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ON THE SUM OF PELL AND JACOBSTHAL NUMBERS BY MATRIX METHOD

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ABSTRACT. In this paper, we define two n-square upper Hessenberg matrices one of which corresponds to the adjacency matrix of a directed pseudo graph. We investigate relations between permanents and determinants of these upper Hessenberg matrices, and sum formulas of the well-known Pell and Jacobsthal sequences. Finally, we present two Maple 13 procedures in order to calculate permanents of these upper Hessenberg matrices.
Keywords: Permanent, Pell sequence, Hessenberg matrix.
MSC(2010): Primary: 15A15; Secondary: 11B83, 15B36.

1. Introduction

The well-known Pell sequence $\{P_n\}_{n=0}^{\infty}$ and Jacobsthal sequence $\{J_n\}_{n=0}^{\infty}$ are defined by the recurrence relation for $n \geq 2$

$$\begin{aligned} P_n &= 2P_{n-1} + P_{n-2}, \\ J_n &= J_{n-1} + 2J_{n-2}, \end{aligned}$$

where $P_0 = J_0 = 0$ and $P_1 = J_1 = 1$.

Binet formulas of these sequences are given by

$$P_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}},$$

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and

$$J_n = \frac{2^n - (-1)^n}{3}.$$

In [2, 3], Horadam obtained sum formulas for the usual Pell and Jacobsthal numbers as

$$\sum_{i=1}^{n} P_i = \frac{P_{n+1} + P_n - 1}{2}, \tag{1.1}$$

and

$$\sum_{i=1}^{n} J_i = \frac{J_{n+2} - 1}{2}. \tag{1.2}$$

The permanent of an n-square matrix $A = [a_{ij}]$ is defined by

$$perA = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n. Let $A = [a_{ij}]$ be an $m \times n$ real matrix with row vectors $r_1, r_2, ..., r_m$. We call A contractible on column (respectively, row) k if column (respectively, row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk} r_i + a_{ik} r_j$, and deleting row j and column k, is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A_{ij:k}^T\right]^T$ is called the contraction of A on row k relative to columns i and j. We say that A can be contracted to a matrix B if either $B = A$ or there exist matrices $A_0, A_1, ..., A$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, ..., t$. Let A be a nonnegative integral matrix of order n for $n > 1$ and let B be a contraction of A. Then in [1], authors proved that

$$perA = perB. \tag{1.3}$$

Authors, in [4], showed the conditions under which the determinant of a Hessenberg matrix becomes its permanent.

There are some relations between permanents of matrices and integer sequences. Let k and n be positive integers. Let $S_n^{(k)} = [s_{ij}]$ be an

n-square $(0,1)$-matrix defined by $s_{ij}=1$ if and only if $-1\le j-i\le k-1$. In [8], Lee defined n-square $(0,1)$-matrix $L^{(n,k)}$ as

$$L^{(n,k)}=\begin{cases} S_n^{(k)}-\sum_{j=2}^{k}E_{1j}+E_{1k+1} & \text{if } k<n,\\ S_n^{(k)}-\sum_{j=2}^{n}E_{1j} & \text{if } n\leqslant k,\end{cases}$$

where E_{ij} denotes the n-square matrix with 1 in the (i,j) position and zeros elsewhere, and showed that $perL^{(n,2)}=L_{n-1}$ and $perL^{(n,k)}=l_{n-1}^{(k)}$ $(2\leqslant n\leqslant k)$, where L_n and $l_n^{(k)}$ are nth Lucas and k-Lucas numbers, respectively.

In [9], Minc defined generalized Fibonacci numbers of order r as

$$f(n,r)=\begin{cases} 0 & \text{if } n<0,\\ 1 & \text{if } n=0,\\ \sum_{k=1}^{r}f(n-k,r) & \text{if } n>0.\end{cases}$$

Minc also defined an n-square super-diagonal $(0,1)$-matrix $F(n,r)=[f_{ij}]$, where

$$f_{ij}=\begin{cases} 1 & \text{if } -1\le j-i\le r-2,\\ 0 & \text{otherwise},\end{cases}$$

and proved that $per(F(n,r))=f(n,r-1)$. In addition, in [7], the authors obtained permanent of the matrix $F(n,r)$ by applying contraction.

In [5], Kilic defined an n-square super-diagonal $(0,1,2)$-matrix and proved that its permanent is equal to the $(n+1)$th generalized k-Pell number.

In [11], Yilmaz and Bozkurt defined two n-square upper Hessenberg matrices and showed that, permanents of these matrices are nth Pell and Perrin numbers.

A pseudo graph $G=(V,E)$ with vertex set $V=\{v_1,v_2,\dots,v_n\}$ and edge set $E=\{e_1,e_2,\dots e_m\}$ is a graph in which loops and multiple edges are allowed. A graph $G=(V,E)$ is called directed graph or digraph if every edge of graph is associated with an ordered pair of vertices. The n-square adjacency matrix $A=[a_{ij}]$ of directed graph $G=(V,E)$ with n vertices is defined as

$$a_{ij}=\begin{cases} 1 & \text{if } (v_i,v_j)\in E,\\ 0 & \text{otherwise}.\end{cases}$$

For more about graph theory see [10].

In [6], authors defined several classes of bipartite graphs whose number of 1-factors are the Fibonacci and Lucas numbers and their sums.

In this paper, we define two n-square upper Hessenberg matrices, one of which corresponds to adjacency matrix of a directed pseudo graph. We investigate relations between permanents and determinants of these upper Hessenberg matrices, and sum formulas of the Pell and Jacobsthal sequences. Finally, we give two Maple 13 procedures in order to calculate their permanents.

Throughout this paper, we denote $P(n) = \sum_{i=1}^{n} P_i$ and $J(n) = \sum_{i=1}^{n} J_i$.

2. Pell numbers and their sums

In this section, we define an n-square upper Hessenberg matrix and show that its permanent is equal to the sum of the first n Pell numbers.

Let $H_n = [h_{i,j}]$ be an n-square upper Hessenberg matrix as

$$H_n = \begin{bmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 1 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}, \tag{2.1}$$

where

$$h_{i,j} = \begin{cases} j & \text{if } i = 1 \text{ and } 1 \le j \le n, \\ 1 & \text{if } 2 \le i \le n \text{ and } -1 \le j-i \le 0, \\ 2 & \text{if } 2 \le i \le n \text{ and } 1 \le j-i \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1. *Let $H_n^r = \left[h_{i,j}^{(r)}\right]$ denotes the rth contraction of the matrix H_n, given in* (2.1)*, for $1 \le r \le n-2$. Then*

$$h_{1,j}^{(r)} = \begin{cases} P(r+1) & \text{if } j = 1, \\ P_{r+2} & \text{if } j = 2, \\ h_{1,j-1}^{(r)} + 1 & \text{if } 3 \le j \le n-r. \end{cases}$$

Proof. We prove the lemma by induction on r. Since the matrix H_n^r can be contracted on column 1, we obtain

$$H_n^1 = \begin{bmatrix} 3 & 5 & 6 & \cdots & n+1 & n+2 \\ 1 & 1 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 1 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix},$$

where, clearly

$$h_{1,j}^{(1)} = \begin{cases} P(2) & \text{if } j = 1, \\ P_3 & \text{if } j = 2, \\ h_{1,j-1}^{(1)} + 1 & \text{if } 3 \leqslant j \leqslant n-1. \end{cases}$$

We assume that the assertion is true for $r-1$, that is,

$$h_{1,j}^{(r-1)} = \begin{cases} P(r) & \text{if } j = 1, \\ P_{r+1} & \text{if } j = 2, \\ h_{1,j-1}^{(r-1)} + 1 & \text{if } 3 \leqslant j \leqslant n-r+1. \end{cases} \tag{2.2}$$

Now, we show that the assertion is true for r. If the matrix H_n^r is contracted on column 1, we get

$$h_{1,1}^{(r)} = h_{1,1}^{(r-1)} + h_{1,2}^{(r-1)} = P(r) + P_{r+1} = P(r+1),$$

and

$$h_{1,2}^{(r)} = 2h_{1,1}^{(r-1)} + h_{1,3}^{(r-1)} = 2P(r) + (1 + P_{r+1}) = 2P_{r+1} + P_r = P_{r+2},$$

by (1.1). By successive calculations, we write

$$\begin{aligned} h_{1,3}^{(r)} &= 2h_{1,1}^{(r-1)} + h_{1,4}^{(r-1)} = 2h_{1,1}^{(r-1)} + h_{1,3}^{(r-1)} + 1 = h_{1,2}^{(r)} + 1, \\ h_{1,4}^{(r)} &= 2h_{1,1}^{(r-1)} + h_{1,5}^{(r-1)} = 2h_{1,1}^{(r-1)} + h_{1,4}^{(r-1)} + 1 = h_{1,3}^{(r)} + 1, \\ &\vdots \\ h_{1,n-r}^{(r)} &= 2h_{1,1}^{(r-1)} + h_{1,n-r+1}^{(r-1)} = 2h_{1,1}^{(r-1)} + h_{1,n-r}^{(r-1)} + 1 = h_{1,n-r-1}^{(r)} + 1, \end{aligned}$$

by (2.2). So, we get

$$h_{1,j}^{(r)} = 2h_{1,1}^{(r-1)} + h_{1,j+1}^{(r-1)} = h_{1,j-1}^{(r)} + 1,$$

for $3 \leqslant j \leqslant n-r$. Thus the proof is completed. $\square$

Theorem 2.2. *Let the matrix H_n be as in (2.1). Then, for $n \geq 2$*

$$perH_n = P(n). \tag{2.3}$$

Proof. Since the matrix H_n is contractible, it is contracted to a 2-square integral matrix by applying successive contractions using Lemma 2.1 and obtain

$$H_n^{n-2} = \begin{bmatrix} P(n-1) & P_n \\ 1 & 1 \end{bmatrix}.$$

So

$$perH_n = perH_n^{n-2} = P(n-1) + P_n = P(n),$$

by (1.3). □

Corollary 2.3. *Let $\widetilde{H}_n$ be an n-square upper Hessenberg matrix obtained from H_n by multiplying all sub-diagonal entries by -1. Then*

$$\det \widetilde{H}_n = P(n).$$

Example 2.4. *For $n = 4$*

$$H_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } \widetilde{H}_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Hence $perH_4 = \det \widetilde{H}_4 = 20 = 1 + 2 + 5 + 12 = P(4)$.

3. The Jacobsthal numbers and their sums

Let us consider a pseudo graph in Figure 1. The adjacency matrix of this graph is an upper Hessenberg matrix. We show that the permanent of this adjacency matrix is equal to sum of the first n Jacobsthal numbers.

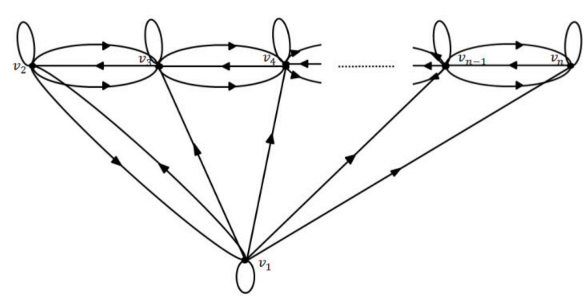

FIGURE 1.

The entries of n-square adjacency matrix $K_n = [k_{i,j}]$ of the pseudo graph in Figure 1 can be written as

$$k_{i,j} = \begin{cases} 1 & \text{if } i = 1, \\ 1 & \text{if } -1 \leq j - i \leq 0 \text{ and } 2 \leq i \leq n, \\ 2 & \text{if } j - i = 1 \text{ and } 2 \leq i \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

that is,

$$K_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 2 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}. \tag{3.1}$$

Theorem 3.1. *Let n-square matrix K_n be as in* (3.1) *for $n \geq 2$. Then*

$$per K_n = J(n). \tag{3.2}$$

Proof. Let K_n^r denotes rth contraction of the matrix K_n for $1 \leq r \leq n-2$. By applying successive contractions to the matrices K_n^r ($K_n^0 = K_n$) for $0 \leqslant r \leqslant n-4$ according to their first columns, we get

$$K_n^{n-2} = \begin{bmatrix} J(n-1) & J_n \\ 1 & 1 \end{bmatrix},$$

by using (1.2). So, we conclude that

$$per K_n = per K_n^{(n-2)} = J\,(n-1) + J_n = J(n),$$

by (1.3). □

Corollary 3.2. *Let $\widetilde{K}_n$ be an n-square upper Hessenberg matrix obtained from K_n by multiplying all sub-diagnal entries by -1. Then*

$$\det \widetilde{K}_n = J(n).$$

4. Maple procedures

Writing matrices and calculating their permanents and determinants are quiet difficult for large orders. For this reason, we give two Maple procedures in this section relevant to the preceeding theorems. Using these Maple procedures, one can calculate permanents of the matrices

H_n and K_n and sum of the first n terms of the Pell and Jacobsthal numbers for any positive integer n.

Procedure 1. This Maple procedure calculates permanent of the matrix H_n, given by (2.3), and sum of the first n Pell numbers $P(n)$, given by (1.1), for any positive integer n.

```
> restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,p,P,r,h,H;
p:=(n)->evalf(((1+sqrt(2))^n-(1-sqrt(2))^n)/(2*sqrt(2)));
P:=(n)->evalf((p(n+1)+p(n)-1)/2);
h:=(i,j)->piecewise(i-j=0,1,i-j=1,1,i=1,j,j-i>=1,2,0);
H:=Matrix(n,n,h):
for r from 0 to n-2 do
print(r,H):
for j from 2 to n-r do
H[1,j]:=H[2,1]*H[1,j]+H[1,1]*H[2,j]:
od:
H:=DeleteRow(DeleteColumn(Matrix(n-r,n-r,H),1),2):
od:
print(r,eval(H),P(n)):
end proc:with(LinearAlgebra):
permanent( );
```

Procedure 2. This Maple procedure calculates permanent of the matrix K_n, given by (3.2), and sum of the first n Jacobsthal numbers $P(n)$, given by (1.2), for any positive integer n.

```
>restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,r,t,J,k,K;
t:=(n)->evalf((2^n-(-1)^n)/3);
J:=(n)->evalf((t(n+2)-1)/2);
k:=(i,j)->piecewise(i=1,1,i-j=1,1,i-j=0,1,j-i=1,2,0);
K:=Matrix(n,n,k):
for r from 0 to n-2 do
print(r,K):
for j from 2 to n-r do
K[1,j]:=K[2,1]*K[1,j]+K[1,1]*K[2,j]:
```

```
od:
K:=DeleteRow(DeleteColumn(Matrix(n-r,n-r,K),1),2):
od:
print(r,eval(K),J(n)):
end proc:with(LinearAlgebra):
permanent( );
```

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