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ON THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. For a polynomial $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu},\ 1 \leq \mu \leq n$ of degree n, having all zeros in $|z| \leq k,\ k \leq 1$, Dewan et al [K. K. Dewan, N. Singh and A. Mir, Extension of some polynomial inequalities to the polar derivative, J. Math. Anal. Appl. 352 (2009) 807-815] proved that

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{1+k^\mu}\{(|\alpha| - A_\mu)\max_{|z|=1}|p(z)| + \frac{|\alpha|k^\mu + A_\mu}{k^n}\min_{|z|=k}|p(z)|\},$$

where $|\alpha| \geq k^\mu$ and $A_\mu = \frac{n(|a_n| - \frac{m}{k^n})k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n| - \frac{m}{k^n})|k^{\mu-1} + \mu|a_{n-\mu}|}$. In this paper we improve and extend the above inequality. Our result generalizes certain well-known polynomial inequalities.
Keywords: Polar derivative, polynomial inequalities, maximum modulus, restricted zeros of polynomials.
MSC(2010): Primary: 30A10; Secondary: 30C10, 30D15.

1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree n, then according to the well known Bernstein's inequality [3] on the derivative of a polynomial, we have that

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

This result is best possible and equality holds for a polynomial that has all zeros at the origin.

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If we restrict to the class of polynomials which have all zeros in $|z| \leq 1$, then it has been proved by Turan [10] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The inequality (1.2) is sharp and equality holds for a polynomial that has all zeros on $|z| = 1$.
As an extension to (1.2) Malik [8] proved that if $p(z)$ has all zeros in $|z| \leq k$, where $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.3}$$

This result is best possible and equality holds for $p(z) = (z-k)^n$.

On the other hand, for the class of polynomials $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu},\ 1 \leq \mu \leq n$ of degree n, having all zeros in $|z| \leq k,\ k \leq 1$, Aziz and Shah [2] demonstrated

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^{\mu}} \{\max_{|z|=1} |p(z)| + \frac{1}{k^{n-\mu}} \min_{|z|=k} |p(z)|\}. \tag{1.4}$$

Let $D_\alpha p(z)$ denote the polar derivative of the polynomial $p(z)$ of degree n with respect to $\alpha \in \mathbb{C}$. Then $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$. The polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} [\frac{D_\alpha p(z)}{\alpha}] = p'(z).$$

Shah [9] extended (1.2) to the polar derivative of $p(z)$ and proved that if all zeros of the polynomial $p(z)$ lie in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$, we have

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{2}(|\alpha| - 1) \max_{|z|=1} |p(z)|. \tag{1.5}$$

This result is best possible and equality holds if $p(z) = (z-1)^n$ with $\alpha \geq 1$.
Aziz and Rather [1] sharpened the inequality (1.5) by proving that if all the zeros of $p(z)$ lie in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$, one obtains

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{2}\{(|\alpha| - 1) \max_{|z|=1} |p(z)| + (|\alpha| + 1) \min_{|z|=1} |p(z)|\}. \tag{1.6}$$

This result is best possible and equality is attained for $p(z) = (z-1)^n$ with $\alpha \geq 1$.

Further, Aziz and Rather [1] generalized the inequality (1.3) to the polar derivative of a polynomial. In fact, they proved that if all zeros of $p(z)$ lie in $|z| \leq k$, $k \leq 1$, then for every α with $|\alpha| \geq k$, we get

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{1+k}(|\alpha| - k) \max_{|z|=1} |p(z)|. \tag{1.7}$$

This result is best possible and equality holds for $p(z) = (z-k)^n$ with $\alpha \geq k$.
As a refinement to inequality (1.7), Govil [6] proved that if $p(z)$ be a polynomial of degree n having all zeros in $|z| \leq k$, where $k \leq 1$, and $L = \frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n| + |a_{n-1}|}$, then for every α with $|\alpha| \geq k$, we have

(1.8)
$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{1+k}\{(|\alpha| - L) \max_{|z|=1} |p(z)| + \frac{(|\alpha|k + L)}{k^n} \min_{|z|=k} |p(z)|\}.$$

As an extension to the inequality (1.8), Dewan et al [4] proved that if $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, be a polynomial of degree n, having all zeros in $|z| \leq k$, $k \leq 1$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \geq k^\mu$, it yields

(1.9)
$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{1+k^\mu}\{(|\alpha| - A_\mu) \max_{|z|=1} |p(z)| + \frac{(|\alpha|k^\mu + A_\mu)}{k^n} \min_{|z|=k} |p(z)|\},$$

where $A_\mu = \frac{n(|a_n| - \frac{m}{k^n})k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n| - \frac{m}{k^n})|k^{\mu-1} + \mu|a_{n-\mu}|}$.
In fact, except the case $\mu = 1$, the inequality (1.9) is always sharper than the inequality (1.8).
The following result, proposes a refinement to inequalities (1.9) and (1.8). In a precise set up, we have:

Theorem 1.1. *Let $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$ be a polynomial of degree n having all zeros in $|z| \leq k$, where $k \leq 1$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \geq A_\mu$,*

(1.10)
$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{1+A_\mu}\{(|\alpha| - A_\mu) \max_{|z|=1} |p(z)| + \frac{(|\alpha|+1)A_\mu}{k^n} \min_{|z|=k} |p(z)|\},$$

where $A_\mu = \frac{n(|a_n|-\frac{m}{k^n})k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n|-\frac{m}{k^n})|k^{\mu-1}+\mu|a_{n-\mu}|}$.

Remark. Theorem 1.1 is in general a refinement to inequality (1.9). To see this, we have to show that

$$\frac{n}{1+k^\mu}\{(|\alpha|-A_\mu)\max_{|z|=1}|p(z)|+\frac{(|\alpha|k^\mu+A_\mu)}{k^n}\min_{|z|=k}|p(z)|\}<$$
$$\frac{n}{1+A_\mu}\{(|\alpha|-A_\mu)\max_{|z|=1}|p(z)|+\frac{(|\alpha|+1)A_\mu}{k^n}\min_{|z|=k}|p(z)|\}.$$

Equivalently,

$$\frac{\min_{|z|=k}|p(z)|}{k^n}(\frac{|\alpha|k^\mu+A_\mu}{1+k^\mu}-\frac{(|\alpha|+1)A_\mu}{1+A_\mu})<$$
$$\frac{(|\alpha|-A_\mu)(k^\mu-A_\mu)}{(1+k^\mu)(1+A_\mu)}\max_{|z|=1}|p(z).$$

Since by the assumption we have $|\alpha|\geq A_\mu$ and Lemma 2.8 proposes $k^\mu\geq A_\mu$, it implies

$$\frac{\min_{|z|=k}|p(z)|}{k^n}<\max_{|z|=1}|p(z), \tag{1.11}$$

but the inequality (1.11) is true by Lemma 2.7 and hence we have the result.

If we take $k=1$ in Theorem 1.1, then inequality (1.10) reduces to inequality (1.6).

Taking $\mu=1$ in Theorem 1.1, gives the following statement parallel to (1.8).

Corollary 1.2. *Let $p(z)$ be a polynomial of degree n, having all zeros in $|z|\leq k$, $k\leq 1$, then for every $\alpha\in\mathbb{C}$ with $|\alpha|\geq k$,*

$$\max_{|z|=1}|D_\alpha p(z)|\geq\frac{n}{1+A_1}\{(|\alpha|-A_1)\max_{|z|=1}|p(z)|+\frac{(|\alpha|+1)A_1}{k^n}\min_{|z|=k}|p(z)|\}, \tag{1.12}$$

where $A_1=\frac{n(|a_n|-\frac{m}{k^n})k^2+|a_{n-1}|}{n(|a_n|-\frac{m}{k^n})|+|a_{n-1}|}$.

If we divide both sides of the inequality in (1.10) by $|\alpha|$ and make $|\alpha|\to\infty$, we obtain the following improvement of inequality (1.4)

Corollary 1.3. *Let* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu},\ 1 \leq \mu \leq n$ *be a polynomial of degree* n *having all zeros in* $|z| \leq k,\ k \leq 1$*, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+A_\mu}\{\max_{|z|=1} |p(z)| + \frac{A_\mu}{k^n} \min_{|z|=k} |p(z)|\}, \tag{1.13}$$

where A_μ *is defined as in Theorem 1.1.*

2. **Lemmas**

For a proof of the theorem, the following lemmas are needed. The first lemma is due to Laguerre [7].

Lemma 2.1. *If all zeros of an* n^{th} *degree polynomial* $p(z)$ *lie in a circular region* C *and* w *is any zero of* $D_\alpha p(z)$*, then at most one of the points* w *and* α *may lie outside* C*.*

Lemma 2.2. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}; 1 \leq \mu \leq n$*, is a polynomial of degree* n *having all its zeros in* $\mid z \mid \leq k \leq 1$ *and* $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$ *then on* $\mid z \mid = 1$

$$\mid q'(z) \mid \leq s_\mu \ \mid p'(z) \mid, \tag{2.1}$$

and

$$\frac{\mu}{n}|\frac{a_{n-\mu}}{a_n}| \leq k^\mu, \tag{2.2}$$

where $s_\mu = \frac{n|a_n|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1}+\mu|a_{n-\mu}|}$.

The above lemma is due to Aziz and Rather [1].

Lemma 2.3. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}; 1 \leq \mu \leq n$*, is a polynomial of degree* n *having all its zeros in* $\mid z \mid \leq k \leq 1$ *then on* $\mid z \mid = 1$

$$|p'(z)| \geq \frac{n}{1+s_\mu}|p(z)| \tag{2.3}$$

Proof. Since $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$, we have

$$q'(z) = nz^{n-1}\overline{p(\frac{1}{\bar{z}})} - z^{n-2}\overline{p'(\frac{1}{\bar{z}})}.$$

Equivalently

$$zq'(z) = nz^n\overline{p(\frac{1}{\bar{z}})} - z^{n-1}\overline{p'(\frac{1}{\bar{z}})},$$

which implies for $|z| = 1$

$$| q'(z) |=| np(z) - zp'(z) | . \tag{2.4}$$

Now using the inequalities (2.1) and (2.4) for $| z |= 1$ we get

$$\begin{aligned}|np(z)| = |np(z) - zp'(z) + zp'(z)| &\leq |np(z) - zp'(z)| + |zp'(z)| \\ &= |q'(z)| + |p'(z)| \leq (s_\mu + 1)|p'(z)|.\end{aligned}$$

The proof is complete. □

Lemma 2.4. *If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu},\ 1 \leq \mu \leq n$, is a polynomial of degree n, having all zeros in the closed disk $| z |\leq k,\ k \leq 1$, then for every real or complex number α with $|\alpha| \geq s_\mu$ and $|z| = 1$, we have*

$$|D_\alpha p(z)| \geq \frac{n}{1 + s_\mu}(|\alpha| - s_\mu)|p(z)|, \tag{2.5}$$

where s_μ is defined as in Lemma 2.2.

Proof. Let $q(z) = z^n\overline{p(1/\overline{z})}$, then $|q'(z)| = |np(z) - zp'(z)|$ on $|z| = 1$. Thus on $|z| = 1$, we get

$$\begin{aligned}|D_\alpha p(z)| = |np(z) + (\alpha - z)p'(z)| = |\alpha p'(z) + np(z) - zp'(z)| \geq \\ |\alpha p'(z)| - |np(z) - zp'(z)|,\end{aligned}$$

which implies that

$$|D_\alpha p(z)| \geq |\alpha||p'(z)| - |q'(z)|. \tag{2.6}$$

By combining (2.1) and (2.6), we obtain

$$|D_\alpha p(z)| \geq (|\alpha| - s_\mu)|p'(z)|,$$

which along with Lemma 2.3, yields

$$|D_\alpha p(z)| \geq \frac{n}{1 + s_\mu}(|\alpha| - s_\mu)|p(z)|.$$

□

Lemma 2.5. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, $p(z) \neq 0$ in $|z| < k$, $(k > 0)$, then $m < |p(z)|$ for $|z| < k$, and in particular $m < |a_0|$, where $m = \min_{|z|=k} |p(z)|$.*

The above lemma is due to Gardner, Govil and Musukula [5].

In the lines of Lemma 2.5, by using Maximum Modulus Principle one can easily prove the following.

Lemma 2.6. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n, $p(z)\neq 0$ in $|z|<k$ $(k\geq 1)$, then*

$$\min_{|z|=k}|p(z)|<\max_{|z|=1}|p(z)|, \tag{2.7}$$

and in particular $\min\limits_{|z|=k}|p(z)|<|a_0|$.

Lemma 2.7. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having all zeros in $|z|\leq k$, $(k\leq 1)$, then*

$$\min_{|z|=k}|p(z)|<k^n\max_{|z|=1}|p(z)|, \tag{2.8}$$

and in particular $\min\limits_{|z|=k}|p(z)|<k^n|a_n|$.

Proof. Since the polynomial $p(z)$ has all zeros in $|z|\leq k$, the polynomial $q(z)=z^n\overline{p(\frac{1}{\bar z})}=\overline{a_n}+\overline{a_{n-1}}z+\cdots+\overline{a_1}z^{n-1}+\overline{a_0}z^n$ has no zero in $|z|<\frac{1}{k}$. Thus by applying Lemma 2.6 for the polynomial $q(z)$, we get

$$\min_{|z|=\frac{1}{k}}|q(z)|<\max_{|z|=1}|q(z)|. \tag{2.9}$$

Since $\min\limits_{|z|=\frac{1}{k}}|q(z)|=\dfrac{1}{k^n}\min\limits_{|z|=k}|p(z)|$ and $\max\limits_{|z|=1}|q(z)|=\max\limits_{|z|=1}|p(z)|$, then (2.9) implies that $\frac{1}{k^n}\min\limits_{|z|=k}|p(z)|<\max\limits_{|z|=1}|p(z)|$. □

The following lemma is due to Dewan el al [4].

Lemma 2.8. *If $p(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leq\mu\leq n$, is a polynomial of degree n having all zeros in $|z|\leq k$, $(k\leq 1)$, then*

$$A_\mu\leq k^\mu, \tag{2.10}$$

where A_μ is as defined in Theorem 1.1.

3. **Proof of the theorem**

Proof of the Theorem 1.1. By the assumptions $p(z)=a_nz^n+\sum\limits_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$ has all its zeros in $|\ z\ |\leq k\leq 1$. If $p(z)$ has a zero on $|\ z\ |=k$ then $m=\min\limits_{|z|=k}|\ p(z)\ |=0$, and in this case $s_\mu=A_\mu$. Thus the result follows from Lemma 2.4. Henceforth, we suppose that all the zeros of $p(z)$ lie in $|z| < k \leq 1$ so that $m > 0$. Now $m \leq |p(z)|$ for $|z| = k$, therefore, if λ is any real or complex number such that $|\lambda| < 1$, then

$$|\frac{\lambda\, mz^n}{k^n}| <| p(z) | \qquad for \qquad | z |= k.$$

Since all the zeros of $p(z)$ lie in $| z |< k$, it follows by Rouche 's Theorem, that all the zeros of

$$F(z) = p(z) - \frac{\lambda\, mz^n}{k^n}$$

also lie in $| z |< k$. Hence, by Lemma 2.1, the polynomial

$$D_\alpha F(z) = D_\alpha p(z) - \frac{\lambda mn\, \alpha\, z^{n-1}}{k^n} \tag{3.1}$$

has all its zeros in $| z |< k$, for $|\alpha| \geq k$. This implies

$$| D_\alpha p(z) | \geq | \frac{mn\, \alpha\, z^{n-1}}{k^n} | \qquad for \qquad | z |\geq k. \tag{3.2}$$

Because, if (3.2) is not true then there is some point $z = z_0$ with $| z_0 |\geq k$, such that

$$| D_\alpha p(z_0) | < | \frac{mn\, \alpha\, z_0^{n-1}}{k^n} | .$$

We choose $\lambda = \dfrac{k^n D_\alpha p(z_0)}{mn\alpha z_0^{n-1}}$, so that $| \lambda |< 1$ and with this choice of λ, from (3.1), we have $D_\alpha F(z_0) = 0$ for $| z_0 |\geq k$, which contradicts the fact that all zeros of $D_\alpha F(z)$ lie in $| z |< k$. Therefore (3.2) must hold. Now consider the polynomial

$$F(z) = p(z) - \frac{\lambda\, mz^n}{k^n} = [a_n - \frac{\lambda\, m}{k^n}]z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$$

that has all zeros in $| z |< k \leq 1$. By applying Lemma 2.4 to $F(z)$ we have for $|\alpha| \geq s'_\mu$

$$| D_\alpha F(z) |\geq n\frac{|\alpha| - s'_\mu}{1 + s'_\mu} | F(z) | \qquad for \quad | z |= 1.$$

Or,

(3.3)

$$| D_\alpha\{p(z) - \frac{\lambda\, mz^n}{k^n}\} |\geq n\frac{|\alpha| - s'_\mu}{1 + s'_\mu} | p(z) - \frac{\lambda\, mz^n}{k^n} | \qquad for \quad | z |= 1,$$

where

$$s'_{\mu} = \frac{n \mid a_n - \dfrac{\lambda\, m}{k^n} \mid k^{2\mu} + \mu \mid a_{n-\mu} \mid k^{\mu-1}}{n \mid a_n - \dfrac{\lambda\, m}{k^n} \mid k^{\mu-1} + \mu \mid a_{n-\mu} \mid}. \tag{3.4}$$

Using Lemma 2.7 we have $\mid a_n \mid > \dfrac{m}{k^n}$, therefore, the term $\mid a_n - \frac{\lambda\, m}{k^n} \mid$ can be substituted by $|a_n| - \frac{m}{k^n}$, because

$$\mid a_n - \frac{\lambda\, m}{k^n} \mid \geq \mid a_n \mid - \frac{\mid \lambda \mid m}{k^n} \geq \mid a_n \mid - \frac{m}{k^n}. \tag{3.5}$$

Now combiningthe (3.4), (3.5), we get

$$s'_{\mu} \leq A_{\mu}. \tag{3.6}$$

By combining (3.3) and (3.6), one can obtain

$$\begin{aligned} \mid D_{\alpha}p(z) - \lambda \frac{mn\alpha z^{n-1}}{k^n} \mid &\geq n\frac{|\alpha| - A_{\mu}}{1 + A_{\mu}} \mid p(z) - \lambda \frac{m\, z^n}{k^n} \mid \\ &\geq n\frac{|\alpha| - A_{\mu}}{1 + A_{\mu}} \{\mid p(z) \mid - \mid \lambda \mid \frac{m \ \mid z^n \mid}{k^n}\} \\ &= n\frac{|\alpha| - A_{\mu}}{1 + A_{\mu}} \{\mid p(z) \mid - \mid \lambda \mid \frac{m}{k^n}\}. \end{aligned} \tag{3.7}$$

Making use of the inequality (3.2), we can take a relevant choice of λ for which

(3.8)

$$\mid D_{\alpha}p(z) - \lambda \frac{mn\alpha\, z^{n-1}}{k^n} \mid = \mid D_{\alpha}p(z) \mid - \mid \lambda \mid \frac{mn \mid \alpha \mid}{k^n} \qquad for \quad \mid z \mid = 1.$$

Now combining the right hand side (3.7) and (3.8), we can rewrite (3.7) as

$$\mid D_{\alpha}p(z) \mid - \mid \lambda \mid \frac{mn \mid \alpha \mid}{k^n} \geq n\frac{|\alpha| - A_{\mu}}{1 + A_{\mu}} \{\mid p(z) \mid - \mid \lambda \mid \frac{m}{k^n} \mid\},$$

or equivalently,

$$\mid D_{\alpha}p(z) \mid \geq n\frac{|\alpha| - A_{\mu}}{1 + A_{\mu}} \mid p(z) \mid + \frac{\mid \lambda \mid mn}{k^n} \{\frac{(|\alpha| + 1)A_{\mu}}{1 + A_{\mu}}\}.$$

Letting $\mid \lambda \mid \to 1$, the result follows. □

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