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# STRONG CONVERGENCE OF A GENERAL IMPLICIT ALGORITHM FOR VARIATIONAL INEQUALITY PROBLEMS AND EQUILIBRIUM PROBLEMS AND A CONTINUOUS REPRESENTATION OF NONEXPANSIVE MAPPINGS 

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#### Abstract

We introduce a general implicit algorithm for finding a common element of the set of solutions of systems of equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings and a continuous representation of nonexpansive mappings. Then we prove the strong convergence of the proposed implicit scheme to the unique solution of the minimization problem on the solution of systems of equilibrium problems and the common fixed points of a sequence of nonexpansive mappings and a continuous representation of nonexpansive mappings. Keywords: Continuous representation, invariant mean, equilibrium problem, nonexpansive mapping, classical variational inequality. MSC(2010): Primary: 47H09; Secondary: 90C33; 47H10.


## 1. Introduction

Throughout this paper, $H$ will denote a real Hilbert space and $C$ will be a closed convex subset of $H$ unless otherwise stated.

Let $G: H \times H \rightarrow \mathbb{R}$ be an equilibrium function, that is,
$G(u, u)=0 \quad$ for every $u \in H$.

[^0]The Equilibrium Problem is defined as follows:

$$
\begin{equation*}
\text { find } \tilde{x} \in H \text { such that } \quad G(\tilde{x}, y) \geq 0 \quad \text { for all } y \in H, \tag{1.1}
\end{equation*}
$$

a solution of (1.1) is said to be an equilibrium point and the set of the equilibrium points is denoted by $\operatorname{SEP}(\mathrm{G})$.

Let $B: C \rightarrow H$ be a nonlinear map. Let $P_{C}$ be the projection of $H$ onto $C$. The classical variational inequality problem, denoted by $V I(C, B)$ is to find $u \in C$ such that

$$
\begin{equation*}
\langle B u, v-u\rangle \geq 0, \tag{1.2}
\end{equation*}
$$

for all $v \in C$. For a given $z \in H, u \in C$ satisfies the inequality

$$
\begin{equation*}
\langle u-z, v-u\rangle \geq 0, \quad(v \in C) \tag{1.3}
\end{equation*}
$$

if and only if $u=P_{C} z$. Therefore

$$
\begin{equation*}
u \in V I(C, B) \Longleftrightarrow u=P_{C}(u-\lambda B u), \tag{1.4}
\end{equation*}
$$

where $\lambda>0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems. It is known that the projection operator $P_{C}$ is nonexpansive. It is also known that $P_{C}$ satisfies

$$
\begin{equation*}
\left\langle x-y, P_{C} x-P_{C} y\right\rangle \geq\left\|P_{C} x-P_{C} y\right\|^{2}, \tag{1.5}
\end{equation*}
$$

for $x, y \in H$.
Recall the following definitions:
(1) a mapping $T$ from $C$ into itself is called nonexpansive if $\|T x-T y\| \leq\|x-y\|$, for all $x, y \in C$,
(2) a mapping $T$ from $C$ into itself is called Lipschitzian if there exists a nonnegative number $k$ such that
$\|T x-T y\| \leq k\|x-y\|$, for all $x, y \in C$,
(3) let $0 \leq \alpha<1$, a mapping $f$ from $C$ into itself is said to be an $\alpha$-contraction if $\|f(x)-f(y)\| \leq \alpha\|x-y\|$, for all $x, y \in H$,
(4) for a map $T$ from $H$ into itself, we denote by
$\operatorname{Fix}(T):=\{\mathrm{x} \in \mathrm{H}: \mathrm{x}=\mathrm{Tx}\}$, the fixed point set of $T$. Note that if $T$ is a nonexpansive mapping, $\operatorname{Fix}(\mathrm{T})$ is closed and convex (see [6]),
(5) a mapping $A$ from $H$ into itself is said to be strongly positive operator with constant $\bar{\gamma}$, if there exists $\bar{\gamma}>0$ such that

$$
\langle A x, x\rangle \geq \bar{\gamma}\|x\|^{2} \quad(x \in H)
$$

(6) a mapping $B$ from $C$ into $H$ is said to be monotone, if

$$
\langle B x-B y, x-y\rangle \geq 0 \text { for all } x, y \in C
$$

(7) a mapping $B$ from $C$ into $H$ is said to be $\eta$-cocoercive, if there exists a constant $\eta>0$ such that

$$
\langle B x-B y, x-y\rangle \geq \eta\|B x-B y\|^{2} \text { for all } x, y \in C
$$

Clearly, every $\eta$-cocoercive map $B$ is $\frac{1}{\eta}$-Lipschitz continuous (see [21] and [22]),
(8) a set-valued mapping $T: H \rightarrow 2^{H}$ is called monotone if for all $x, y \in H, f \in T x$ and $g \in T y$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T: H \rightarrow 2^{H}$ is maximal if the graph $G(T)$ of $T$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal if and only if for $(x, f) \in H \times H,\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in T x$. Let $B$ be a monotone map of $C$ into $H$ and let $N_{C} v$ be the normal cone to $C$ at $v \in C$, i.e., $N_{C} v=\{w \in H:\langle v-u, w\rangle \geq 0,(u \in C)\}$ and define

$$
T v= \begin{cases}B v+N_{C} v, & v \in C \\ \emptyset, & v \notin C\end{cases}
$$

Then $T$ is maximal monotone and $0 \in T v$ if and only if $v \in$ $V I(C, B)($ see [14] $)$,
(9) a semitopological semigroup is a semigroup $S$ with a Hausdorff topology such that for each $a \in S$ the mappings $s \rightarrow a . s$ and $s \rightarrow s . a$ from $S$ to $S$ are continuous,
(10) let $S$ be a semitopological semigroup. A family $\mathcal{S}=\left\{T_{s}: s \in\right.$ $S\}$ of mappings from $C$ into itself is said to be a continuous representation of $S$ as nonexpansive mapping on $C$ into itself if $\mathcal{S}$ satisfies the following conditions:
(1) $T_{s t} x=T_{s} T_{t} x$ for all $s, t \in S$ and $x \in C$;
(2) for every $x \in C$, the mapping $s \mapsto T_{s} x$ from $S$ into $C$ is continuous;
(3) for every $s \in S$ the mapping $T_{s}: C \rightarrow C$ is nonexpansive.

We denote by $\operatorname{Fix}(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, that is $\operatorname{Fix}(\mathcal{S})=\left\{x \in C: T_{s} x=x, \quad(s \in S)\right\}$,
(11) let $C$ be a nonempty convex subset of a Banach space, $\left\{T_{i}\right\}_{i \in \mathbb{N}}$ a sequence of nonexpansive mappings of $C$ into itself and $\left\{\lambda_{i}\right\}$ a real sequence such that $0 \leq \lambda_{i} \leq 1$ for every $i \in \mathbb{N}$. Following
[16], for any $n \geq 1$, we define a mapping $W_{n}$ of $C$ into itself as follows,

$$
\begin{aligned}
& U_{n, n+1}:=I \\
& U_{n, n}:=\lambda_{n} T_{n} U_{n, n+1}+\left(1-\lambda_{n}\right) I, \\
& \vdots \\
& U_{n, k}:=\lambda_{k} T_{k} U_{n, k+1}+\left(1-\lambda_{k}\right) I, \\
& \vdots \\
& U_{n, 2}:=\lambda_{2} T_{2} U_{n, 3}+\left(1-\lambda_{2}\right) I \\
& W_{n}:=U_{n, 1}:=\lambda_{1} T_{1} U_{n, 2}+\left(1-\lambda_{1}\right) I,
\end{aligned}
$$

(12) let $S$ be a semitopological semigroup. We denote by $B(S)$ the Banach space of all bounded real-valued functions defined on $S$ with supremum norm and let $C(S)$ be the subspace of $B(S)$ which consists of all bounded, continuous real-valued functions on $S$. For each $s \in S$ and $f \in B(S)$ we define $l_{s}$ and $r_{s}$ in $B(S)$ by

$$
\left(l_{s} f\right)(t)=f(s t), \quad\left(r_{s} f\right)(t)=f(t s), \quad(t \in S)
$$

Let $X$ be a subspace of $C(S)$ containing 1 and let $X^{*}$ be its topological dual. An element $\mu$ of $X^{*}$ is said to be a mean on $X$ if $\|\mu\|=\mu(1)=1$. We often write $\mu_{t}(f(t))$ instead of $\mu(f)$ for $\mu \in X^{*}$ and $f \in X$. Let $X$ be left invariant (resp. right invariant), i.e, $l_{s}(X) \subset X\left(\right.$ resp. $\left.r_{s}(X) \subset X\right)$ for each $s \in S$. A mean $\mu$ on $X$ is said to be left invariant (resp. right invariant) if $\mu\left(l_{s} f\right)=\mu(f)\left(\right.$ resp. $\left.\mu\left(r_{s} f\right)=\mu(f)\right)$ for each $s \in S$ and $f \in X$. Let $X$ be invariant i.e, $X$ be both left and right invariant, a mean $\mu$ on $X$ is said to be invariant if it is both left and right invariant,
(13) let $T: C \rightarrow H$ be a mapping. Then $T$ is said to be demiclosed at $v \in H$ if for any sequence $\left\{x_{n}\right\}$ in $C$, the following implication holds:
$x_{n} \rightharpoonup u \in C, \quad T x_{n} \rightarrow v$ imply $T u=v$, where $\rightarrow($ resp. $\rightarrow)$ denotes strong (resp. weak) convergence,
(14) a vector space $X$ is said to satisfy Opial's condition, if for each sequence $\left\{x_{n}\right\}$ in $X$ which converges weakly to point $x \in X$,
$\liminf _{n \rightarrow \infty}\left\|x_{n}-x\right\|<\liminf _{n \rightarrow \infty}\left\|x_{n}-y\right\| \quad(y \in X, y \neq x)$.

Note that every Hilbert space satisfies the Opial's condition (see [10] and [13]),
(15) let $K$ be a nonempty subset of a Banach space $X$ and $\left\{x_{n}\right\}$ be a sequence in $K$. The set of the asymptotic center of $\left\{x_{n}\right\}$ with respect to $K$, defined by

$$
A\left(\left\{x_{n}\right\}\right)=\left\{x \in K: \limsup _{n \rightarrow \infty}\left\|x_{n}-x\right\|=\inf _{y \in K} \limsup _{n \rightarrow \infty}\left\|x_{n}-y\right\|\right\}
$$

Let $f$ be an $\alpha$-contraction on $H$, and $A$ be a bounded linear operator on $H$. The following variational inequality problem with viscosity is of great interest [8, 9]:
find $x^{*}$ in $C$ such that

$$
\begin{equation*}
\left\langle(A-\gamma f) x^{*}, x-x^{*}\right\rangle \geq 0 \quad(x \in C) \tag{1.7}
\end{equation*}
$$

which is the optimality condition for the minimization problem

$$
\min _{x \in C}\left(\frac{1}{2}\langle A x, x\rangle+h(x)\right),
$$

where $\gamma$ satisfies $\|I-A\| \leq 1-\alpha \gamma$ and $h$ is a potential function for $\gamma f$ (that is $h^{\prime}(x)=\gamma f(x)$ ).

Plubtieng and Punpaeng in [12] proved a strong convergence theorem for an implicit sequence $\left\{x_{n}\right\}$ obtained from the viscosity approximation method for finding a common element in $\operatorname{SEP}(\mathrm{G}) \cap \operatorname{Fix}(\mathrm{T})$ which satisfies the variational inequality (1.7) (see also [19]):

Theorem 1.1. Let $G$ be a bifunction from $H \times H$ into $\mathbb{R}$ satisfying $\left(A_{1}\right) G(x, x)=0$ for all $x \in C$;
$\left(A_{2}\right) G$ is monotone, i.e., $G(x, y)+G(y, x) \leq 0$ for all $x, y \in C$;
$\left(A_{3}\right)$ For all $x, y, z \in C$,

$$
\limsup _{t \rightarrow 0} G(t z+(1-t) x, y) \leq G(x, y)
$$

$\left(A_{4}\right)$ For all $x \in C, y \mapsto G(x, y)$ is convex and lower semicontinuous.
For $x \in H$ and $r>0$, set $S_{r}: H \rightarrow C$ to be the resolvent of $G$, i.e., $S_{r}(x)$ is the unique $z \in C$ for which

$$
G(z, y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad(y \in C)
$$

Let $T$ be a nonexpansive mapping on $H$ such that $\operatorname{SEP}(\mathrm{G}) \cap \operatorname{Fix}(\mathrm{T}) \neq \emptyset$. Let $f$ be a contraction of $H$ into itself with $\alpha \in(0,1)$ and let $A$ be a
strongly positive bounded linear operator on $H$ with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{\gamma}{\alpha}$. Let $\left\{x_{n}\right\}$ be the sequence generated by

$$
\begin{cases}x_{n}=\alpha_{n} \gamma f\left(x_{n}\right)+\left(I-\alpha_{n} A\right) T u_{n}, & (n \in \mathbb{N}), \\ G\left(u_{n}, y\right)+\frac{1}{r_{n}}\left\langle y-u_{n}, u_{n}-x_{n}\right\rangle \geq 0 & (y \in H),\end{cases}
$$

where $u_{n}=S_{r_{n}} x_{n},\left\{r_{n}\right\} \subset(0, \infty)$ and $\alpha_{n} \subset[0,1]$ satisfying $\lim _{n \rightarrow \infty} \alpha_{n}=0$ and $\liminf _{n \rightarrow \infty} r_{n}>0$. Then $\left\{x_{n}\right\}$ and $\left\{u_{n}\right\}$ converge strongly to a point $z$ in $\operatorname{Fix}(\mathrm{T}) \cap \operatorname{SEP}(\mathrm{G})$ which solves the variational inequality

$$
\langle(A-\gamma f) z, z-x\rangle \leq 0, \quad x \in \operatorname{Fix}(\mathrm{~T}) \cap \operatorname{SEP}(\mathrm{G}) .
$$

In this paper, motivated by Lau, Miyake and Takahashi [7], Atsushiba and Takahashi [2], Shimizu and Takahashi [15] and Takahashi [20], we introduce the following general implicit algorithm for finding a common element of the set of solutions of a system of equilibrium problems $\operatorname{SEP}(\mathcal{G})$ for a family $\mathcal{G}=\left\{G_{k} ; k=1,2 \cdots, K\right\}$ of bifunctions and of the set of fixed points of a family $\left\{T_{i}\right\}_{i \in \mathbb{N}}$ of nonexpansive mappings from $C$ into itself and a continuous representation $\mathcal{S}=\left\{T_{t}: t \in S\right\}$ of a semitopological semigroup $S$ as nonexpansive mappings from $C$ into itself, with respect to $W$-mappings and a sequence $\left\{\mu_{n}\right\}$ of invariant means defined on an appropriate subspace of bounded, continuous real-valued functions of the semigroup:

$$
\begin{array}{r}
z_{n}=\epsilon_{n} \gamma f\left(z_{n}\right)+\left(I-\epsilon_{n} A\right) T_{\mu_{n}} W_{n} P_{C}\left(I-r_{n} B\right) S_{r_{1, n}}^{1} S_{r_{2, n}}^{2} \cdots S_{r_{K, n}}^{K} z_{n} \\
(n \in \mathbb{N})
\end{array}
$$

Our goal is to prove a result of strong convergence for the above implicit scheme to approach a unique solution $x^{*} \in \bigcap_{n \in \mathbb{N}} \operatorname{Fix}\left(\mathrm{~T}_{\mathrm{n}}\right) \cap \operatorname{Fix}(\mathcal{S}) \cap \operatorname{SEP}(\mathcal{G}) \cap \mathrm{VI}(\mathrm{C}, \mathrm{B})$ of the problem (1.7).

## 2. Preliminaries

The projection operator $P_{C}$ assigns to each $x \in H$, the unique point $P_{C} x \in C$ satisfying the condition

$$
\left\|x-P_{C} x\right\|=\min _{y \in C}\|x-y\| .
$$

The following Lemma characterizes the projection $P_{C}$ :
Lemma 2.1. ([18]). Let $x \in H$ and $y \in C$. Then $P_{C} x=y$ if and only if it satisfies the inequality

$$
\langle x-y, y-z\rangle \geq 0 \quad(z \in C)
$$

Lemma 2.2. ([8]). Let $A$ be a strongly positive linear bounded operator on $H$ with coefficient $\bar{\gamma}$ and $0<\rho \leq\|A\|^{-1}$ Then $\|I-\rho A\| \leq 1-\rho \bar{\gamma}$.

The following result generalizes Theorem 3.3.3 of [18].
Theorem 2.3. Let $S$ be a semitopological semigroup such that $C(S)$ has an invariant mean $\mu$. Let $\mathcal{S}=\left\{T_{s}: s \in S\right\}$ be a continuous representation of $S$ as nonexpansive mappings on $C$ into itself and suppose $\operatorname{Fix}(\mathcal{S}) \neq \emptyset$. If we write $T_{\mu} x$ instead of $\int T_{t} x d \mu(t)$, then the following hold:
(i) $T_{\mu} T_{s}=T_{s} T_{\mu}=T_{\mu}$ for all $s \in S$;
(ii) $T_{\mu}$ is a nonexpansive retraction of $C$ onto $\operatorname{Fix}(\mathcal{S})$, i.e.,

$$
\left\|T_{\mu} x-T_{\mu} y\right\| \leq\|x-y\| \quad \text { for all } \quad x, y \in C \quad \text { and } \quad T_{\mu}^{2}=T_{\mu}
$$

(iii) $T_{\mu} x \in \overline{\operatorname{co}}\left\{T_{s} x: s \in S\right\}$ for each $x \in C$;
(iv) $T_{\mu} x=x$ for each $x \in \operatorname{Fix}(\mathcal{S})$.

Proof. For proving (i)-(iii), see the proof of Theorem 3.3.3 of [18]. (iv) is clear, since for every $x \in \operatorname{Fix}(\mathcal{S}), T_{s} x=x$ for all $s \in S$. Thus $\overline{\text { co }}\left\{T_{s} x: s \in S\right\}=\{x\}$. Hence by (iii), $T_{\mu} x=x$ for each $x \in \operatorname{Fix}(\mathcal{S})$.

Theorem 2.4. ([5]). Let $G: H \times H \rightarrow \mathbb{R}$ satisfy,
$\left(A_{1}\right) G(x, x)=0$ for all $x \in C$;
$\left(A_{2}\right) G$ is monotone, i.e., $G(x, y)+G(y, x) \leq 0$ for all $x, y \in C$;
$\left(A_{3}\right)$ For all $x, y, z \in C$,

$$
\limsup _{t \rightarrow 0} G(t z+(1-t) x, y) \leq G(x, y) ;
$$

$\left(A_{4}\right)$ For all $x \in C, y \mapsto G(x, y)$ is convex and lower semicontinuous. For $x \in H$ and $r>0$, set $S_{r}: H \rightarrow C$ to be
$S_{r}(x):=\left\{z \in C: G(z, y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad(y \in C)\right\}$,
then $S_{r}$ is well defined and the followings are valid:
(i) $S_{r}$ is single-valued;
(ii) $S_{r}$ is firmly nonexpansive, i.e.,
$\left\|S_{r} x-S_{r} y\right\|^{2} \leq\left\langle S_{r} x-S_{r} y, x-y\right\rangle$,
for all $x, y \in H$;
(iii) Fix $S_{r}=\operatorname{SEP}(G)$;
(iv) $\operatorname{SEP}(G)$ is closed and convex.

Theorem 2.5. ([4]). Let $\left\{r_{n}\right\} \subset(0, \infty)$ be a sequence converging to $r>0$. For a bifunction $G: H \times H \rightarrow \mathbb{R}$, satisfying conditions $\left(A_{1}\right)$ $\left(A_{4}\right)$, define $S_{r}$ and $S_{r_{n}}$ for $n \in \mathbb{N}$ as in Theorem 2.4, then for every $x \in H$, we have

$$
\lim _{n}\left\|S_{r_{n}}-S_{r}\right\|=0
$$

Lemma 2.6. ([1]). Suppose that $T: C \rightarrow H$ is nonexpansive. Then, the mapping $I-T$ is demiclosed at zero.

Lemma 2.7. ([1]). Let $X$ be a uniformly convex Banach space satisfying the Opial's condition and let $K$ be a nonempty closed convex subset of $X$. If a sequence $\left\{z_{n}\right\} \subset K$ converges weakly to a point $z_{0}$, then $\left\{z_{0}\right\}$ is the asymptotic center of $\left\{z_{n}\right\}$ with respect to $K$.

Remark 2.8. Every Hilbert space is a uniformly convex Banach space, and therefore is a strictly convex Banach space (see pages 95, 98 of [18]).

The following results hold for the mappings $W_{n}$.
Theorem 2.9. ([16]). Let $C$ be a nonempty closed convex subset of a strictly convex Banach space. Let $\left\{T_{i}\right\}_{i \in \mathbb{N}}$ be a sequence of nonexpansive mappings of $C$ into itself such that $\bigcap_{i=1}^{\infty} \operatorname{Fix}\left(\mathrm{T}_{\mathrm{i}}\right) \neq \emptyset$, and let $\left\{\lambda_{i}\right\}$ be a real sequence such that $0 \leq \lambda_{i} \leq b<1$ for every $i \in \mathbb{N}$. Then
(1) $W_{n}$ is nonexpansive and $\operatorname{Fix}\left(\mathrm{W}_{\mathrm{n}}\right)=\bigcap_{\mathrm{i}=1}^{\mathrm{n}} \operatorname{Fix}\left(\mathrm{T}_{\mathrm{i}}\right)$ for each $\mathrm{n} \geq 1$,
(2) for each $x \in C$ and for each positive integer $j$, the limit $\lim _{n \rightarrow \infty} U_{n, j} x$ exists.
(3) The mapping $W: C \rightarrow C$ defined by

$$
W x:=\lim _{n \rightarrow \infty} W_{n} x=\lim _{n \rightarrow \infty} U_{n, 1} \quad(\mathrm{x} \in \mathrm{C})
$$

is a nonexpansive mapping satisfying $\operatorname{Fix}(\mathrm{W})=\bigcap_{\mathrm{i}=1}^{\infty} \operatorname{Fix}\left(\mathrm{T}_{\mathrm{i}}\right)$. Such $a$ mapping is called the $W$-mapping generated by $\left\{T_{i}\right\}_{i \in \mathbb{N}}$, and $\left\{\lambda_{i}\right\}_{i \in \mathbb{N}}$.

Theorem 2.10. ([11]). Let $\left\{T_{i}\right\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings of $C$ into itself such that $\bigcap_{i=1}^{\infty} \operatorname{Fix}\left(\mathrm{T}_{\mathrm{i}}\right) \neq \emptyset,\left\{\lambda_{\mathrm{i}}\right\}$ a real sequence such that $0<\lambda_{i} \leq b<1,(i \geq 1)$. If $D$ is any bounded subset of $C$, then

$$
\lim _{n \rightarrow \infty} \sup _{x \in D}\left\|W x-W_{n} x\right\|=0
$$

Throughout the rest of this paper, the open ball of radius $r$ centered at 0 is denoted by $B_{r}$. For $\epsilon>0$ and a mapping $T: D \rightarrow H$, we let $F_{\epsilon}(T ; D)$ be the set of $\epsilon$-approximate fixed points of $T$, i.e., $F_{\epsilon}(T ; D)=$ $\{x \in D:\|x-T x\| \leq \epsilon\}$.

## 3. Main results

In this section, we deal with the strong convergence approximation scheme for finding a common element of the set of solutions of a system of an equilibrium problem and the set of common fixed points of a sequence of nonexpansive mappings and a continuous representation. This result improves the main result of [4] and many others.

Theorem 3.1. Let $S$ be a semitopological semigroup. Suppose that $\mathcal{S}=\left\{T_{s}: s \in S\right\}$ be a continuous representation of $S$ as nonexpansive mappings of $C$ into itself. Let $X$ be an amenable subspace of $C(S)$ such that $1 \in X$, and the function $t \mapsto\left\langle T_{t} x, y\right\rangle$ is an element of $X$ for each $x \in C$ and $y \in H$. Let $\left\{\mu_{n}\right\}$ be a sequence of invariant means on $X$. Let $\left\{T_{i}\right\}_{i \in \mathbb{N}}$ be a sequence of nonexpansive mappings from $C$ into itself such that $T_{i}(\operatorname{Fix}(\mathcal{S})) \subseteq \operatorname{Fix}(\mathcal{S})$ for every $i \in \mathbb{N}$, and $\mathcal{G}=\left\{G_{k}: k=1,2, \cdots K\right\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$. Suppose that $A$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma}$ such that $\|A\| \leq 1$ and let $B$ be an $\eta$-cocoercive mapping from $C$ into $H$, and $f$ is an $\alpha$-contraction on $H$. Moreover, let $\left\{r_{k, n}\right\}_{k=1}^{K},\left\{r_{n}\right\},\left\{\epsilon_{n}\right\}$ and $\left\{\lambda_{n}\right\}$ be real sequences such that $r_{k, n}>0, r_{n}>0,0<\epsilon_{n}<1$ and $0<\lambda_{n} \leq c<1$ for some $c$, and let $\gamma$ be a real number such that $0<\gamma<\frac{\bar{\gamma}}{\alpha}$. Assume that,
(i) for every $k \in\{1,2, \cdots, K\}$, the function $G_{k}$ satisfies $\left(A_{1}\right)-\left(\mathrm{A}_{4}\right)$ of Theorem 2.4,
(ii) $\lim _{n} \epsilon_{n}=0$ and,
(iii) for every $k \in\{1,2, \cdots, K\}, \lim _{n} r_{k, n}$ exists and is a positive real number,
(iv) $\left\{r_{n}\right\} \subset[a, b]$ for some positive real numbers $a, b$ such that $b^{2}<2 \eta a<\eta^{2}+b^{2}$,
(v) $\mathfrak{F}:=\bigcap_{n \in \mathbb{N}} \operatorname{Fix}\left(\mathrm{~T}_{\mathrm{n}}\right) \cap \operatorname{Fix}(\mathcal{S}) \cap \operatorname{SEP}(\mathcal{G}) \cap \operatorname{VI}(\mathrm{C}, \mathrm{B}) \neq \emptyset$.

For every $n \in \mathbb{N}$, let $W_{n}$ be the mapping generated by $\left\{T_{i}\right\}$ and $\left\{\lambda_{n}\right\}$ as in (1.6), for every $k \in\{1,2, \cdots, K\}$ and $n \in \mathbb{N}$. Let $S_{r_{k, n}}^{k}$ be the resolvent generated by $G_{k}$ and $r_{k, n}$ as in Theorem 2.4. Let $\left\{z_{n}\right\}$ be the sequence generated by

$$
z_{n}=\epsilon_{n} \gamma f\left(z_{n}\right)+\left(I-\epsilon_{n} A\right) T_{\mu_{n}} W_{n} P_{C}\left(I-r_{n} B\right) S_{r_{1, n}}^{1} S_{r_{2, n}}^{2} \cdots S_{r_{K, n}}^{K} z_{n}
$$

$$
\begin{equation*}
(n \in \mathbb{N}) \tag{3.1}
\end{equation*}
$$

then there exists a unique element $u^{*} \in \mathfrak{F}$ such that $\left\{z_{n}\right\}$ strongly converges to $u^{*}$ which is:
(1) the unique solution of the variational inequality:

$$
\begin{equation*}
\left\langle(A-\gamma f) u^{*}, x-u^{*}\right\rangle \geq 0 \quad(x \in \mathfrak{F}) \tag{3.2}
\end{equation*}
$$

or equivalently,

$$
u^{*}=P_{\mathfrak{F}}(I-(A-\gamma f)) u^{*},
$$

(2) the unique solution of the minimization problem

$$
\min _{x \in \mathfrak{F}} \frac{1}{2}\langle A x, x\rangle+h(x),
$$

where $h$ is a potential function for $\gamma f$.
Proof. Since $\epsilon_{n} \rightarrow 0$, we may assume that $\epsilon_{n} \leq\|A\|^{-1}$. We show that $\left\langle\left(I-\epsilon_{n} A\right) x, x\right\rangle \geq 0$, for all $x \in H$. We may assume that $\|x\|=1$, so we have

$$
\left\langle\left(I-\epsilon_{n} A\right) x, x\right\rangle=1-\epsilon_{n}\langle A x, x\rangle \geq 1-\epsilon_{n}\|A\| \geq 0
$$

By Lemma 2.2, we have

$$
\left\|I-\epsilon_{n} A\right\| \leq 1-\epsilon_{n} \bar{\gamma}
$$

We show that $I-r_{n} B$ is nonexpansive. Indeed, since $B$ is $\eta$-cocoercive, by condition (iv), we have

$$
\begin{aligned}
\|\left(I-r_{n} B\right) x- & \left(I-r_{n} B\right) y \|^{2} \\
= & \left\|(x-y)-r_{n}(B x-B y)\right\|^{2} \\
= & \|x-y\|^{2}-2 r_{n}\langle x-y, B x-B y\rangle \\
& +r_{n}^{2}\|B x-B y\|^{2} \\
\leq & \|x-y\|^{2}-2 r_{n} \eta\|B x-B y\|^{2}+r_{n}^{2}\|B x-B y\|^{2} \\
\leq & \|x-y\|^{2}+\left(r_{n}^{2}-2 \eta r_{n}\right)\|B x-B y\|^{2} \\
\leq & \|x-y\|^{2}+\left(b^{2}-2 \eta a\right)\|B x-B y\|^{2} \\
\leq & \|x-y\|^{2},
\end{aligned}
$$

for each $x, y \in C$, which implies that the mapping $I-r_{n} B$ is nonexpansive.

We put $S_{n}^{k}:=S_{r_{1, n}}^{1} S_{r_{2, n}}^{2} \cdots S_{r_{k, n}}^{k}$ for every $k \in\{1,2, \cdots, K\}$ and $\rho_{n}=$ $P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}$. Let $p \in \mathfrak{F}$. Since $P_{C}\left(I-r_{n} B\right) p=p$, we have

$$
\begin{aligned}
\left\|\rho_{n}-p\right\| & =\left\|P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}-P_{C}\left(I-r_{n} B\right) p\right\| \\
& \leq\left\|S_{n}^{K} z_{n}-p\right\| \leq\left\|z_{n}-p\right\| .
\end{aligned}
$$

Putting $\mu_{1}=\mu$, by [18,Lemma 3.4.3], we have $T_{\mu_{n}}=T_{\mu}$ for all $n \in \mathbb{N}$. Therefore, we have

$$
z_{n}=\epsilon_{n} \gamma f\left(z_{n}\right)+\left(I-\epsilon_{n} A\right) T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n} \quad(n \in \mathbb{N}) .
$$

We divide the rest of the proof into eleven steps.
Step 1. The existence of $z_{n}$ which satisfies (3.1).
Proof. This follows immediately from the fact that for every $n \in \mathbb{N}$, the mapping $N_{n}$ given by

$$
N_{n} x:=\epsilon_{n} \gamma f(x)+\left(I-\epsilon_{n} A\right) T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} x \quad(x \in H),
$$

is a contraction. To see this, put $\beta_{n}=1+\epsilon_{n} \gamma \alpha-\epsilon_{n} \bar{\gamma}$, then $0 \leq \beta_{n}<1(n \in \mathbb{N})$. Using Lemma 2.2, we have

$$
\begin{aligned}
\left\|N_{n} x-N_{n} y\right\| \leq & \epsilon_{n} \gamma\|f(x)-f(y)\| \\
& +\left(1-\epsilon_{n} \bar{\gamma}\right) \| T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} x \\
& -T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} y \| \\
\leq & \epsilon_{n} \gamma \alpha\|x-y\|+\left(1-\epsilon_{n} \bar{\gamma}\right)\|x-y\| \\
= & \left(1+\epsilon_{n} \gamma \alpha-\epsilon_{n} \bar{\gamma}\right)\|x-y\| \\
= & \beta_{n}\|x-y\| .
\end{aligned}
$$

Therefore, by Banach Contraction Principle ([18],p.4), there exists a unique point $z_{n}$ such that $N_{n} z_{n}=z_{n}$.

Step 2. $\left\{z_{n}\right\}$ is bounded.
Proof. Let $p \in \mathfrak{F}$. Since $P_{C}\left(I-r_{n} B\right) p=p$, we have

$$
\begin{aligned}
\left\|z_{n}-p\right\|^{2}= & \left\langle\epsilon_{n} \gamma f\left(z_{n}\right)\right. \\
& \left.+\left(I-\epsilon_{n} A\right) T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}-p, z_{n}-p\right\rangle \\
= & \epsilon_{n} \gamma\left\langle f\left(z_{n}\right)-f(p), z_{n}-p\right\rangle+\epsilon_{n}\left\langle\gamma f(p)-A p, z_{n}-p\right\rangle \\
& +\left\langle( I - \epsilon _ { n } A ) \left( T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}\right.\right. \\
& \left.\left.-T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} p\right), z_{n}-p\right\rangle \\
\leq & \epsilon_{n} \gamma \alpha\left\|z_{n}-p\right\|^{2}+\left(1-\epsilon_{n} \bar{\gamma}\right)\left\|z_{n}-p\right\|^{2} \\
& +\epsilon_{n}\left\langle\gamma f(p)-A p, z_{n}-p\right\rangle .
\end{aligned}
$$

Thus,

$$
\begin{equation*}
\left\|z_{n}-p\right\|^{2} \leq \frac{1}{\bar{\gamma}-\alpha \gamma}\left\langle\gamma f(p)-A p, z_{n}-p\right\rangle . \tag{3.3}
\end{equation*}
$$

Hence,

$$
\left\|z_{n}-p\right\| \leq \frac{1}{\bar{\gamma}-\alpha \gamma}\|\gamma f(p)-A p\| .
$$

That is, the sequence $\left\{z_{n}\right\}$ is bounded.
Step 3. For every fixed $k \in\{1,2, \cdots, K\}$, we have

$$
\begin{equation*}
\lim _{n}\left\|z_{n}-S_{r_{k, n}}^{k} z_{n}\right\|=0 \tag{3.4}
\end{equation*}
$$

Proof. Let $k \in\{1,2, \cdots, K\}$. Since by (ii) of Theorem 2.4, $S_{r_{k, n}}^{k}$ is firmly nonexpansive, we conclude that

$$
\begin{aligned}
\left\|S_{r_{k, n}}^{k} z_{n}-p\right\|^{2} & =\left\|S_{r_{k, n}}^{k} z_{n}-S_{r_{k, n}}^{k} p\right\|^{2} \\
& \leq\left\langle S_{r_{k, n}}^{k} z_{n}-S_{r_{k, n}}^{k} p, z_{n}-p\right\rangle \\
& =\frac{1}{2}\left(\left\|S_{r_{k, n}}^{k} z_{n}-p\right\|^{2}+\left\|z_{n}-p\right\|^{2}-\left\|z_{n}-S_{r_{k, n}}^{k} z_{n}\right\|^{2}\right) .
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
\left\|S_{r_{k, n}}^{k} z_{n}-p\right\|^{2} \leq\left\|z_{n}-p\right\|^{2}-\left\|z_{n}-S_{r_{k, n}}^{k} z_{n}\right\|^{2} . \tag{3.5}
\end{equation*}
$$

It follows that

$$
\begin{aligned}
\left\|z_{n}-p\right\|^{2}= & \left\|\epsilon_{n}\left(\gamma f\left(z_{n}\right)-A p\right)+\left(I-\epsilon_{n} A\right)\left(T_{\mu} W_{n} \rho_{n}-p\right)\right\|^{2} \\
\leq & \left(\epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|+\left(1-\epsilon_{n} \bar{\gamma}\right)\left\|\rho_{n}-p\right\|\right)^{2} \\
\leq & \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|^{2}+\left(1-\epsilon_{n} \bar{\gamma}\right)\left\|\rho_{n}-p\right\|^{2} \\
& +2 \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|\left\|\rho_{n}-p\right\| \\
\leq & \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|^{2}+\left(1-\epsilon_{n} \bar{\gamma}\right)\left\|S_{r_{K, n}}^{K} z_{n}-p\right\|^{2} \\
& +2 \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|\left\|\rho_{n}-p\right\| \\
\leq & \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|^{2}+\left\|z_{n}-p\right\|^{2} \\
& -\left(1-\epsilon_{n} \bar{\gamma}\right)\left\|z_{n}-S_{r_{K, n}}^{K} z_{n}\right\|^{2} \\
& +2 \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\| \rho_{n}-p \| .
\end{aligned}
$$

That is,

$$
\begin{aligned}
\left(1-\epsilon_{n} \bar{\gamma}\right)\left\|z_{n}-S_{r_{K, n}}^{K} z_{n}\right\|^{2} \leq & \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|^{2} \\
& +2 \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|\left\|\rho_{n}-p\right\| .
\end{aligned}
$$

From (ii) and that $\left\{f\left(z_{n}\right)\right\}$ and $\left\{\rho_{n}\right\}$ are bounded sequences, we conclude

$$
\lim _{n}\left\|z_{n}-S_{r_{K, n}}^{K} z_{n}\right\|=0
$$

Now by induction we assume that (3.4) holds for every $k>\bar{k}$, and we prove it for $\bar{k}$.
If we put $L_{n}:=2\left\langle\gamma f\left(z_{n}\right)-A T_{\mu} W_{n} \rho_{n}, z_{n}-p\right\rangle$, then by using the inequality

$$
\begin{equation*}
\|x+y\|^{2} \leq\|x\|^{2}+2\langle y, x+y\rangle \tag{3.6}
\end{equation*}
$$

we obtain

$$
\begin{align*}
\left\|z_{n}-p\right\|^{2} & =\left\|\epsilon_{n} \gamma f\left(z_{n}\right)+\left(I-\epsilon_{n} A\right) T_{\mu} W_{n} \rho_{n}-p\right\|^{2} \\
& =\left\|T_{\mu} W_{n} \rho_{n}-p+\epsilon_{n}\left(\gamma f\left(z_{n}\right)-A T_{\mu} W_{n} \rho_{n}\right)\right\|^{2} \\
& \leq\left\|T_{\mu} W_{n} \rho_{n}-p\right\|^{2}+\epsilon_{n} L_{n} \\
& \leq\left\|\rho_{n}-p\right\|^{2}+\epsilon_{n} L_{n} \\
& \leq\left\|S_{n}^{K} z_{n}-p\right\|^{2}+\epsilon_{n} L_{n} \\
& \leq\left\|S_{r_{1, n}}^{1} S_{r_{2, n}}^{2} \cdots S_{r_{K, n}}^{K} z_{n}-p\right\|^{2}+\epsilon_{n} L_{n} \\
& \leq\left\|S_{r_{\bar{k}, n}}^{\bar{k}} \cdots S_{r_{K, n}}^{K} z_{n}-p\right\|^{2}+\epsilon_{n} L_{n} . \tag{3.7}
\end{align*}
$$

Observe that

$$
\begin{aligned}
\left\|S_{r_{\bar{k}, n}}^{\bar{k}} \cdots S_{r_{K, n}}^{K} z_{n}-p\right\|= & \left\|S_{r_{\bar{k}, n}}^{\bar{k}} \cdots S_{r_{K, n}}^{K} z_{n}-S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}+S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}-p\right\| \\
\leq & \left\|S_{r_{\bar{k}+1, n}}^{\bar{k}+1} \cdots S_{r_{K, n}}^{K} z_{n}-z_{n}\right\|+\left\|S_{r_{\bar{k}, n}}^{k} z_{n}-p\right\| \\
\leq & \left\|S_{r_{\bar{k}+1, n}}^{\bar{k}+1} \cdots S_{r_{K, n}}^{K} z_{n}-S_{r_{\bar{k}+1, n}}^{\bar{k}+1} z_{n}\right\| \\
& +\left\|S_{r_{\bar{k}+1, n}+1}^{k} z_{n}-z_{n}\right\|+\left\|S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}-p\right\| \\
\leq & \left\|S_{r_{\bar{k}+2, n}}^{\bar{k}+2} \cdots S_{r_{K, n}}^{K} z_{n}-z_{n}\right\| \\
& +\left\|S_{r_{\bar{k}+1, n}}^{\bar{k}+1} z_{n}-z_{n}\right\|+\left\|S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}-p\right\| \\
& \vdots \\
\leq & \left\|S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}-p\right\|+\sum_{k=\bar{k}+1}^{K}\left\|S_{r_{k, n}}^{k} z_{n}-z_{n}\right\| .
\end{aligned}
$$

Inequality (3.7) gives,

$$
\begin{aligned}
\left\|z_{n}-p\right\|^{2} \leq & \left(\sum_{k=\bar{k}+1}^{K}\left\|S_{r_{k, n}}^{k} z_{n}-z_{n}\right\|+2\left\|S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}-p\right\|\right) \\
& \left(\sum_{k=\bar{k}+1}^{K}\left\|S_{r_{k, n}}^{k} z_{n}-z_{n}\right\|\right)+\left\|S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}-p\right\|^{2}+\epsilon_{n} L_{n} .
\end{aligned}
$$

From this inequality and (3.5), we obtain

$$
\begin{aligned}
\left\|z_{n}-S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}\right\|^{2} \leq & \left(\sum_{k=\bar{k}+1}^{K}\left\|S_{r_{k, n}}^{k} z_{n}-z_{n}\right\|+2\left\|S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}-p\right\|\right) \\
& \left(\sum_{k=\bar{k}+1}^{K}\left\|S_{r_{k, n}}^{k} z_{n}-z_{n}\right\|\right)+\epsilon_{n} L_{n} .
\end{aligned}
$$

Since by assumption,

$$
\lim _{n} \sum_{k=\bar{k}+1}^{K}\left\|S_{r_{k, n}}^{k} z_{n}-z_{n}\right\|=0
$$

then, from (ii) and that $\left\{L_{n}\right\}$ is a bounded sequence, we conclude

$$
\lim _{n}\left\|z_{n}-S_{r_{\bar{k}, n}}^{\bar{k}} z_{n}\right\|=0
$$

as required.

Step 4. $\lim _{n}\left\|S_{n}^{K} z_{n}-z_{n}\right\|=0$.

## Proof. Observe that

$$
\begin{aligned}
\left\|S_{n}^{K} z_{n}-z_{n}\right\| & =\left\|S_{r_{1, n}}^{1} \cdots S_{r_{K, n}}^{K} z_{n}-z_{n}\right\| \\
& \leq\left\|S_{r_{1, n}}^{1} \cdots S_{r_{K, n}}^{K} z_{n}-S_{r_{1, n}}^{1} z_{n}\right\|+\left\|S_{r_{1, n}}^{1} z_{n}-z_{n}\right\| \\
\leq & \left\|S_{r_{2, n}}^{2} \cdots S_{r_{K, n}}^{K} z_{n}-z_{n}\right\|+\left\|S_{r_{1, n}}^{1} z_{n}-z_{n}\right\| \\
& \vdots \\
\leq & \sum_{k=1}^{K}\left\|S_{r_{k, n}}^{k} z_{n}-z_{n}\right\|
\end{aligned}
$$

Therefore by using (3.4), we have $\lim _{n}\left\|S_{n}^{K} z_{n}-z_{n}\right\|=0$.

Step 5. $\lim _{n}\left\|B S_{n}^{K} z_{n}-B p\right\|=0$.

Proof. Observe that for $p \in \mathfrak{F}$, since $B$ is $\eta$-cocoercive, we have

$$
\begin{aligned}
\left\|\rho_{n}-p\right\|^{2}= & \left\|P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}-P_{C}\left(I-r_{n} B\right) p\right\|^{2} \\
\leq & \left\|\left(S_{n}^{K} z_{n}-p\right)-r_{n}\left(B S_{n}^{K} z_{n}-B p\right)\right\|^{2} \\
= & \left\|S_{n}^{K} z_{n}-p\right\|^{2}-2 r_{n}\left\langle S_{n}^{K} z_{n}-p, B S_{n}^{K} z_{n}-B p\right\rangle \\
& +r_{n}^{2}\left\|B S_{n}^{K} z_{n}-B p\right\|^{2} \\
\leq \leq & \left\|z_{n}-p\right\|^{2}-2 r_{n} \eta\left\|B S_{n}^{K} z_{n}-B p\right\|^{2}+r_{n}^{2}\left\|B S_{n}^{K} z_{n}-B p\right\|^{2} \\
\leq & \left\|z_{n}-p\right\|^{2}+\left(r_{n}^{2}-2 r_{n} \eta\right)\left\|B S_{n}^{K} z_{n}-B p\right\|^{2}
\end{aligned}
$$

Observe that

$$
\begin{align*}
\left\|z_{n}-p\right\|^{2}= & \left\|\epsilon_{n}\left(\gamma f\left(z_{n}\right)-A p\right)+\left(I-\epsilon_{n} A\right)\left(T_{\mu} W_{n} \rho_{n}-p\right)\right\|^{2} \\
\leq & \left(\epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|+\left\|I-\epsilon_{n} A\right\|\left\|T_{\mu} W_{n} \rho_{n}-p\right\|\right)^{2} \\
\leq & \left(\epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|+\left(1-\epsilon_{n} \bar{\gamma}\right)\left\|\rho_{n}-p\right\|\right)^{2} \\
\leq & \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|^{2}+\left\|\rho_{n}-p\right\|^{2} \\
& +2 \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|\left\|\rho_{n}-p\right\| \tag{3.9}
\end{align*}
$$

Substituting (3.8) into (3.9), we have

$$
\begin{aligned}
\left\|z_{n}-p\right\|^{2} \leq & \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|^{2}+\left\|z_{n}-p\right\|^{2} \\
& +\left(r_{n}^{2}-2 r_{n} \eta\right)\left\|B S_{n}^{K} z_{n}-B p\right\|^{2} \\
& +2 \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|\left\|\rho_{n}-p\right\|
\end{aligned}
$$

It follows from the condition (iv) that

$$
\begin{aligned}
\left(2 a \eta-b^{2}\right)\left\|B S_{n}^{K} z_{n}-B p\right\|^{2} \leq & \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|^{2} \\
& +2 \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|\left\|\rho_{n}-p\right\|
\end{aligned}
$$

¿From condition (ii), we have

$$
\lim _{n}\left\|B S_{n}^{K} z_{n}-B p\right\|=0
$$

Step 6. $\lim _{n}\left\|\rho_{n}-S_{n}^{K} z_{n}\right\|=0$.

Proof. Observe that, by using (1.5), we have

$$
\begin{aligned}
\left\|\rho_{n}-p\right\|^{2}= & \left\|P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}-P_{C}\left(I-r_{n} B\right) p\right\|^{2} \\
\leq & \left\langle\left(I-r_{n} B\right) S_{n}^{K} z_{n}-\left(I-r_{n} B\right) p, \rho_{n}-p\right\rangle \\
= & \frac{1}{2}\left\{\left\|\left(I-r_{n} B\right) S_{n}^{K} z_{n}-\left(I-r_{n} B\right) p\right\|^{2}+\left\|\rho_{n}-p\right\|^{2}\right. \\
& \left.-\left\|\left(I-r_{n} B\right) S_{n}^{K} z_{n}-\left(I-r_{n} B\right) p-\left(\rho_{n}-p\right)\right\|^{2}\right\} \\
\leq & \frac{1}{2}\left\{\left\|S_{n}^{K} z_{n}-p\right\|^{2}+\left\|\rho_{n}-p\right\|^{2}\right. \\
& \left.-\left\|\left(S_{n}^{K} z_{n}-\rho_{n}\right)-r_{n}\left(B S_{n}^{K} z_{n}-B p\right)\right\|^{2}\right\} \\
\leq & \frac{1}{2}\left\{\left\|z_{n}-p\right\|^{2}+\left\|\rho_{n}-p\right\|^{2}-\left\|S_{n}^{K} z_{n}-\rho_{n}\right\|^{2}\right. \\
& -r_{n}^{2}\left\|B S_{n}^{K} z_{n}-B p\right\|^{2} \\
& \left.+2 r_{n}\left\langle S_{n}^{K} z_{n}-\rho_{n}, B S_{n}^{K} z_{n}-B p\right\rangle\right\},
\end{aligned}
$$

which yields that

$$
\begin{align*}
\left\|\rho_{n}-p\right\|^{2} \leq & \left\|z_{n}-p\right\|^{2}-\left\|S_{n}^{K} z_{n}-\rho_{n}\right\|^{2} \\
& +2 r_{n}\left\|S_{n}^{K} z_{n}-\rho_{n}\right\|\left\|B S_{n}^{K} z_{n}-B p\right\| . \tag{3.10}
\end{align*}
$$

Substituting (3.10) into (3.9) yields that

$$
\begin{aligned}
\left\|z_{n}-p\right\|^{2} \leq & \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|^{2}+\left\|z_{n}-p\right\|^{2}-\left\|S_{n}^{K} z_{n}-\rho_{n}\right\|^{2} \\
& +2 r_{n}\left\|S_{n}^{K} z_{n}-\rho_{n}\right\|\left\|B S_{n}^{K} z_{n}-B p\right\| \\
& +2 \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|\left\|\rho_{n}-p\right\| .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left\|S_{n}^{K} z_{n}-\rho_{n}\right\|^{2} \leq & \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|^{2} \\
& +2 r_{n}\left\|S_{n}^{K} z_{n}-\rho_{n}\right\|\left\|B S_{n}^{K} z_{n}-B p\right\| \\
& +2 \epsilon_{n}\left\|\gamma f\left(z_{n}\right)-A p\right\|\left\|\rho_{n}-p\right\| .
\end{aligned}
$$

¿From condition (ii) and Step 5, we have

$$
\lim _{n}\left\|\rho_{n}-S_{n}^{K} z_{n}\right\|=0
$$

Step 7. $\lim _{n}\left\|z_{n}-T_{\mu} W_{n} z_{n}\right\|=0$.

Proof. To see this, put

$$
M_{n}:=2\left\langle\gamma f\left(z_{n}\right)-A T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}, z_{n}-T_{\mu} W_{n} z_{n}\right\rangle .
$$

It is obvious that $\left\{M_{n}\right\}_{n \in \mathbb{N}}$ is a bounded sequence. By using (3.6), we have

$$
\begin{aligned}
& \left\|z_{n}-T_{\mu} W_{n} z_{n}\right\|^{2} \\
& \quad=\left\|\epsilon_{n} \gamma f\left(z_{n}\right)+\left(I-\epsilon_{n} A\right) T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}-T_{\mu} W_{n} z_{n}\right\|^{2} \\
& \quad \leq\left\|T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}-T_{\mu} W_{n} z_{n}\right\|^{2}+\epsilon_{n} M_{n}, \\
& \quad \leq\left\|\rho_{n}-z_{n}\right\|^{2}+\epsilon_{n} M_{n} \leq\left(\left\|\rho_{n}-S_{n}^{K} z_{n}\right\|+\left\|S_{n}^{K} z_{n}-z_{n}\right\|\right)^{2}+\epsilon_{n} M_{n} .
\end{aligned}
$$

Therefore, by Step 4, Step 6, and the fact that $\left\{M_{n}\right\}_{n \in \mathbb{N}}$ is a bounded sequence, we can conclude that,

$$
\begin{aligned}
\lim _{n}\left\|z_{n}-T_{\mu} W_{n} z_{n}\right\|^{2} \leq & \left(\lim _{n}\left\|\rho_{n}-S_{n}^{K} z_{n}\right\|+\lim _{n}\left\|S_{n}^{K} z_{n}-z_{n}\right\|\right)^{2} \\
& +\lim _{n} \epsilon_{n} M_{n}=0 .
\end{aligned}
$$

Step 8. $\lim _{n \rightarrow \infty}\left\|z_{n}-T_{t} z_{n}\right\|=0$, for all $t \in S$.

Proof. Let $p \in \mathfrak{F}$ and put

$$
M_{0}=\frac{\|\gamma f(p)-A p\|}{\bar{\gamma}-\alpha \gamma} .
$$

Let $D=\left\{y \in H:\|y-p\| \leq M_{0}\right\}$. It is clear that $D$ is a bounded closed convex set, and $\left\{z_{n}: n \in \mathbb{N}\right\} \subseteq D$. It is also obvious that $D$ is invariant under $\left\{S_{r_{k, n}}^{k}: k=1,2, \ldots K, n \in \mathbb{N}\right\}, W_{n}$ for every $n \in \mathbb{N}, \mathcal{S}$, and $P_{C}\left(I-r_{n} B\right)$ for every $n \in \mathbb{N}$.
Since $S$ is a semitopological semigroup, by (i) of Theorem 2.3, we have

$$
\begin{equation*}
T_{t} T_{\mu} y=T_{\mu} y \quad(\mathrm{t} \in \mathrm{~S}, \mathrm{y} \in \mathrm{D}) \tag{3.11}
\end{equation*}
$$

Let $\epsilon>0$. By [3,Theorem 1.2], there exists $\delta>0$ such that

$$
\begin{equation*}
\overline{\operatorname{co}} F_{\delta}\left(T_{t} ; D\right)+B_{\delta} \subseteq F_{\epsilon}\left(T_{t} ; D\right) \quad(\mathrm{t} \in \mathrm{~S}) \tag{3.12}
\end{equation*}
$$

Take $L_{0}=(1+\gamma \alpha) M_{0}+\|\gamma f(p)-A p\|$. Now from (3.11) and condition (ii) there exists a natural number $N_{1}$ such that $T_{\mu} y \in F_{\delta}\left(T_{t} ; D\right)$ for all
$y \in D$ and $\epsilon_{n}<\frac{\delta}{2 L_{0}}$ for all $n \geq N_{1}$. We note that

$$
\begin{aligned}
\epsilon_{n} \| \gamma f\left(z_{n}\right)- & A T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n} \| \\
\leq & \epsilon_{n}\left(\left\|\gamma f\left(z_{n}\right)-\gamma f(p)\right\|+\|\gamma f(p)-A p\|\right. \\
& +\| A T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} p \\
& \left.-A T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n} \|\right) \\
\leq & \epsilon_{n}\left(\gamma \alpha\left\|z_{n}-p\right\|+\|\gamma f(p)-A p\|+\|A\|\left\|z_{n}-p\right\|\right) \\
\leq & \epsilon_{n}\left(\gamma \alpha\left\|z_{n}-p\right\|+\|\gamma f(p)-A p\|+\left\|z_{n}-p\right\|\right) \\
\leq & \epsilon_{n}\left((1+\gamma \alpha)\left\|z_{n}-p\right\|+\|\gamma f(p)-A p\|\right) \\
\leq & \epsilon_{n}\left((1+\gamma \alpha) M_{0}+\|\gamma f(p)-A p\|\right) \\
= & \epsilon_{n} L_{0} \leq \frac{\delta}{2},
\end{aligned}
$$

for all $n \geq N_{1}$. Observe that

$$
\begin{aligned}
z_{n} & =\epsilon_{n} \gamma f\left(z_{n}\right)+\left(I-\epsilon_{n} A\right) T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n} \\
& =T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n} \\
& +\epsilon_{n}\left(\gamma f\left(z_{n}\right)-A T_{\mu} W_{n} P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}\right) \\
& \in F_{\delta}\left(T_{t} ; D\right)+B_{\frac{\delta}{2}} \\
& \subseteq F_{\delta}\left(T_{t} ; D\right)+B_{\delta} \\
& \subseteq F_{\epsilon}\left(T_{t} ; D\right) .
\end{aligned}
$$

for all $n \geq N_{1}$. This shows that

$$
\left\|z_{n}-T_{t} z_{n}\right\| \leq \epsilon \quad\left(\mathrm{n} \geq \mathrm{N}_{1}\right)
$$

Since $\epsilon>0$ is arbitrary, we get $\lim _{n \rightarrow \infty}\left\|z_{n}-T_{t} z_{n}\right\|=0$.
Step 9. The weak limit set of $\left\{z_{n}\right\}$ which is denoted by $\omega_{\omega}\left\{z_{n}\right\}$ is a subset of $\mathfrak{F}$.

Proof. Let $x^{*} \in \omega_{\omega}\left\{z_{n}\right\}$ and let $\left\{z_{n_{j}}\right\}$ be a subsequence of $\left\{z_{n}\right\}$ such that $z_{n_{j}} \rightharpoonup x^{*}$. We need to show that $x^{*} \in \mathfrak{F}$. In terms of Lemma 2.6 and Step 8, we conclude that $x^{*} \in \operatorname{Fix}(\mathcal{S})$.

By Theorems 2.9, 2.10, the mapping $W: C \rightarrow C$, given by $W x:=$ $\lim _{n} W_{n} x$ satisfies

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\|W_{n} x^{*}-W x^{*}\right\|=0 \tag{3.13}
\end{equation*}
$$

Putting $\lim _{n} r_{k, n}=\hat{r}_{k}$ for every $k \in\{1,2, \cdots, K\}$, by Theorem 2.5, we have

$$
\begin{equation*}
S_{\tilde{r}_{k}}^{k} x=\lim _{n} S_{r_{k, n}}^{k} x \quad(x \in H) . \tag{3.14}
\end{equation*}
$$

Since $x^{*} \in \operatorname{Fix}(\mathcal{S})$, by our assumption, we have $T_{i} x^{*} \in \operatorname{Fix}(\mathcal{S})$ for all $i \in$ $\mathbb{N}$ and then $W_{n} x^{*} \in \operatorname{Fix}(\mathcal{S})$. Hence, by (iv) of Theorem 2.3, $T_{\mu} W_{n} x^{*}=$ $W_{n} x^{*}$.
Consider the set of the asymptotic center $A\left(z_{n_{j}}\right)$ of $\left\{z_{n_{j}}\right\}$ with respect to $H$. Since $z_{n_{j}} \rightharpoonup x^{*}$, Lemma 2.7 implies that $A\left(z_{n_{j}}\right)=\left\{x^{*}\right\}$. By the definition of $A\left(z_{n_{j}}\right)$, we have

$$
\limsup _{j \rightarrow \infty}\left\|z_{n_{j}}-z\right\| \leq \limsup _{j \rightarrow \infty}\left\|z_{n_{j}}-T_{t} z_{n_{j}}\right\| \quad(t \in S)
$$

for all $z \in A\left(z_{n_{j}}\right)$. Since $A\left(z_{n_{j}}\right)=\left\{x^{*}\right\}$, by Step 8, we have $z_{n_{j}} \rightarrow x^{*}$. Using (3.13) and Step 7, we have

$$
\begin{aligned}
\limsup _{j \rightarrow \infty}\left\|z_{n_{j}}-W x^{*}\right\| \leq & \limsup _{j \rightarrow \infty}\left\|z_{n_{j}}-T_{\mu} W_{n_{j}} z_{n_{j}}\right\| \\
& +\quad \limsup _{j \rightarrow \infty}\left\|T_{\mu} W_{n_{j}} z_{n_{j}}-T_{\mu} W_{n_{j}} x^{*}\right\| \\
& +\limsup _{j \rightarrow \infty}\left\|T_{\mu} W_{n_{j}} x^{*}-W x^{*}\right\| \\
\leq & \limsup _{j \rightarrow \infty}\left\|z_{n_{j}}-T_{\mu} W_{n_{j}} z_{n_{j}}\right\|+\underset{j \rightarrow \infty}{\limsup }\left\|z_{n_{j}}-x^{*}\right\| \\
& +\limsup _{j \rightarrow \infty}\left\|W_{n_{j}} x^{*}-W x^{*}\right\|=0 .
\end{aligned}
$$

This implies that $W\left(x^{*}\right)=x^{*}$.
Using Theorem 2.5 and (3.14) and Step 3, we have

$$
\begin{aligned}
\limsup _{j \rightarrow \infty}\left\|z_{n_{j}}-S_{\hat{r}_{k}}^{k} x^{*}\right\| \leq & \limsup _{j \rightarrow \infty}\left\|z_{n_{j}}-S_{r_{k}, n_{j}}^{k} z_{n_{j}}\right\| \\
& +\limsup _{j \rightarrow \infty}\left\|S_{r_{k}, n_{j}}^{k} z_{n_{j}}-S_{r_{k}, n_{j}}^{k} x^{*}\right\| \\
& +\limsup _{j \rightarrow \infty}\left\|S_{r_{k}, n_{j}}^{k} x^{*}-S_{\hat{r}_{k}}^{k} x^{*}\right\| \\
\leq & \limsup _{j \rightarrow \infty}\left\|z_{n_{j}}-x^{*}\right\|=0
\end{aligned}
$$

This implies that $S_{\hat{r}_{k}}^{k}\left(x^{*}\right)=x^{*}$ for every $k \in\{1,2, \cdots, K\}$.
Therefore, we have $x^{*} \in \operatorname{Fix}(\mathrm{~W}) \cap\left(\bigcap_{\mathrm{k}=1}^{\mathrm{K}} \operatorname{Fix}\left(\mathrm{S}_{\mathrm{r}_{\mathrm{k}}}^{\mathrm{k}}\right)\right)$. In terms of Theorems 2.9 and 2.4 , we have $x^{*} \in\left(\bigcap_{i=1}^{\infty} \operatorname{Fix}\left(\mathrm{T}_{\mathrm{i}}\right)\right) \cap \operatorname{SEP}(\mathcal{G})$. Since $x^{*} \in \operatorname{Fix}(\mathcal{S})$, we have $x^{*} \in\left(\bigcap_{i=1}^{\infty} \operatorname{Fix}\left(\mathrm{T}_{\mathrm{i}}\right)\right) \cap \operatorname{SEP}(\mathcal{G}) \cap \operatorname{Fix}(\mathcal{S})$.

Now, let us show that $x^{*} \in V I(C, B)$. Let $U: H \rightarrow 2^{H}$ be a set-valued mapping defined by

$$
U x= \begin{cases}B x+N_{C} x, & x \in C \\ \emptyset, & x \notin C .\end{cases}
$$

¿From condition (iv) and this fact that $B$ is $\eta$-cocoercive, we have

$$
\langle B x-B y, x-y\rangle \geq \eta\|B x-B y\|^{2} \geq 0,
$$

which yields that $B$ is monotone, thus $U$ is maximal monotone. Let $\left(x_{1}, x_{2}\right) \in G(U)$. Since $x_{2}-B x_{1} \in N_{C} x_{1}$ and $\rho_{n} \in C$, we have

$$
\left\langle x_{1}-\rho_{n}, x_{2}-B x_{1}\right\rangle \geq 0 .
$$

Moreover, since $\rho_{n}=P_{C}\left(I-r_{n} B\right) S_{n}^{K} z_{n}$, from (1.3) we have

$$
\left\langle x_{1}-\rho_{n}, \rho_{n}-\left(I-r_{n} B\right) S_{n}^{K} z_{n}\right\rangle \geq 0
$$

and hence

$$
\left\langle x_{1}-\rho_{n}, \frac{\rho_{n}-S_{n}^{K} z_{n}}{r_{n}}+B S_{n}^{K} z_{n}\right\rangle \geq 0
$$

Therefore,

$$
\begin{aligned}
\left\langle x_{1}-\rho_{n_{j}}, x_{2}\right\rangle \geq \geq & \left\langle x_{1}-\rho_{n_{j}}, B x_{1}\right\rangle \\
\geq & \left\langle x_{1}-\rho_{n_{j}}, B x_{1}\right\rangle \\
& -\left\langle x_{1}-\rho_{n_{j}}, \frac{\rho_{n_{j}}-S_{n_{j}}^{K} z_{n_{j}}}{r_{n_{j}}}+B S_{n_{j}}^{K} z_{n_{j}}\right\rangle \\
= & \left\langle x_{1}-\rho_{n_{j}}, B x_{1}-\frac{\rho_{n_{j}}-S_{n_{j}}^{K} z_{n_{j}}}{r_{n_{j}}}-B S_{n_{j}}^{K} z_{n_{j}}\right\rangle \\
= & \left\langle x_{1}-\rho_{n_{j}}, B x_{1}-B \rho_{n_{j}}\right\rangle \\
& +\left\langle x_{1}-\rho_{n_{j}}, B \rho_{n_{j}}-B S_{n_{j}}^{K} z_{n_{j}}\right\rangle \\
& -\left\langle x_{1}-\rho_{n_{j}}, \frac{\rho_{n_{j}}-S_{n_{j}}^{K} z_{n_{j}}}{r_{n_{j}}}\right\rangle \\
\geq & \left\langle x_{1}-\rho_{n_{j}}, B \rho_{n_{j}}-B S_{n_{j}}^{K} z_{n_{j}}\right\rangle \\
& -\left\langle x_{1}-\rho_{n_{j}}, \frac{\rho_{n_{j}}-S_{n_{j}}^{K} z_{n_{j}}}{r_{n_{j}}}\right\rangle
\end{aligned}
$$

therefore, by Step 6 and that $B$ is a $\frac{1}{\eta}$-Lipschitz mapping, we have $\left\langle x_{1}-x^{*}, x_{2}\right\rangle \geq 0$. Thus, by (8), we have $x^{*} \in U^{-1} 0$ and hence, by (8), $x^{*} \in V I(C, B)$. Therefore, $x^{*} \in \mathfrak{F}$.

Step 10. There exists a unique element $u^{*} \in \mathfrak{F}$ that satisfies in the following inequality

$$
\begin{equation*}
\Gamma:=\limsup _{n}\left\langle(\gamma f-A) u^{*}, z_{n}-u^{*}\right\rangle \leq 0 \tag{3.15}
\end{equation*}
$$

Proof. From Lemma 2.2 we have

$$
\begin{aligned}
& \left\|P_{\mathfrak{F}}(I-(A-\gamma f)) x-P_{\mathfrak{F}}(I-(A-\gamma f)) y\right\| \\
& \quad \leq\|(I-(A-\gamma f)) x-(I-(A-\gamma f)) y\| \\
& \quad=\|((I-A)(x-y)+(\gamma f(x)-\gamma f(y)) \| \\
& \quad \leq(1-\bar{\gamma})\|x-y\|+\gamma \alpha\|x-y\| \\
& \quad=(1-\bar{\gamma}+\gamma \alpha)\|x-y\|
\end{aligned}
$$

since $1-\bar{\gamma}+\gamma \alpha<1, P_{\mathfrak{F}}(I-(A-\gamma f))$ is a contraction. So, by Banach Contraction Principle, there exists a unique point $u^{*} \in \mathfrak{F}$ such that
$P_{\mathfrak{F}}(I-(A-\gamma f)) u^{*}=u^{*}$ or equivalently, $u^{*}$ is the unique solution of the variational inequality:

$$
\begin{equation*}
\left\langle(A-\gamma f) u^{*}, x-u^{*}\right\rangle \geq 0 \quad(x \in \mathfrak{F}), \tag{3.16}
\end{equation*}
$$

The existence of $\Gamma$ follows from the fact that $\left\{z_{n}\right\}$ is a bounded sequence. So we can select a subsequence $\left\{z_{n_{j}}^{\prime}\right\}$ of $\left\{z_{n}\right\}$ such that $\lim _{j}\left\langle(\gamma f-A) u^{*}, z_{n_{j}}^{\prime}-u^{*}\right\rangle=\Gamma$. There is a subsequence of $\left\{z_{n_{j}}^{\prime}\right\}$ which we denote it again by $\left\{z_{n_{j}}^{\prime}\right\}$ that converges weakly to a point $y^{*}$. By Step $9, y^{*} \in \mathfrak{F}$ and from (3.16) we have

$$
\Gamma=\lim _{j}\left\langle(\gamma f-A) u^{*}, z_{n_{j}}^{\prime}-u^{*}\right\rangle=\left\langle(\gamma f-A) u^{*}, y^{*}-u^{*}\right\rangle \leq 0 .
$$

Step 11. $\left\{z_{n}\right\}$ converges strongly to $u^{*}$ and $u^{*}=y^{*}$.

Proof. Indeed, from (3.3), (3.15), we conclude

$$
\limsup _{n}\left\|z_{n}-u^{*}\right\|^{2} \leq \frac{1}{\bar{\gamma}-\alpha \gamma} \limsup _{n}\left\langle(\gamma f-A) u^{*}, z_{n}-u^{*}\right\rangle \leq 0 .
$$

That is $z_{n} \rightarrow u^{*}$. Therefore, $z_{n} \rightharpoonup u^{*}$. Hence $z_{n_{j}}^{\prime} \rightharpoonup u^{*}$. Now as in the proof of Step 10, $z_{n_{j}}^{\prime} \rightharpoonup y^{*}$, so we conclude that $u^{*}=y^{*}$.

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