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STRONG CONVERGENCE OF A GENERAL IMPLICIT ALGORITHM FOR VARIATIONAL INEQUALITY PROBLEMS AND EQUILIBRIUM PROBLEMS AND A CONTINUOUS REPRESENTATION OF NONEXPANSIVE MAPPINGS

M. LASHKARIZADEH BAMI AND E. SOORI*

(Communicated by Behzad Djafari-Rouhani)

Abstract. We introduce a general implicit algorithm for finding a common element of the set of solutions of systems of equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings and a continuous representation of nonexpansive mappings. Then we prove the strong convergence of the proposed implicit scheme to the unique solution of the minimization problem on the solution of systems of equilibrium problems and the common fixed points of a sequence of nonexpansive mappings and a continuous representation of nonexpansive mappings.

Keywords: Continuous representation, invariant mean, equilibrium problem, nonexpansive mapping, classical variational inequality.
MSC(2010): Primary: 47H09; Secondary: 90C33; 47H10.

1. **Introduction**

Throughout this paper, H will denote a real Hilbert space and C will be a closed convex subset of H unless otherwise stated.

Let $G : H \times H \to \mathbb{R}$ be an equilibrium function, that is,

$$G(u,u) = 0 \quad \text{for every } u \in H.$$

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The Equilibrium Problem is defined as follows:

$$\text{find } \tilde{x} \in H \text{ such that} \quad G(\tilde{x}, y) \geq 0 \quad \text{for all } y \in H, \tag{1.1}$$

a solution of (1.1) is said to be an equilibrium point and the set of the equilibrium points is denoted by SEP(G).

Let $B : C \to H$ be a nonlinear map. Let P_C be the projection of H onto C. The classical variational inequality problem, denoted by $VI(C, B)$ is to find $u \in C$ such that

$$\langle Bu, v - u \rangle \geq 0, \tag{1.2}$$

for all $v \in C$. For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad (v \in C), \tag{1.3}$$

if and only if $u = P_C z$. Therefore

$$u \in VI(C, B) \Longleftrightarrow u = P_C(u - \lambda Bu), \tag{1.4}$$

where $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems. It is known that the projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \tag{1.5}$$

for $x, y \in H$.

Recall the following definitions:

1. a mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$,
2. a mapping T from C into itself is called Lipschitzian if there exists a nonnegative number k such that $\|Tx - Ty\| \leq k\|x - y\|$, for all $x, y \in C$,
3. let $0 \leq \alpha < 1$, a mapping f from C into itself is said to be an α-contraction if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, for all $x, y \in H$,
4. for a map T from H into itself, we denote by $\mathrm{Fix(T)} := \{\mathrm{x \in H : x = Tx}\}$, the fixed point set of T. Note that if T is a nonexpansive mapping, Fix(T) is closed and convex (see [6]),
5. a mapping A from H into itself is said to be strongly positive operator with constant $\overline{\gamma}$, if there exists $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2 \qquad (x \in H),$$

(6) a mapping B from C into H is said to be monotone, if

$$\langle Bx - By, x - y\rangle \geq 0 \text{ for all } x, y \in C,$$

(7) a mapping B from C into H is said to be η-cocoercive, if there exists a constant $\eta > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \eta\|Bx - By\|^2 \text{ for all } x, y \in C.$$

Clearly, every η-cocoercive map B is $\frac{1}{\eta}$-Lipschitz continuous (see [21] and [22]),

(8) a set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y\ ,\ f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y\ ,\ f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let N_Cv be the normal cone to C at $v \in C$, i.e., $N_Cv = \{w \in H : \langle v - u, w\rangle \geq 0,\ (u \in C)\}$ and define

$$Tv = \begin{cases} Bv + N_Cv, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$ (see [14]),

(9) a semitopological semigroup is a semigroup S with a Hausdorff topology such that for each $a \in S$ the mappings $s \to a.s$ and $s \to s.a$ from S to S are continuous,

(10) let S be a semitopological semigroup. A family $\mathcal{S} = \{T_s : s \in S\}$ of mappings from C into itself is said to be a continuous representation of S as nonexpansive mapping on C into itself if $\mathcal{S}$ satisfies the following conditions:
(1) $T_{st}x = T_sT_tx$ for all $s, t \in S$ and $x \in C$;
(2) for every $x \in C$, the mapping $s \mapsto T_sx$ from S into C is continuous;
(3) for every $s \in S$ the mapping $T_s : C \to C$ is nonexpansive.
We denote by Fix($\mathcal{S}$) the set of common fixed points of $\mathcal{S}$, that is Fix($\mathcal{S}$)=$\{x \in C : T_sx = x,\ \ (s \in S)\}$,

(11) let C be a nonempty convex subset of a Banach space, $\{T_i\}_{i\in\mathbb{N}}$ a sequence of nonexpansive mappings of C into itself and $\{\lambda_i\}$ a real sequence such that $0 \leq \lambda_i \leq 1$ for every $i \in \mathbb{N}$. Following

[16], for any $n \geq 1$, we define a mapping W_n of C into itself as follows,

$$\begin{aligned}
&U_{n,n+1} := I,\\
&U_{n,n} := \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&\quad\vdots\\
&U_{n,k} := \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \qquad (1.6)\\
&\quad\vdots\\
&U_{n,2} := \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n := U_{n,1} := \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I,
\end{aligned}$$

(12) let S be a semitopological semigroup. We denote by $B(S)$ the Banach space of all bounded real-valued functions defined on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ which consists of all bounded, continuous real-valued functions on S. For each $s \in S$ and $f \in B(S)$ we define l_s and r_s in $B(S)$ by

$$(l_s f)(t) = f(st), \qquad (r_s f)(t) = f(ts), \qquad (t \in S).$$

Let X be a subspace of $C(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e, $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. Let X be invariant i.e, X be both left and right invariant, a mean μ on X is said to be invariant if it is both left and right invariant,

(13) let $T : C \to H$ be a mapping. Then T is said to be demiclosed at $v \in H$ if for any sequence $\{x_n\}$ in C, the following implication holds:
$x_n \rightharpoonup u \in C$, $Tx_n \to v$ imply $Tu = v$, where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence,

(14) a vector space X is said to satisfy Opial's condition, if for each sequence $\{x_n\}$ in X which converges weakly to point $x \in X$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad (y \in X,\ y \neq x).$$

Note that every Hilbert space satisfies the Opial's condition (see [10] and [13]),

(15) let K be a nonempty subset of a Banach space X and $\{x_n\}$ be a sequence in K. The set of the asymptotic center of $\{x_n\}$ with respect to K, defined by

$$A(\{x_n\}) = \left\{ x \in K : \limsup_{n\to\infty} \|x_n - x\| = \inf_{y\in K} \limsup_{n\to\infty} \|x_n - y\| \right\}.$$

Let f be an α-contraction on H, and A be a bounded linear operator on H. The following variational inequality problem with viscosity is of great interest [8, 9]:
find x^* in C such that

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \qquad (x \in C), \tag{1.7}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \left(\frac{1}{2}\langle Ax, x\rangle + h(x) \right),$$

where γ satisfies $\|I - A\| \leq 1 - \alpha\gamma$ and h is a potential function for γf (that is $h'(x) = \gamma f(x)$).

Plubtieng and Punpaeng in [12] proved a strong convergence theorem for an implicit sequence $\{x_n\}$ obtained from the viscosity approximation method for finding a common element in SEP(G)∩Fix(T) which satisfies the variational inequality (1.7) (see also [19]):

Theorem 1.1. *Let G be a bifunction from $H \times H$ into $\mathbb{R}$ satisfying*
(A_1) $G(x, x) = 0$ *for all* $x \in C$;
(A_2) *G is monotone, i.e.,* $G(x, y) + G(y, x) \leq 0$ *for all* $x, y \in C$;
(A_3) *For all* $x, y, z \in C$,

$$\limsup_{t\to 0} G(tz + (1-t)x, y) \leq G(x, y);$$

(A_4) *For all* $x \in C$, $y \mapsto G(x, y)$ *is convex and lower semicontinuous.*
For $x \in H$ *and* $r > 0$, *set* $S_r : H \to C$ *to be the resolvent of G, i.e.,* $S_r(x)$ *is the unique* $z \in C$ *for which*

$$G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \qquad (y \in C).$$

Let T be a nonexpansive mapping on H such that SEP(G)∩Fix(T) $\neq \emptyset$. *Let f be a contraction of H into itself with* $\alpha \in (0, 1)$ *and let A be a* *strongly positive bounded linear operator on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. *Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T u_n, & (n \in \mathbb{N}), \\ G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 & (y \in H), \end{cases}$$

where $u_n = S_{r_n} x_n$, $\{r_n\} \subset (0, \infty)$ *and* $\alpha_n \subset [0,1]$ *satisfying* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* z *in* $\mathrm{Fix(T)} \cap \mathrm{SEP(G)}$ *which solves the variational inequality*

$$\langle (A - \gamma f)z, z - x\rangle \leq 0, \quad x \in \mathrm{Fix(T)} \cap \mathrm{SEP(G)}.$$

In this paper, motivated by Lau, Miyake and Takahashi [7], Atsushiba and Takahashi [2], Shimizu and Takahashi [15] and Takahashi [20], we introduce the following general implicit algorithm for finding a common element of the set of solutions of a system of equilibrium problems $\mathrm{SEP}(\mathcal{G})$ for a family $\mathcal{G} = \{G_k; k = 1, 2 \cdots, K\}$ of bifunctions and of the set of fixed points of a family $\{T_i\}_{i\in\mathbb{N}}$ of nonexpansive mappings from C into itself and a continuous representation $\mathcal{S} = \{T_t : t \in S\}$ of a semitopological semigroup S as nonexpansive mappings from C into itself, with respect to W-mappings and a sequence $\{\mu_n\}$ of invariant means defined on an appropriate subspace of bounded, continuous real-valued functions of the semigroup:

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_{\mu_n} W_n P_C (I - r_n B) S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}} z_n \qquad (n \in \mathbb{N}),$$

Our goal is to prove a result of strong convergence for the above implicit scheme to approach a unique solution $x^* \in \bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \mathrm{Fix}(\mathcal{S}) \cap \mathrm{SEP}(\mathcal{G}) \cap \mathrm{VI(C,B)}$ of the problem (1.7).

2. **Preliminaries**

The projection operator P_C assigns to each $x \in H$, the unique point $P_C x \in C$ satisfying the condition

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

The following Lemma characterizes the projection P_C:

Lemma 2.1. ([18]). *Let* $x \in H$ *and* $y \in C$. *Then* $P_C x = y$ *if and only if it satisfies the inequality*

$$\langle x - y, y - z\rangle \geq 0 \qquad (z \in C).$$

Lemma 2.2. ([8]). *Let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$ Then $\|I - \rho A\| \leq 1 - \rho\,\overline{\gamma}$.*

The following result generalizes Theorem 3.3.3 of [18].

Theorem 2.3. *Let S be a semitopological semigroup such that $C(S)$ has an invariant mean μ. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings on C into itself and suppose* $\text{Fix}(\mathcal{S}) \neq \emptyset$*. If we write $T_\mu x$ instead of $\int T_t x\, d\mu(t)$, then the following hold:*
(i) *$T_\mu T_s = T_s T_\mu = T_\mu$ for all $s \in S$;*
(ii) *T_μ is a nonexpansive retraction of C onto* $\text{Fix}(\mathcal{S})$*, i.e.,*

$$\|T_\mu x - T_\mu y\| \leq \|x - y\| \quad for\ all\ \ x, y \in C \quad and \quad T_\mu^2 = T_\mu;$$

(iii) *$T_\mu x \in \overline{\text{co}}\,\{T_s x : s \in S\}$ for each $x \in C$;*
(iv) *$T_\mu x = x$ for each $x \in$* $\text{Fix}(\mathcal{S})$.

Proof. For proving (i)-(iii), see the proof of Theorem 3.3.3 of [18]. (iv) is clear, since for every $x \in \text{Fix}(\mathcal{S})$, $T_s x = x$ for all $s \in S$. Thus $\overline{\text{co}}\,\{T_s x : s \in S\} = \{x\}$. Hence by (iii), $T_\mu x = x$ for each $x \in \text{Fix}(\mathcal{S})$.

□

Theorem 2.4. ([5]). *Let $G : H \times H \to \mathbb{R}$ satisfy,*
(A_1) *$G(x, x) = 0$ for all $x \in C$;*
(A_2) *G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;*
(A_3) *For all $x, y, z \in C$,*

$$\limsup_{t \to 0} G(tz + (1 - t)x, y) \leq G(x, y);$$

(A_4) *For all $x \in C$, $y \mapsto G(x, y)$ is convex and lower semicontinuous. For $x \in H$ and $r > 0$, set $S_r : H \to C$ to be*

$$S_r(x) := \{z \in C : G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ (y \in C)\},$$

then S_r is well defined and the followings are valid:
(i) *S_r is single-valued;*
(ii) *S_r is firmly nonexpansive, i.e.,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle,$$

for all $x, y \in H$;
(iii) Fix S_r =SEP(G);
(iv) SEP(G) is closed and convex.

Theorem 2.5. ([4]). *Let* $\{r_n\} \subset (0,\infty)$ *be a sequence converging to* $r > 0$. *For a bifunction* $G : H \times H \to \mathbb{R}$, *satisfying conditions* (A_1)-(A_4), *define* S_r *and* S_{r_n} *for* $n \in \mathbb{N}$ *as in* Theorem 2.4, then for every $x \in H$, we have

$$\lim_n \|S_{r_n} - S_r\| = 0.$$

Lemma 2.6. ([1]). *Suppose that* $T : C \to H$ *is nonexpansive. Then, the mapping* $I - T$ *is demiclosed at zero.*

Lemma 2.7. ([1]). *Let* X *be a uniformly convex Banach space satisfying the Opial's condition and let* K *be a nonempty closed convex subset of* X. *If a sequence* $\{z_n\} \subset K$ *converges weakly to a point* z_0, *then* $\{z_0\}$ *is the asymptotic center of* $\{z_n\}$ *with respect to* K.

Remark 2.8. *Every Hilbert space is a uniformly convex Banach space, and therefore is a strictly convex Banach space (see pages 95, 98 of* [18]*).*

The following results hold for the mappings W_n.

Theorem 2.9. ([16]). *Let* C *be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i\in\mathbb{N}}$ *be a sequence of nonexpansive mappings of* C *into itself such that* $\bigcap_{\mathrm{i}=1}^{\infty} \mathrm{Fix(T_i)} \neq \emptyset$, *and let* $\{\lambda_i\}$ *be a real sequence such that* $0 \leq \lambda_i \leq b < 1$ *for every* $i \in \mathbb{N}$. *Then*
(1) W_n *is nonexpansive and* $\mathrm{Fix(W_n)} = \bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{Fix(T_i)}$ for each $\mathrm{n} \geq 1$,
(2) *for each* $x \in C$ *and for each positive integer* j, *the limit* $\lim\limits_{n\to\infty} U_{n,j}x$ *exists.*
(3) *The mapping* $W : C \to C$ *defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} \quad (\mathrm{x} \in \mathrm{C}),$$

is a nonexpansive mapping satisfying $\mathrm{Fix(W)} = \bigcap_{\mathrm{i}=1}^{\infty} \mathrm{Fix(T_i)}$. *Such a mapping is called the* W*-mapping generated by* $\{T_i\}_{i\in\mathbb{N}}$, *and* $\{\lambda_i\}_{i\in\mathbb{N}}$.

Theorem 2.10. ([11]). *Let* $\{T_i\}_{i=1}^{\infty}$ *be a sequence of nonexpansive mappings of* C *into itself such that* $\bigcap_{i=1}^{\infty} \mathrm{Fix(T_i)} \neq \emptyset, \{\lambda_\mathrm{i}\}$ *a real sequence such that* $0 < \lambda_i \leq b < 1$, $(i \geq 1)$. *If* D *is any bounded subset of* C, *then*

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

Throughout the rest of this paper, the open ball of radius r centered at 0 is denoted by B_r. For $\epsilon > 0$ and a mapping $T : D \to H$, we let $F_\epsilon(T; D)$ be the set of ϵ-approximate fixed points of T, i.e., $F_\epsilon(T; D) = \{x \in D : \|x - Tx\| \leq \epsilon\}$.

3. Main results

In this section, we deal with the strong convergence approximation scheme for finding a common element of the set of solutions of a system of an equilibrium problem and the set of common fixed points of a sequence of nonexpansive mappings and a continuous representation. This result improves the main result of [4] and many others.

Theorem 3.1. *Let S be a semitopological semigroup. Suppose that $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings of C into itself. Let X be an amenable subspace of $C(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, y\rangle$ is an element of X for each $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a sequence of invariant means on X. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix}(\mathcal{S})) \subseteq \mathrm{Fix}(\mathcal{S})$ for every $i \in \mathbb{N}$, and $\mathcal{G} = \{G_k : k = 1, 2, \cdots K\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ such that $\|A\| \le 1$ and let B be an η-cocoercive mapping from C into H, and f is an α-contraction on H. Moreover, let $\{r_{k,n}\}_{k=1}^{K}$, $\{r_n\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $r_n > 0$, $0 < \epsilon_n < 1$ and $0 < \lambda_n \le c < 1$ for some c, and let γ be a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that,*
(i) *for every $k \in \{1, 2, \cdots, K\}$, the function G_k satisfies $(A_1) - (\mathrm{A}_4)$ of* Theorem 2.4,
(ii) $\lim\limits_{n} \epsilon_n = 0$ and,
(iii) for every $k \in \{1, 2, \cdots, K\}$, $\lim\limits_{n} r_{k,n}$ exists and is a positive real number,
(iv) $\{r_n\} \subset [a, b]$ for some positive real numbers a, b such that
$b^2 < 2\eta a < \eta^2 + b^2$,
(v) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(\mathrm{T_n}) \cap \mathrm{Fix}(\mathcal{S}) \cap \mathrm{SEP}(\mathcal{G}) \cap \mathrm{VI(C,B)} \ne \emptyset$.

For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (1.6), for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$. Let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in Theorem 2.4. Let $\{z_n\}$ be the sequence generated by

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_{\mu_n} W_n P_C (I - r_n B) S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}} z_n \qquad (n \in \mathbb{N}), \tag{3.1}$$

then there exists a unique element $u^* \in \mathfrak{F}$ such that $\{z_n\}$ strongly converges to u^* which is:

(1) the unique solution of the variational inequality:

$$\langle (A-\gamma f)u^*, x-u^*\rangle \geq 0 \quad (x\in \mathfrak{F}), \tag{3.2}$$

or equivalently,

$$u^* = P_{\mathfrak{F}}(I-(A-\gamma f))u^*,$$

(2) the unique solution of the minimization problem

$$\min_{x\in\mathfrak{F}} \frac{1}{2}\langle Ax, x\rangle + h(x),$$

where h is a potential function for γf.

Proof. Since $\epsilon_n \to 0$, we may assume that $\epsilon_n \leq \|A\|^{-1}$. We show that $\langle (I-\epsilon_n A)x\ ,\ x\rangle \geq 0$, for all $x \in H$. We may assume that $\|x\| = 1$, so we have

$$\langle (I-\epsilon_n A)x\ ,\ x\rangle = 1-\epsilon_n\langle Ax\ ,\ x\rangle \geq 1-\epsilon_n\|A\| \geq 0.$$

By Lemma 2.2, we have

$$\|I-\epsilon_n A\| \leq 1-\epsilon_n\bar{\gamma}.$$

We show that $I-r_nB$ is nonexpansive. Indeed, since B is η-cocoercive, by condition (iv), we have

$$\begin{aligned}
&\|(I-r_nB)x-(I-r_nB)y\|^2\\
&\qquad = \|(x-y)-r_n(Bx-By)\|^2\\
&\qquad = \|x-y\|^2 - 2r_n\langle x-y, Bx-By\rangle\\
&\qquad\quad + r_n^2\|Bx-By\|^2\\
&\qquad \leq \|x-y\|^2 - 2r_n\eta\|Bx-By\|^2 + r_n^2\|Bx-By\|^2\\
&\qquad \leq \|x-y\|^2 + (r_n^2-2\eta r_n)\|Bx-By\|^2\\
&\qquad \leq \|x-y\|^2 + (b^2-2\eta a)\|Bx-By\|^2\\
&\qquad \leq \|x-y\|^2,
\end{aligned}$$

for each $x, y \in C$, which implies that the mapping $I-r_nB$ is nonexpansive.

We put $S_n^k := S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^k_{r_{k,n}}$ for every $k \in \{1,2,\cdots,K\}$ and $\rho_n = P_C(I-r_nB)S_n^K z_n$. Let $p \in \mathfrak{F}$. Since $P_C(I-r_nB)p = p$, we have

$$\begin{aligned}
\|\rho_n - p\| &= \|P_C(I-r_nB)S_n^K z_n - P_C(I-r_nB)p\|\\
&\leq \|S_n^K z_n - p\| \leq \|z_n - p\|.
\end{aligned}$$

Putting $\mu_1 = \mu$, by [18,Lemma 3.4.3], we have $T_{\mu_n} = T_\mu$ for all $n \in \mathbb{N}$. Therefore, we have

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_\mu W_n P_C (I - r_n B) S_n^K z_n \quad (n \in \mathbb{N}).$$

We divide the rest of the proof into eleven steps.

Step 1. The existence of z_n which satisfies (3.1).

Proof. This follows immediately from the fact that for every $n \in \mathbb{N}$, the mapping N_n given by

$$N_n x := \epsilon_n \gamma f(x) + (I - \epsilon_n A) T_\mu W_n P_C (I - r_n B) S_n^K x \qquad (x \in H),$$

is a contraction. To see this, put $\beta_n = 1 + \epsilon_n \gamma \alpha - \epsilon_n \bar{\gamma}$, then $0 \le \beta_n < 1$ $(n \in \mathbb{N})$. Using Lemma 2.2, we have

$$\begin{aligned} \|N_n x - N_n y\| \le & \epsilon_n \gamma \|f(x) - f(y)\| \\ & + (1 - \epsilon_n \bar{\gamma}) \|T_\mu W_n P_C (I - r_n B) S_n^K x \\ & - T_\mu W_n P_C (I - r_n B) S_n^K y\| \\ \le & \epsilon_n \gamma \alpha \|x - y\| + (1 - \epsilon_n \bar{\gamma}) \|x - y\| \\ = & (1 + \epsilon_n \gamma \alpha - \epsilon_n \bar{\gamma}) \|x - y\| \\ = & \beta_n \|x - y\|. \end{aligned}$$

Therefore, by Banach Contraction Principle ([18],p.4), there exists a unique point z_n such that $N_n z_n = z_n$. □

Step 2. $\{z_n\}$ is bounded.

Proof. Let $p \in \mathfrak{F}$. Since $P_C(I - r_n B)p = p$, we have

$$\begin{aligned} \|z_n - p\|^2 = & \Big\langle \epsilon_n \gamma f(z_n) \\ & + (I - \epsilon_n A) T_\mu W_n P_C (I - r_n B) S_n^K z_n - p \, , \, z_n - p \Big\rangle \\ = & \epsilon_n \gamma \Big\langle f(z_n) - f(p), z_n - p \Big\rangle + \epsilon_n \Big\langle \gamma f(p) - Ap \, , \, z_n - p \Big\rangle \\ & + \Big\langle (I - \epsilon_n A) \Big(T_\mu W_n P_C (I - r_n B) S_n^K z_n \\ & - T_\mu W_n P_C (I - r_n B) S_n^K p \Big) \, , \, z_n - p \Big\rangle \\ \le & \epsilon_n \gamma \alpha \|z_n - p\|^2 + (1 - \epsilon_n \overline{\gamma}) \|z_n - p\|^2 \\ & + \epsilon_n \Big\langle \gamma f(p) - Ap \, , \, z_n - p \Big\rangle . \end{aligned}$$

Thus,

$$\|z_n - p\|^2 \le \frac{1}{\overline{\gamma} - \alpha\gamma} \left\langle \gamma f(p) - Ap\, ,\ z_n - p \right\rangle . \tag{3.3}$$

Hence,

$$\|z_n - p\| \le \frac{1}{\overline{\gamma} - \alpha\gamma} \|\gamma f(p) - Ap\|.$$

That is, the sequence $\{z_n\}$ is bounded. □

Step 3. For every fixed $k \in \{1, 2, \cdots, K\}$, we have

$$\lim_n \|z_n - S^k_{r_{k,n}} z_n\| = 0. \tag{3.4}$$

Proof. Let $k \in \{1, 2, \cdots, K\}$. Since by (ii) of Theorem 2.4, $S^k_{r_{k,n}}$ is firmly nonexpansive, we conclude that

$$\begin{aligned}\|S^k_{r_{k,n}} z_n - p\|^2 &= \|S^k_{r_{k,n}} z_n - S^k_{r_{k,n}} p\|^2 \\ &\le \left\langle S^k_{r_{k,n}} z_n - S^k_{r_{k,n}} p\, ,\ z_n - p \right\rangle \\ &= \frac{1}{2}\left(\|S^k_{r_{k,n}} z_n - p\|^2 + \|z_n - p\|^2 - \|z_n - S^k_{r_{k,n}} z_n\|^2 \right).\end{aligned}$$

Therefore,

$$\|S^k_{r_{k,n}} z_n - p\|^2 \le \|z_n - p\|^2 - \|z_n - S^k_{r_{k,n}} z_n\|^2. \tag{3.5}$$

It follows that

$$\begin{aligned}\|z_n - p\|^2 =& \|\epsilon_n(\gamma f(z_n) - Ap) + (I - \epsilon_n A)(T_\mu W_n \rho_n - p)\|^2 \\ \le& \big(\epsilon_n \|\gamma f(z_n) - Ap\| + (1 - \epsilon_n \overline{\gamma})\|\rho_n - p\|\big)^2 \\ \le& \epsilon_n \|\gamma f(z_n) - Ap\|^2 + (1 - \epsilon_n \overline{\gamma})\|\rho_n - p\|^2 \\ &+ 2\epsilon_n \|\gamma f(z_n) - Ap\| \|\rho_n - p\| \\ \le& \epsilon_n \|\gamma f(z_n) - Ap\|^2 + (1 - \epsilon_n \overline{\gamma})\|S^K_{r_{K,n}} z_n - p\|^2 \\ &+ 2\epsilon_n \|\gamma f(z_n) - Ap\| \|\rho_n - p\| \\ \le& \epsilon_n \|\gamma f(z_n) - Ap\|^2 + \|z_n - p\|^2 \\ &- (1 - \epsilon_n \overline{\gamma})\|z_n - S^K_{r_{K,n}} z_n\|^2 \\ &+ 2\epsilon_n \|\gamma f(z_n) - Ap\| \|\rho_n - p\|.\end{aligned}$$

That is,

$$(1-\epsilon_n\overline{\gamma})\|z_n - S^K_{r_{K,n}} z_n\|^2 \leq \epsilon_n \|\gamma f(z_n) - Ap\|^2 + 2\epsilon_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|.$$

From (ii) and that $\{f(z_n)\}$ and $\{\rho_n\}$ are bounded sequences, we conclude

$$\lim_n \|z_n - S^K_{r_{K,n}} z_n\| = 0.$$

Now by induction we assume that (3.4) holds for every $k > \overline{k}$, and we prove it for $\overline{k}$.
If we put $L_n := 2\left\langle \gamma f(z_n) - AT_\mu W_n\rho_n\,,\, z_n - p\right\rangle$, then by using the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\left\langle y, x+y\right\rangle, \tag{3.6}$$

we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|\epsilon_n \gamma f(z_n) + (I - \epsilon_n A)T_\mu W_n \rho_n - p\|^2 \\
&= \|T_\mu W_n\rho_n - p + \epsilon_n\left(\gamma f(z_n) - AT_\mu W_n\rho_n\right)\|^2 \\
&\leq \|T_\mu W_n \rho_n - p\|^2 + \epsilon_n L_n \\
&\leq \|\rho_n - p\|^2 + \epsilon_n L_n \\
&\leq \|S^K_n z_n - p\|^2 + \epsilon_n L_n \\
&\leq \|S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n \\
&\leq \|S^{\overline{k}}_{r_{\overline{k},n}} \cdots S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n. \qquad (3.7)
\end{aligned}$$

Observe that

$$\begin{aligned}
\|S^{\overline{k}}_{r_{\overline{k},n}}\cdots S^{K}_{r_{K,n}}z_n-p\| =&\|S^{\overline{k}}_{r_{\overline{k},n}}\cdots S^{K}_{r_{K,n}}z_n-S^{\overline{k}}_{r_{\overline{k},n}}z_n+S^{\overline{k}}_{r_{\overline{k},n}}z_n-p\|\\
\le&\|S^{\overline{k}+1}_{r_{\overline{k}+1,n}}\cdots S^{K}_{r_{K,n}}z_n-z_n\|+\|S^{\overline{k}}_{r_{\overline{k},n}}z_n-p\|\\
\le&\|S^{\overline{k}+1}_{r_{\overline{k}+1,n}}\cdots S^{K}_{r_{K,n}}z_n-S^{\overline{k}+1}_{r_{\overline{k}+1,n}}z_n\|\\
&+\|S^{\overline{k}+1}_{r_{\overline{k}+1,n}}z_n-z_n\|+\|S^{\overline{k}}_{r_{\overline{k},n}}z_n-p\|\\
\le&\|S^{\overline{k}+2}_{r_{\overline{k}+2,n}}\cdots S^{K}_{r_{K,n}}z_n-z_n\|\\
&+\|S^{\overline{k}+1}_{r_{\overline{k}+1,n}}z_n-z_n\|+\|S^{\overline{k}}_{r_{\overline{k},n}}z_n-p\|\\
&\vdots\\
\le&\|S^{\overline{k}}_{r_{\overline{k},n}}z_n-p\|+\sum_{k=\overline{k}+1}^{K}\|S^{k}_{r_{k,n}}z_n-z_n\|.
\end{aligned}$$

Inequality (3.7) gives,

$$\begin{aligned}
\|z_n-p\|^2\le&\Big(\sum_{k=\overline{k}+1}^{K}\|S^{k}_{r_{k,n}}z_n-z_n\|+2\|S^{\overline{k}}_{r_{\overline{k},n}}z_n-p\|\Big)\\
&\Big(\sum_{k=\overline{k}+1}^{K}\|S^{k}_{r_{k,n}}z_n-z_n\|\Big)+\|S^{\overline{k}}_{r_{\overline{k},n}}z_n-p\|^2+\epsilon_nL_n.
\end{aligned}$$

From this inequality and (3.5), we obtain

$$\begin{aligned}
\|z_n-S^{\overline{k}}_{r_{\overline{k},n}}z_n\|^2\le&\Big(\sum_{k=\overline{k}+1}^{K}\|S^{k}_{r_{k,n}}z_n-z_n\|+2\|S^{\overline{k}}_{r_{\overline{k},n}}z_n-p\|\Big)\\
&\Big(\sum_{k=\overline{k}+1}^{K}\|S^{k}_{r_{k,n}}z_n-z_n\|\Big)+\epsilon_nL_n.
\end{aligned}$$

Since by assumption,

$$\lim_n\sum_{k=\overline{k}+1}^{K}\|S^{k}_{r_{k,n}}z_n-z_n\|=0,$$

then, from (ii) and that $\{L_n\}$ is a bounded sequence, we conclude

$$\lim_n \|z_n - S^{\bar{k}}_{r_{\bar{k},n}} z_n\| = 0,$$

as required. □

Step 4. $\lim\limits_n \|S^K_n z_n - z_n\| = 0.$

Proof. Observe that

$$\begin{aligned}\|S^K_n z_n - z_n\| =& \|S^1_{r_{1,n}} \cdots S^K_{r_{K,n}} z_n - z_n\| \\ \leq& \|S^1_{r_{1,n}} \cdots S^K_{r_{K,n}} z_n - S^1_{r_{1,n}} z_n\| + \|S^1_{r_{1,n}} z_n - z_n\| \\ \leq& \|S^2_{r_{2,n}} \cdots S^K_{r_{K,n}} z_n - z_n\| + \|S^1_{r_{1,n}} z_n - z_n\| \\ & \vdots \\ \leq& \sum_{k=1}^{K} \|S^k_{r_{k,n}} z_n - z_n\|.\end{aligned}$$

Therefore by using (3.4), we have $\lim\limits_n \|S^K_n z_n - z_n\| = 0.$ □

Step 5. $\lim\limits_n \|BS^K_n z_n - Bp\| = 0.$ □

Proof. Observe that for $p \in \mathfrak{F}$, since B is η-cocoercive, we have

$$\begin{aligned}\|\rho_n - p\|^2 =& \|P_C(I - r_n B)S^K_n z_n - P_C(I - r_n B)p\|^2 \\ \leq& \|(S^K_n z_n - p) - r_n(BS^K_n z_n - Bp)\|^2 \\ =& \|S^K_n z_n - p\|^2 - 2r_n\langle S^K_n z_n - p\,,\, BS^K_n z_n - Bp\rangle \\ & + r_n^2\|BS^K_n z_n - Bp\|^2 \\ \leq& \|z_n - p\|^2 - 2r_n\eta\|BS^K_n z_n - Bp\|^2 + r_n^2\|BS^K_n z_n - Bp\|^2 \\ \leq& \|z_n - p\|^2 + (r_n^2 - 2r_n\eta)\|BS^K_n z_n - Bp\|^2. \qquad (3.8)\end{aligned}$$

Observe that

$$\begin{aligned}\|z_n-p\|^2 =&\|\epsilon_n(\gamma f(z_n)-Ap)+(I-\epsilon_nA)(T_\mu W_n\rho_n-p)\|^2\\ \leq&\Big(\epsilon_n\|\gamma f(z_n)-Ap\|+\|I-\epsilon_nA\|\|T_\mu W_n\rho_n-p\|\Big)^2\\ \leq&\Big(\epsilon_n\|\gamma f(z_n)-Ap\|+(1-\epsilon_n\bar{\gamma})\|\rho_n-p\|\Big)^2\\ \leq&\epsilon_n\|\gamma f(z_n)-Ap\|^2+\|\rho_n-p\|^2\\ &+2\epsilon_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|.\end{aligned} \tag{3.9}$$

Substituting (3.8) into (3.9), we have

$$\begin{aligned}\|z_n-p\|^2 \leq&\epsilon_n\|\gamma f(z_n)-Ap\|^2+\|z_n-p\|^2\\ &+(r_n^2-2r_n\eta)\|BS_n^Kz_n-Bp\|^2\\ &+2\epsilon_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|.\end{aligned}$$

It follows from the condition (iv) that

$$\begin{aligned}(2a\eta-b^2)\|BS_n^Kz_n-Bp\|^2 \leq&\epsilon_n\|\gamma f(z_n)-Ap\|^2\\ &+2\epsilon_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|.\end{aligned}$$

¿From condition (ii), we have

$$\lim_n\|BS_n^Kz_n-Bp\|=0.$$

□

Step 6. $\lim_n\|\rho_n-S_n^Kz_n\|=0.$

Proof. Observe that, by using (1.5), we have

$$\begin{aligned}
\|\rho_n - p\|^2 =& \|P_C(I - r_nB)S_n^K z_n - P_C(I - r_nB)p\|^2 \\
\leq& \left\langle (I - r_nB)S_n^K z_n - (I - r_nB)p \;,\; \rho_n - p \right\rangle \\
=& \frac{1}{2}\{\|(I - r_nB)S_n^K z_n - (I - r_nB)p\|^2 + \|\rho_n - p\|^2 \\
& - \|(I - r_nB)S_n^K z_n - (I - r_nB)p - (\rho_n - p)\|^2\} \\
\leq& \frac{1}{2}\{\|S_n^K z_n - p\|^2 + \|\rho_n - p\|^2 \\
& - \|(S_n^K z_n - \rho_n) - r_n(BS_n^K z_n - Bp)\|^2\} \\
\leq& \frac{1}{2}\{\|z_n - p\|^2 + \|\rho_n - p\|^2 - \|S_n^K z_n - \rho_n\|^2 \\
& - r_n^2\|BS_n^K z_n - Bp\|^2 \\
& + 2r_n \left\langle S_n^K z_n - \rho_n, BS_n^K z_n - Bp \right\rangle\},
\end{aligned}$$

which yields that

$$\begin{aligned}
\|\rho_n - p\|^2 \leq& \|z_n - p\|^2 - \|S_n^K z_n - \rho_n\|^2 \\
& + 2r_n\|S_n^K z_n - \rho_n\|\|BS_n^K z_n - Bp\|.
\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9) yields that

$$\begin{aligned}
\|z_n - p\|^2 \leq& \epsilon_n\|\gamma f(z_n) - Ap\|^2 + \|z_n - p\|^2 - \|S_n^K z_n - \rho_n\|^2 \\
& + 2r_n\|S_n^K z_n - \rho_n\|\|BS_n^K z_n - Bp\| \\
& + 2\epsilon_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|S_n^K z_n - \rho_n\|^2 \leq& \epsilon_n\|\gamma f(z_n) - Ap\|^2 \\
& + 2r_n\|S_n^K z_n - \rho_n\|\|BS_n^K z_n - Bp\| \\
& + 2\epsilon_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|.
\end{aligned}$$

¿From condition (ii) and Step 5, we have

$$\lim_n \|\rho_n - S_n^K z_n\| = 0.$$

□

Step 7. $\lim_n \|z_n - T_\mu W_n z_n\| = 0.$

Proof. To see this, put

$$M_n := 2\left\langle \gamma f(z_n) - AT_\mu W_n P_C(I - r_n B)S_n^K z_n\,,\, z_n - T_\mu W_n z_n \right\rangle.$$

It is obvious that $\{M_n\}_{n\in\mathbb{N}}$ is a bounded sequence. By using (3.6), we have

$$\begin{aligned}
&\|z_n - T_\mu W_n z_n\|^2 \\
&\quad= \|\epsilon_n \gamma f(z_n) + (I - \epsilon_n A)T_\mu W_n P_C(I - r_n B)S_n^K z_n - T_\mu W_n z_n\|^2 \\
&\quad\le \|T_\mu W_n P_C(I - r_n B)S_n^K z_n - T_\mu W_n z_n\|^2 + \epsilon_n M_n, \\
&\quad\le \|\rho_n - z_n\|^2 + \epsilon_n M_n \le \left(\|\rho_n - S_n^K z_n\| + \|S_n^K z_n - z_n\|\right)^2 + \epsilon_n M_n.
\end{aligned}$$

Therefore, by Step 4, Step 6, and the fact that $\{M_n\}_{n\in\mathbb{N}}$ is a bounded sequence, we can conclude that,

$$\begin{aligned}
\lim_n \|z_n - T_\mu W_n z_n\|^2 \le \left(\lim_n \|\rho_n - S_n^K z_n\| + \lim_n \|S_n^K z_n - z_n\|\right)^2 \\
+ \lim_n \epsilon_n M_n = 0.
\end{aligned}$$

□

Step 8. $\lim_{n\to\infty} \|z_n - T_t z_n\| = 0$, for all $t \in S$.

Proof. Let $p \in \mathfrak{F}$ and put

$$M_0 = \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \alpha\gamma}.$$

Let $D = \{y \in H : \|y - p\| \le M_0\}$. It is clear that D is a bounded closed convex set, and $\{z_n : n \in \mathbb{N}\} \subseteq D$. It is also obvious that D is invariant under $\{S_{r_{k,n}}^k : k = 1, 2, \ldots K, n \in \mathbb{N}\}$, W_n for every $n \in \mathbb{N}$, $\mathcal{S}$, and $P_C(I - r_n B)$ for every $n \in \mathbb{N}$.
Since S is a semitopological semigroup, by (i) of Theorem 2.3, we have

$$T_t T_\mu y = T_\mu y \quad (\mathrm{t \in S, y \in D}). \tag{3.11}$$

Let $\epsilon > 0$. By [3,Theorem 1.2], there exists $\delta > 0$ such that

$$\overline{\mathrm{co}} F_\delta(T_t; D) + B_\delta \subseteq F_\epsilon(T_t; D) \quad (\mathrm{t \in S}). \tag{3.12}$$

Take $L_0 = (1 + \gamma\alpha)M_0 + \|\gamma f(p) - Ap\|$. Now from (3.11) and condition (ii) there exists a natural number N_1 such that $T_\mu y \in F_\delta(T_t; D)$ for all

$y \in D$ and $\epsilon_n < \frac{\delta}{2L_0}$ for all $n \geq N_1$. We note that

$$\begin{aligned}
\epsilon_n\|\gamma f(z_n) - &AT_\mu W_n P_C(I - r_n B)S_n^K z_n\| \\
\leq& \epsilon_n \Big(\|\gamma f(z_n) - \gamma f(p)\| + \|\gamma f(p) - Ap\| \\
&+ \|AT_\mu W_n P_C(I - r_n B)S_n^K p \\
&- AT_\mu W_n P_C(I - r_n B)S_n^K z_n\| \Big) \\
\leq& \epsilon_n \Big(\gamma\alpha\|z_n - p\| + \|\gamma f(p) - Ap\| + \|A\|\|z_n - p\| \Big) \\
\leq& \epsilon_n \left(\gamma\alpha\|z_n - p\| + \|\gamma f(p) - Ap\| + \|z_n - p\| \right) \\
\leq& \epsilon_n \left((1 + \gamma\alpha)\|z_n - p\| + \|\gamma f(p) - Ap\| \right) \\
\leq& \epsilon_n \left((1 + \gamma\alpha)M_0 + \|\gamma f(p) - Ap\| \right) \\
=& \epsilon_n L_0 \leq \frac{\delta}{2},
\end{aligned}$$

for all $n \geq N_1$. Observe that

$$\begin{aligned}
z_n &= \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)T_\mu W_n P_C(I - r_n B)S_n^K z_n \\
&= T_\mu W_n P_C(I - r_n B)S_n^K z_n \\
&+ \epsilon_n \Big(\gamma f(z_n) - AT_\mu W_n P_C(I - r_n B)S_n^K z_n \Big) \\
&\in F_\delta(T_t; D) + B_{\frac{\delta}{2}} \\
&\subseteq F_\delta(T_t; D) + B_\delta \\
&\subseteq F_\epsilon(T_t; D).
\end{aligned}$$

for all $n \geq N_1$. This shows that

$$\|z_n - T_t z_n\| \leq \epsilon \quad (\mathrm{n} \geq \mathrm{N}_1).$$

Since $\epsilon > 0$ is arbitrary, we get $\lim_{n\to\infty} \|z_n - T_t z_n\| = 0$. □

Step 9. The weak limit set of $\{z_n\}$ which is denoted by $\omega_\omega\{z_n\}$ is a subset of $\mathfrak{F}$.

Proof. Let $x^* \in \omega_\omega\{z_n\}$ and let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_j} \rightharpoonup x^*$. We need to show that $x^* \in \mathfrak{F}$. In terms of Lemma 2.6 and Step 8, we conclude that $x^* \in \mathrm{Fix}(\mathcal{S})$.

By Theorems 2.9, 2.10, the mapping $W : C \to C$, given by $Wx := \lim_n W_n x$ satisfies

$$\limsup_{n\to\infty} \|W_n x^* - Wx^*\| = 0. \tag{3.13}$$

Putting $\lim_n r_{k,n} = \hat{r}_k$ for every $k \in \{1, 2, \cdots, K\}$, by Theorem 2.5, we have

$$S^k_{\hat{r}_k} x = \lim_n S^k_{r_{k,n}} x \qquad (x \in H). \tag{3.14}$$

Since $x^* \in \mathrm{Fix}(\mathcal{S})$, by our assumption, we have $T_i x^* \in \mathrm{Fix}(\mathcal{S})$ for all $i \in \mathbb{N}$ and then $W_n x^* \in \mathrm{Fix}(\mathcal{S})$. Hence, by (iv) of Theorem 2.3, $T_\mu W_n x^* = W_n x^*$.
Consider the set of the asymptotic center $A(z_{n_j})$ of $\{z_{n_j}\}$ with respect to H. Since $z_{n_j} \rightharpoonup x^*$, Lemma 2.7 implies that $A(z_{n_j}) = \{x^*\}$. By the definition of $A(z_{n_j})$, we have

$$\limsup_{j\to\infty} \|z_{n_j} - z\| \leq \limsup_{j\to\infty} \|z_{n_j} - T_t z_{n_j}\| \qquad (t \in S).$$

for all $z \in A(z_{n_j})$. Since $A(z_{n_j}) = \{x^*\}$, by Step 8, we have $z_{n_j} \to x^*$. Using (3.13) and Step 7, we have

$$\begin{aligned}
\limsup_{j\to\infty} \|z_{n_j} - Wx^*\| \leq & \limsup_{j\to\infty} \|z_{n_j} - T_\mu W_{n_j} z_{n_j}\| \\
& + \limsup_{j\to\infty} \|T_\mu W_{n_j} z_{n_j} - T_\mu W_{n_j} x^*\| \\
& + \limsup_{j\to\infty} \|T_\mu W_{n_j} x^* - Wx^*\| \\
\leq & \limsup_{j\to\infty} \|z_{n_j} - T_\mu W_{n_j} z_{n_j}\| + \limsup_{j\to\infty} \|z_{n_j} - x^*\| \\
& + \limsup_{j\to\infty} \|W_{n_j} x^* - Wx^*\| = 0.
\end{aligned}$$

This implies that $W(x^*) = x^*$.
Using Theorem 2.5 and (3.14) and Step 3, we have

$$\begin{aligned}
\limsup_{j\to\infty} \|z_{n_j} - S^k_{\hat{r}_k} x^*\| \leq & \limsup_{j\to\infty} \|z_{n_j} - S^k_{r_k,n_j} z_{n_j}\| \\
& + \limsup_{j\to\infty} \|S^k_{r_k,n_j} z_{n_j} - S^k_{r_k,n_j} x^*\| \\
& + \limsup_{j\to\infty} \|S^k_{r_k,n_j} x^* - S^k_{\hat{r}_k} x^*\| \\
\leq & \limsup_{j\to\infty} \|z_{n_j} - x^*\| = 0.
\end{aligned}$$

This implies that $S^k_{\hat{r}_k}(x^*) = x^*$ for every $k \in \{1, 2, \cdots, K\}$.
Therefore, we have $x^* \in \mathrm{Fix(W)} \cap (\bigcap_{\mathrm{k=1}}^{\mathrm{K}} \mathrm{Fix}(\mathrm{S}^{\mathrm{k}}_{\hat{\mathrm{r}}_{\mathrm{k}}}))$. In terms of Theorems 2.9 and 2.4, we have $x^* \in (\bigcap_{i=1}^{\infty} \mathrm{Fix(T_i)}) \cap \mathrm{SEP}(\mathcal{G})$. Since $x^* \in \mathrm{Fix}(\mathcal{S})$, we have $x^* \in (\bigcap_{i=1}^{\infty} \mathrm{Fix(T_i)}) \cap \mathrm{SEP}(\mathcal{G}) \cap \mathrm{Fix}(\mathcal{S})$.

Now, let us show that $x^* \in VI(C, B)$. Let $U : H \to 2^H$ be a set-valued mapping defined by

$$Ux = \begin{cases} Bx + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

¿From condition (iv) and this fact that B is η-cocoercive, we have

$$\langle Bx - By \ , \ x - y \rangle \geq \eta \|Bx - By\|^2 \geq 0,$$

which yields that B is monotone, thus U is maximal monotone. Let $(x_1, x_2) \in G(U)$. Since $x_2 - Bx_1 \in N_C x_1$ and $\rho_n \in C$, we have

$$\langle x_1 - \rho_n \ , \ x_2 - Bx_1 \rangle \geq 0.$$

Moreover, since $\rho_n = P_C(I - r_n B) S_n^K z_n$, from (1.3) we have

$$\langle x_1 - \rho_n \ , \ \rho_n - (I - r_n B) S_n^K z_n \rangle \geq 0,$$

and hence

$$\langle x_1 - \rho_n \ , \ \frac{\rho_n - S_n^K z_n}{r_n} + B S_n^K z_n \rangle \geq 0.$$

Therefore,

$$\begin{aligned}\langle x_1-\rho_{n_j}\ ,\ x_2\rangle \geq&\langle x_1-\rho_{n_j}\ ,\ Bx_1\rangle\\ \geq&\langle x_1-\rho_{n_j}\ ,\ Bx_1\rangle\\ &-\left\langle x_1-\rho_{n_j}\ ,\ \frac{\rho_{n_j}-S^K_{n_j}z_{n_j}}{r_{n_j}}+BS^K_{n_j}z_{n_j}\right\rangle\\ =&\left\langle x_1-\rho_{n_j}\ ,\ Bx_1-\frac{\rho_{n_j}-S^K_{n_j}z_{n_j}}{r_{n_j}}-BS^K_{n_j}z_{n_j}\right\rangle\\ =&\langle x_1-\rho_{n_j}\ ,\ Bx_1-B\rho_{n_j}\rangle\\ &+\langle x_1-\rho_{n_j}\ ,\ B\rho_{n_j}-BS^K_{n_j}z_{n_j}\rangle\\ &-\left\langle x_1-\rho_{n_j}\ ,\ \frac{\rho_{n_j}-S^K_{n_j}z_{n_j}}{r_{n_j}}\right\rangle\\ \geq&\langle x_1-\rho_{n_j}\ ,\ B\rho_{n_j}-BS^K_{n_j}z_{n_j}\rangle\\ &-\left\langle x_1-\rho_{n_j}\ ,\ \frac{\rho_{n_j}-S^K_{n_j}z_{n_j}}{r_{n_j}}\right\rangle,\end{aligned}$$

therefore, by Step 6 and that B is a $\frac{1}{\eta}$-Lipschitz mapping, we have $\langle x_1-x^*\ ,\ x_2\rangle\geq 0$. Thus, by (8), we have $x^*\in U^{-1}0$ and hence, by (8), $x^*\in VI(C,B)$. Therefore, $x^*\in\mathfrak{F}$. □

Step 10. There exists a unique element $u^*\in\mathfrak{F}$ that satisfies in the following inequality

$$\Gamma:=\limsup_n\langle(\gamma f-A)u^*,z_n-u^*\rangle\leq 0. \tag{3.15}$$

Proof. From Lemma 2.2 we have

$$\begin{aligned}&\|P_{\mathfrak{F}}(I-(A-\gamma f))x-P_{\mathfrak{F}}(I-(A-\gamma f))y\|\\ &\leq\|(I-(A-\gamma f))x-(I-(A-\gamma f))y\|\\ &=\|((I-A)(x-y)+(\gamma f(x)-\gamma f(y))\|\\ &\leq(1-\overline{\gamma})\|x-y\|+\gamma\alpha\|x-y\|\\ &=(1-\overline{\gamma}+\gamma\alpha)\|x-y\|,\end{aligned}$$

since $1-\overline{\gamma}+\gamma\alpha<1$, $P_{\mathfrak{F}}(I-(A-\gamma f))$ is a contraction. So, by Banach Contraction Principle, there exists a unique point $u^*\in\mathfrak{F}$ such that

$P_{\mathfrak{F}}(I-(A-\gamma f))u^* = u^*$ or equivalently, u^* is the unique solution of the variational inequality:

$$\langle (A-\gamma f)u^*, x-u^*\rangle \geq 0 \quad (x\in\mathfrak{F}), \tag{3.16}$$

The existence of Γ follows from the fact that $\{z_n\}$ is a bounded sequence. So we can select a subsequence $\{z^{'}_{n_j}\}$ of $\{z_n\}$ such that

$\lim_j \left\langle (\gamma f - A)u^*, z^{'}_{n_j} - u^*\right\rangle = \Gamma$. There is a subsequence of $\{z^{'}_{n_j}\}$ which we denote it again by $\{z^{'}_{n_j}\}$ that converges weakly to a point y^*.
By Step 9, $y^*\in\mathfrak{F}$ and from (3.16) we have

$$\Gamma = \lim_j \left\langle (\gamma f - A)u^*, z^{'}_{n_j} - u^*\right\rangle = \langle (\gamma f - A)u^*, y^* - u^*\rangle \leq 0.$$

□

Step 11. $\{z_n\}$ converges strongly to u^* and $u^* = y^*$.

Proof. Indeed, from (3.3), (3.15), we conclude

$$\limsup_n \|z_n - u^*\|^2 \leq \frac{1}{\overline{\gamma}-\alpha\gamma}\limsup_n \langle (\gamma f - A)u^*, z_n - u^*\rangle \leq 0.$$

That is $z_n \to u^*$. Therefore, $z_n \rightharpoonup u^*$. Hence $z^{'}_{n_j} \rightharpoonup u^*$. Now as in the proof of Step 10, $z^{'}_{n_j} \rightharpoonup y^*$, so we conclude that $u^* = y^*$. □

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