are at least two maximal subfields whose intersection strictly contains F.

Case 1. By the Lemma we have $N(T^*) = T^*$. Put $\overline{D^*} = D^*/F^*$ and $\overline{T^*} = T^*/F^*$. If for an element $d \in D$, we have $dTd^{-1} \cap T \neq F$, then since dTd^{-1} is a maximal subfield of D the equality $dTd^{-1} = T$ holds. This implies that $d \in T$. So for all $d \in D \setminus T$ we have $dTd^{-1} \cap T = F$ or $\overline{dT^*d^{-1}} \cap \overline{T^*} = \overline{F^*} = \overline{1}$. This equality shows that the main conditions for $\overline{D^*}$ to be a Frobenius group hold (see Theorem **A**). So, we may consider $\overline{D^*} = \overline{K}\ \overline{T^*}$ to be the Frobenius representation of $\overline{D^*}$ with kernel $\overline{K}$, where $\overline{T^*} \cap \overline{K} = \{\overline{1}\}$. Since $\overline{K} \neq \{\overline{1}\}$ there exists a nontrivial element $\overline{z} \in \overline{K}$. Let S be the maximal subfield of D containing z. The same process for S instead of T leads to a Frobenius representation $\overline{D^*} = \overline{L}\ \overline{S^*}$ with kernel $\overline{L}$ where $\overline{S^*} \cap \overline{L} = \{\overline{1}\}$. By Proposition **B** above, we have $\overline{K} = Fit(\overline{D^*}) = \overline{L}$. Consequently $\overline{1} \neq \overline{z} \in \overline{S^*} \cap \overline{K} = \overline{S^*} \cap \overline{L} = \{\overline{1}\}$. This is a clear contradiction.

Case 2. Let S and T be two maximal subfields of D with $S \cap T \neq F$. Consider $z \in (S \cap T) \setminus F$. Since $S \neq T$ there exist at least two elements $x \in T$ and $y \in S$, where $xy \neq yx$ and the induction process yields $D = F(x, y)$. Now we have $zx = xz$ and $zy = yz$. This implies that $z \in F$ which is a contradiction. $\square$

Now, we would like to pose some open problems for readers interested in pursuing the concepts introduced in this note. We believe the following question is considerable: whether in a non commutative division ring all maximal subfields are equal to their normalizers. We propose the following.

Conjecture. Let D be a division ring. Then D is commutative if and only if for all maximal subfields T of D, we have $N(T^*) = T^*$.

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