ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 40 (2014), No. 5, pp. 1067–1086

Title:

A two-phase free boundary problem for a semilinear elliptic equation

Author(s):

A. Aghajani

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 5, pp. 1067–1086
Online ISSN: 1735-8515

A TWO-PHASE FREE BOUNDARY PROBLEM FOR A SEMILINEAR ELLIPTIC EQUATION

A. AGHAJANI

(Communicated by Rahim Zaare-Nahandi)

Abstract. In this paper we study a two-phase free boundary problem for a semilinear elliptic equation on a bounded domain $D \subset \mathbb{R}^n$ with smooth boundary. We give some results on the growth of solutions and characterize the free boundary points in terms of homogeneous harmonic polynomials using a fundamental result of Caffarelli and Friedman regarding the representation of functions whose Laplacians enjoy a certain inequality. We show that in dimension $n = 2$, solutions have optimal growth at non-isolated singular points, and the same result holds for $n \geq 3$ under an $(n-1)$-dimensional density condition. Furthermore, we prove that the set of points in the singular set that the solution does not have optimal growth is locally countably $(n-2)$-rectifiable.
Keywords: Free boundary problems, optimal growth, regularity, singular set.
MSC(2010): Primary: 35R35; Secondary: 35J60.

1. Introduction

1.1. **Problem statement:** Given a bounded domain $D \subset \mathbb{R}^n$ with smooth boundary and $u_0 \in W^{1,2}(D) \cap L^\infty(D)$ consider the following minimization problem:

Minimize

$$J(u) = \int_D \left(\frac{|\nabla u|^2}{2} + F(u)\right) dx \tag{1.1}$$

Article electronically published on October 27, 2014.
Received: 14 February 2013, Accepted: 27 July 2013.

©2014 Iranian Mathematical Society

over the set

$$\mathcal{A}_{u_0} = \{u \in W^{1,2}(D); u - u_0 \in W_0^{1,2}(D)\}.$$

Here

$$F(u) = \frac{\lambda^+}{q}(u^+)^q - \frac{\lambda^-}{q}(u^-)^q,$$

where $u^{\pm} = max\{\pm u, 0\}$, $\lambda^{\pm} > 0$ and $1 < q < 2$. By the direct method of the calculus of variations we will show the existence of minimizers u of J which are in the class $C^{2,q-1}_{loc}(D)$ and satisfy the corresponding Euler-Lagrange equation

$$\Delta u = \lambda^+(u^+)^{q-1} + \lambda^-(u^-)^{q-1} \ in\ D, \tag{1.2}$$

in the classical sense. Since the functional J is not convex, there might be more than one minimizer with given boundary value u_0. Also, since we are not imposing any sign constraint on u_0, any minimizer u may take both positive and negative values.

We use the notation

$$\Omega^+(u) = \{u > 0\},\ \Omega^-(u) = \{u < 0\}$$

and their interfaces

$$\Gamma^{\pm}(u) = \partial\Omega^{\pm}(u) \cap D,$$

which we also call free boundaries, as they are a priori unknown. Note that because of the continuity of minimizers, $\Omega^+(u)$ and $\Omega^-(u)$ are open sets, hence u is real analytic in $\Omega^{\pm}(u)$.

1.2. **Known results.** For the cases $q = 0, 1$ with $\lambda^+ > 0 \geq \lambda^-$, the minimization problem (1.1) has been studied extensively in the last three decades using a wide variety of methods, among them the powerful monotonicity formula of Alt-Caffarelli-Friedman [1] as well as some of its generalizations [6]. For the particular case where $q = 0$, Caffarelli, Jerison and Kenig in [6] have shown the optimal Lipschitz regularity of minimizers and the C^1 regularity of the free boundary in dimension two. When $q = 1$, problem (1.1) corresponds to the obstacle type problem. The one-phase obstacle problem has been studied intensively, and it has been shown that minimizers have the optimal $C^{1,1}$ regularity. For the two-phase version of the problem, i.e., with no sign constraint, Shahgholian [19] and Uraltseva [22] proved the optimal $C^{1,1}_{loc}$ regularity of solutions, and in [20] Shahgholian, Uraltseva and Weiss showed the C^1 regularity of the free boundary near the so-called branching points

(points where the gradient vanishes); also see [21].
The case $0 < q < 1$, $\lambda^+ > 0 \geq \lambda^-$, has also received a great attention in the literature. In this case, the one phase version of the problem (1.1) has been well studied by Phillips [17, 18] and Alt-Phillips [2], among others. It has been established in [17] that nonnegative minimizers enjoy the optimal regularity $C^{1,\beta}$, $\beta = \frac{2}{2-q}$. Also, Giaquinta and Giusti in [9] proved the Hölder continuity of the gradient of minimizers. Weiss in [23] considered the two-phase version of this problem and studied the size and the structure of the singular set of the free boundary, i.e., the set of free boundary points at which no outer normal exists and presented some results on the partial regularity of weak solutions. Also, E. Lindgren and A. Petrosyan in ([16], Theorem 1.1) showed the C^1 regularity of the free boundary in two dimension. Recently, and for a more general two-phase variational free boundary problems, a rather complete description of the regularity of solutions and the Hölder continuity of the gradient of solutions together with the asymptotic interior regularity are given by Leitão, Queiroz and Teixeira in [15].
When $1 < q < 2$, problem (1.1) has been considered in the literature mostly with the sign condition $u \geq 0$. D. Phillips in [18] considered the problem of minimizing the functional

$$J(u) = \int_D |\nabla u|^2 + \chi_{\{u \geq 0\}} u^q dx, \ 1 < q < 2, \tag{1.3}$$

on the convex set $\mathbb{K} = \{u \in H^1(D), u = u_0 \geq 0 \ on \ \partial D\}$. He proved that the minimizers are subharmonic in D and are in the class $C^{[\beta],\beta-[\beta]}(G)$ for $\bar{G} \subset D$, where $[\beta]$ is the greatest integer less than or equal to β, and satisfy the Euler equation

$$\Delta u = q u^{q-1} \ in \ D,$$

$$u = |\nabla u| = 0 \ on \ D \cap \partial\{u > 0\}.$$

Then he demonstrates a number of measure estimates for the free boundary. Later, Alt and Phillips [2] considered positive solutions of a more general semilinear Dirichlet problem and investigated the nature of the free boundary (also see [8]). L. Bonorino [3] considered the above problem (1.3) and proved that the points of the free boundary where the zero set has no density lie in a Lipschitz surface. Furthermore, he proved that the singular points that have some $(n-1)$-density lie locally in a C^1 surface ([3], Theorem 4.13).

In this paper, we study the minimizers u of the energy functional J given in (1.1) on the admissible set $\mathcal{A}_{u_0}$, without any sign restriction on the function u. In Section 2, with a standard argument we prove the existence of solution for this problem, which is locally $C^{2,q-1}$ by the regularity theory and satisfy the Euler-Lagrange equation (1.2) in the classical sense. In Section 3, we give a one sided nondegeneracy result and a growth estimate for the solution u away from (and near) the free boundary points lie on $\Gamma^+ \setminus \Gamma^-$. However, because of the structure of our problem and since u changes sign on every neighborhood of a free boundary point lies on $\Gamma^+ \cap \Gamma^-$, it is difficult to analyze the growth of solutions away from the free boundary. In fact, the Harnack inequality as well as the techniques used in [17, 18] and [2] are no longer applicable here. As one can see in a rather similar structure in ([7], Lemma 9), Caffarelli and Salazar used the powerful monotonicity lemma to estimate a quadratic growth of the solutions of $\Delta u = cu$, in $\{|\nabla u| \neq 0\}$. To overcome this difficulty, in Section 3 we employ an interesting result of Caffarelli and Friedman ([5], Lemma 3.1) to give a representation for the free boundary set $\Gamma(u)$, and then show an optimal growth estimate for u away from and near the set of non-isolating points of the singular part of the free boundary in two-dimension. Also, in higher dimensions we prove the same result at non-isolated singular points under an $(n-1)$-dimensional density condition. Finally, in Section 4 by invoking a technique used in [11] to study the structure of the singular set of solutions of homogeneous elliptic differential equations of the second order, we prove that the set of singular points where u does not have optimal growth is countably $(n-2)$-rectifiable.

2. Existence and $C^{2,q-1}_{loc}$ regularity of global minimizers

In this section by the direct method of calculus of variations we prove the existence of minimizers u of J, which are in the class $C^{2,q-1}_{loc}(D)$ and satisfy the corresponding Euler-Lagrange equation (1.2).

Proposition 2.1. *There exists at lease one minimizer $u \in W^{1,2}(D)$ of the functional J which satisfies (1.2) in the sense of distributions.*

Proof. We show that J is weakly coercive and weakly sequentially lower semicontinuous (wslsc) on the set $\mathcal{A}_{u_0}$. First note that for $u \in \mathcal{A}_{u_0}$ we have

$$J(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda^+ + \lambda^-}{q}\|u\|_q^q. \tag{2.1}$$

Taking $u-g=w$ and $\lambda := \frac{\lambda^+ + \lambda^-}{q}$, (2.1) yeilds,

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|\nabla w+\nabla g\|_2^2 - \lambda\|w+g\|_q^q \\ &\geq \frac{1}{2} \mid \|\nabla w\|_2 - \|\nabla g\|_2 \mid^2 - \lambda(\|w\|_q + \|g\|_q)^q. \end{aligned}$$

Using the inequality $(a+b)^q \leq 2^{q-1}(a^q+b^q)$ for $a,b \geq 0$, and the above inequality follows

$$(2.2)\quad J(u) \geq \frac{1}{2}\|\nabla w\|_2^2 + \frac{1}{2}\|\nabla g\|_2^2 - \|\nabla w\|_2\|\nabla g\|_2 - 2^{q-1}\lambda(\|w\|_q^q + \|g\|_q^q).$$

Since $1 < q < 2 < 2^* = \frac{2n}{n-2}$, by the Sobolev inequality there exists a constant C such that $\|w\|_q \leq C\|\nabla w\|_2$, therefore (1.4) implies

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|\nabla w\|_2^2 - 2^{q-1}\lambda C\|\nabla w\|_2^q - \|\nabla w\|_2\|\nabla g\|_2 + \frac{1}{2}\|\nabla g\|_2^2 \\ &- 2^{q-1}\lambda\|g\|_q^q \to \infty \ as\ \|u\|_{W^{1,2}} \to \infty. \end{aligned}$$

To show that J is wslsc on $\mathcal{A}_{u_0}$, it is enough to prove it for the functional $G : W^{1,2}(D) \to \mathbb{R}$ defined by

$$G(u) = \int_D F(u)dx,$$

which is an easy task using the compact embedding $W_0^{1,2} \hookrightarrow\hookrightarrow L^q$. Now let $\xi \in C_c^\infty(D)$ and consider the function $I(\varepsilon) = J(u+\varepsilon\xi)$ for $\varepsilon \in \mathbb{R}$. Since the integrand of functional J, $f(p,z) = |p|^2 + \frac{\lambda^+}{q}(z^+)^q - \frac{\lambda^-}{q}(z^-)^q$ is a $C^{1,q-1}(\mathbb{R}^n \times \mathbb{R})$ function, $I(\varepsilon)$ is differentiable at $\varepsilon = 0$ thus

$$0 = I'(0) = J'(u+\varepsilon\xi)|_{\varepsilon=0} = \int_D (\nabla u.\nabla\xi - F'(u)\xi)dx.$$

Hence u satisfies (1.2) in the sense of distributions. □

Proposition 2.2. *Let $u \in L^2(D)$ be a solution of (1.2) in the sense of distributions. Then $u \in C_{loc}^{2,q-1}(D)$, i.e., u is a classical solution.*

Proof. Let $u \in L^2(D)$ be a solution of (1.2), then $F(u) \in L^{\frac{2}{q-1}}(D)$. Consider the Newtonian potential of $F(u)$, i.e.,

$$U(x) = \int_D \Phi_n(x-y)F(u(y))dy,$$

where Φ_n is the fundamental solution of the Laplacian in $\mathbb{R}^n$, i.e., $\Delta\Phi_n = \delta$ in the sense of distributions. Then it is readily verified that U is a weak solution of $\Delta u = F(u)$ in D. Thus $w = u - U$ is harmonic in D, and consequently belongs to $C^\infty(D)$. From the standard estimates for singular integrals (see e.g., [10], Theorem. 9.9), it can be shown that U is in $W_{loc}^{2,\frac{2}{q-1}}(D)$, and so $u = w + U \in W_{loc}^{2,\frac{2}{q-1}}(D)$. Next we show that $u \in W_{loc}^{2,\frac{2}{(q-1)^2}}(D)$, etc. Therefore,

$$\forall j \in \mathbb{N}, u \in W_{loc}^{2,\frac{2}{(q-1)^j}}(D).$$

Hence, $u \in W^{2,p}_{loc}$, for every $1 \le p < \infty$, and the Sobolev embedding theorem $W^{2,p}_{loc} \hookrightarrow C^{1,\alpha}_{loc}$, $\alpha = 1 - \frac{n}{p}$ implies that

$$\forall\ 0 < \alpha < 1, u \in W^{2,p}_{loc}(D) \cap C^{1,\alpha}_{loc}(D).$$

Now, to prove $u \in C^{2,q-1}_{loc}(D)$, by the classical elliptic regularity theory it suffices to show that $F(u) \in C^{0,q-1}_{loc}$. Let K be a compact subset of D. Since $u \in C^{1,\alpha}(K)$, thus u is a Lipschitz function on K, and since functions $(x^+)^{q-1}$ and $(x^-)^{q-1}$ are Hölder continuous of order $q-1$, as a consequence $F(u) \in C^{0,q-1}(K)$. □

3. Analysis of the free boundary

Let $u \in C^{2,q-1}_{loc}(D)$ be a solution of (1.2) and set $\Gamma^+(u) = \partial\Omega^+(u) \cap D$ and $\Gamma^-(u) = \partial\Omega^-(u) \cap D$. Then, due to the subharmonicity of solutions of (1.2) we have

$$\Gamma^-(u) \subseteq \Gamma^+(u). \tag{3.1}$$

Indeed, if there exists an $x_0 \in \Gamma^-(u) \setminus \Gamma^+(u)$, then

$$\exists r_0 > 0;\ u(x) \le 0 \ \textit{in}\ B_{r_0}(x_0)\ \textit{and}\ u(x_0) = 0.$$

But u is subharmonic so by the strong maximum principle u can not attain a local maximum and hence $u(x) \equiv 0$ in $B_{r_0}(x_0)$, contradicting with $x_0 \in \Gamma^-(u)$.
It is noteworthy that if $0 \in D$ with $B_{r_0} \subset D$, there is a radial positive solution of (1.2) in B_{r_0}. Indeed, it is easy to find a suitable $\gamma_q > 0$ so that

$$U_0(x) = \gamma_q |x|^\beta,\ \beta = \frac{2}{2-q} \tag{3.2}$$

is a solution of (1.2) in B_{r_0}.

Lemma 3.1. *Let u be a solution of (1.2). Then there are constants $C_\pm = C(n, q, \lambda_\pm)$, such that*
(i) if $x_0 \in \Omega^+(u)$,

$$\sup_{B_r(x_0)\cap\Omega^+} u(x) \geq u(x_0) + Cr^\beta$$

for any $r > 0$ such that $B_r(x_0) \subset\subset D$ and
(ii) if $x \in \Omega^-(u)$, then

$$u(x) \leq -C(dist(x, \Gamma^-))^\beta.$$

Proof. Let $x_0 \in \Omega^+(u)$ and $U_0(x)$ given in (3.2) be the radial solution of (1.2). Define

$$\Lambda = \{x \in B_r(x_0) \cap \Omega^+, u(x) > U_0(x - x_0)\}.$$

Since $x_0 \in \Lambda$, Λ is a nonempty open set. Next we define the auxiliary function

$$w(x) = u(x) - u(x_0) - U_0(x - x_0),$$

and take $y \in \overline{\Lambda}$ such that

$$w(y) = \sup_{x\in\overline{\Lambda}} w(x).$$

For $x \in \Lambda$ we have

$$\Delta w(x) = \Delta u(x) - \Delta U_0(x - x_0) = \lambda^+(u(x)^{q-1} - U_0(x - x_0)^{q-1}) > 0,$$

hence, w is subharmonic on Λ. Suppose that $y \in \Lambda$. Also, let $y \in A \subseteq \Lambda$, where A is a connected component of Λ. If y is an interior point of A, then the maximum principle says that w is constant on A. So $w \equiv w(y)$ on A, gives $u(x) = w(y) + u(x_0) + U_0(x - x_0)$ for $x \in A$, and consequently $\Delta u = \Delta U_0(x - x_0)$ on A. The last equality yields $\lambda^+ u(x)^{q-1} = \lambda^+ U_0(x-x_0)^{q-1}$ or $u(x) = U_0(x-x_0)$ for $x \in A$. Therefore, $w(y) = -u(x_0) < 0 = w(x_0)$, a contradiction. Therefore, we must have $y \in \partial A$ and due to the standard fact of point-set topology $\partial A \subset \partial\Lambda$, gives $y \in \partial\Lambda$. Now, we have the three possible cases, $y \in \partial B_r(x_0) \cap \Omega^+$, $u(y) = U_0(y - x_0)$ or $y \in \partial\Omega^+$. It is easy to see that the two later cases lead to the contradiction $w(y) < 0$, so $y \in \partial B_r(x_0) \cap \Omega^+$, and thus

$$\sup_{\partial B_r(x_0)\cap\Omega^+} u(x) - u(x_0) - Cr^\beta = \sup_{\partial B_r(x_0)\cap\Omega^+} w(x)$$

$$\geq \sup_{\partial B_r(x_0)\cap\Omega^+\cap\bar{\Lambda}} w(x) = \sup_{\bar{\Lambda}} w(x) = w(y) \geq w(x_0) = 0$$

which implies the required inequality in (i).
For the proof of statement (ii) let $x \in \Omega^-(u)$ and take $r_x := dist(x, \Gamma^-)$. Set $v = -u$, then $v > 0$ in $B_{r_x}(x)$ and $-\Delta v = \lambda^- v^{q-1}$. Now, define the auxiliary function

$$w(y) = \frac{v(y)^{2-q}}{2-q} - \frac{\lambda^-}{2n}(r_x^2 - |y-x|^2),$$

and compute

$$\Delta w(y) = (1-q)\frac{|\nabla v|^2}{v^q} + v^{1-q}\Delta v + \lambda^- = -(q-1)\frac{|\nabla v|^2}{v^q} < 0.$$

Since on $\partial B_{r_x}(x)$ we have $w(y) = \frac{v(y)^{2-q}}{2-q} \geq 0$ and w is superharmonic we get $w(x) = \frac{v(x)^{2-q}}{2-q} - \frac{\lambda^-}{2n} r_x^2 \geq 0$, which gives the desired estimate. □

Remark 3.2. *(Nondegeneracy) Note that from the statement (i), if $x_0 \in \Gamma^+$ by the continuity of u we get*

$$\sup_{B_r(x_0)\cap\Omega^+} u(x) \geq Cr^\beta$$

for any $r > 0$ such that $B_r(x_0) \subset D$. Also, if $x_0 \in \Gamma^-$ is an exterior point of Ω^+, i.e., there exist $x_1 \in \Omega^-$ and $r_0 > 0$ such that $|x_0 - x_1| = r_0$ and $B_{r_0(x_1)} \subset \Omega^-$, then the statement (ii) of the above lemma gives

$$\inf_{B_r(x_0)\cap\Omega^-} u(x) \leq -Cr^\beta$$

for any $r < r_0$. Indeed if $r < r_0$, we can take a point x_r in the line segment $\overline{x_0 x_1}$ such that $|x_0 - x_r| = r$, then

$$\inf_{B_r(x_0)\cap\Omega^-} u(x) \leq u(x_r) \leq -C(dist(x_r, \Gamma^-))^\beta = -C|x_0 - x_r|^\beta = -Cr^\beta.$$

The next lemma says that a solution u of (1.2) will have the optimal growth β at free boundary points lieing on $\Gamma^+ - \Gamma^-$. Note that since u is non-negative in a neighborhood of a point on $\Gamma^+ - \Gamma^-$, it can be deduced from the results of [18] (also see [[2], Corollary 1.11]), but we give a different proof using the comparison principle and Harnack's inequality.

Lemma 3.3. *Let $u \in W^{1,2}(D)$ be a solution of (1.2) and $x_0 \in \Gamma^+ - \Gamma^-$. Then there exists $r_0 = r_0(x_0) > 0$ such that*

$$\sup_{B_r(x_0)} u(x) \leq Cr^\beta, \ \textit{for every } r \leq r_0,$$

where $\beta = \frac{2}{2-q}$ and C is a constant depends on n, q and λ^+.

Proof. Let $x_0 \in \Gamma^+ - \Gamma^-$, then $u(x_0) = 0$ and there exists $r_0 > 0$ such that $u(x) \geq 0$ in $B_{r_0}(x_0)$, so from (1.2) we obtain

$$\Delta u(x) = \lambda^+ u^{q-1} \text{ in } B_{r_0}(x_0). \tag{3.3}$$

Define

$$M_r = \sup_{B_r(x_0)} |u(x)|, \ r < r_0.$$

Now split u into the sum $v + H$ in $B_r(x_0)$, where

$$\Delta v = \lambda^+ u^{q-1}, \ \Delta H = 0 \ in \ B_r(x_0),$$

$$v = 0, \ H = u \ on \ \partial B_r(x_0).$$

We estimate v and H separately.
To estimate v consider the auxiliary function $w(x) = \frac{1}{2n}(r^2 - |x - x_0|^2)$, which satisfies

$$\Delta w = -1, \ in \ B_r(x_0),$$

$$w = 0, \ on \ \partial B_r(x_0).$$

From (3.3) and the definition of M_r we have

$$0 \leq \Delta v = \lambda^+ u^{q-1} \leq \lambda^+ M_r^{q-1} \ in \ B_r(x_0),$$

and by the comparison principle we get

$$-\lambda^+ M_r^{q-1} w(x) \leq v(x) \leq 0, \ in \ B_r(x_0),$$

which implies

$$-\frac{\lambda^+}{2n} M_r^{q-1} r^2 \leq v(x) \leq 0, \ in \ B_r(x_0). \tag{3.4}$$

To estimate H observe that H is a nonnegative harmonic function in $B_r(x_0)$ and $H = u$ on $\partial B_r(x_0)$. Therefore, the Harnack's inequality gives

$$H(x) \leq C_n H(x_0) = -C_n v(x_0) \leq C_n \lambda^+ M_r^{q-1} r^2, \ x \in B_{\frac{r}{2}}(x_0).$$

Combining the estimates for v and H we get

$$u(x) \leq C M_r^{q-1} r^2, \ x \in B_{\frac{r}{2}}(x_0),$$

and

$$M_{\frac{r}{2}} \leq C M_r^{q-1} r^2 \ for \ every \ r < r_0.$$

Using the equality $r^2 = \frac{r^\beta}{r^{\beta(q-1)}}$ in the last inequality we have

$$\tilde{M}_{\frac{r}{2}} \leq C_q \tilde{M_r}^{q-1}, \ where \ \tilde{M}_r := \frac{M_r}{r^\beta} \ and \ C_q = 2^\beta C. \tag{3.5}$$

Taking $r_j := \frac{r_0}{2^j}$ and $\tilde{M}_j := \tilde{M}_{r_j}$ for $j = 0, 1, 2, ...,$ from (3.5) we obtain

$$\tilde{M}_{j+1} \leq C_q \tilde{M}_j^{q-1} \leq C_q C_q^{q-1} \tilde{M_{j-1}}^{(q-1)^2} \leq ... \leq C_q^{(1+(q-1)^2+...+(q-1)^j)} \tilde{M}_0^{(q-1)^j}.$$

Since $(q-1)^j \to 0$ and $1 + (q-1)^2 + ... + (q-1)^j \to \frac{1}{2-q}$ when $j \to \infty$, then $\{\tilde{M}_j\}$ is a bounded sequence, hence we find a positive constant C such that $M_j \leq C r_j^\beta$ for every $j \in \mathbb{N}$ which is enough to conclude that $M_r \leq C r^\beta$ for any $r < r_0$ and the proof is complete. □

Let u be a solution of (1.2) and $x_0 \in \Gamma^+ - \Gamma^-$, then from lemma (3.3) there exists $r_0 > 0$ such that $u(x) \geq 0$ in $B_{r_0}(x_0)$ and

$$\sup_{B_r(x_0)} u(x) \leq C r^\beta, \ for \ every \ r \leq r_0.$$

Now, following [18] for $r < r_0$

$$|D^\alpha u(x)| \leq C(u(x))^{\frac{\beta-|\alpha|}{\beta}}, \ x \in \Omega^+(u) \cap B_{\frac{r}{3}}(x_0),$$

where $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $u \in C^{[\beta], \beta-[\beta]}(B_{\frac{r}{3}}(x_0))$.
Note that if $x_0 \in \Gamma^+ \cap \Gamma^-$, then u changes sign in every ball $B_r(x_0)$, $r > 0$ and the techniques used in the previous works mentioned in Section 1.2 can not be used here to get the above result.
In the sequel, to study the free boundary we frequently use the following fundamental lemma of Caffarelli and Friedman [5]. Note that this lemma in [5] is proved for the case $n = 3$, but as the authors indicated in the introduction the proof is valid for any dimension. A related result in two dimensions was proved by Hartman and Wintner by complex variables methods [14].

Lemma 3.4. *([5], Lemma 3.1) Let γ be a positive non-integer, $\gamma \geq \gamma_0 > 0$, and let $v(x)$ be a function satisfying*

$$|\Delta v(x)| \leq C_\gamma |x|^\gamma \ in \ B_1, \ C_\gamma \geq 2^\gamma. \tag{3.6}$$

Then

$$v(x) = P(x) + \Gamma(x) \ in \ B_1, \tag{3.7}$$

where $P(x)$ is a harmonic polynomial of degree $[\gamma] + 2$ and

$$|\Gamma(x)| \leq C C_\gamma \frac{\gamma}{\langle \gamma \rangle} |x|^{\gamma+2} \ in \ B_1, \tag{3.8}$$

$$|\nabla \Gamma(x)| \leq C C_\gamma \frac{\gamma^3}{\langle \gamma \rangle} |x|^{\gamma+1} \ in \ B_1, \tag{3.9}$$

where $\langle\gamma\rangle = \min\{\gamma-[\gamma], 1+[\gamma]-\gamma\}$ *and* C *is a constant depending only on* γ_0*, and on upper bounds on* $|v(x)|$ *and* $|\nabla v(x)|$ *for* $x \in \partial B_1$*.*

For a function u define the singular set as

$$\mathcal{S}(u) = \{x \in D;\ u(x) = |\nabla u(x)| = 0\},$$

and set

$$\mathcal{O}_b(u) = \{x_0; \exists C, \delta > 0; |u(x)| \leq C|x-x_0|^b,\ in\ B_\delta(x_0)\}.$$

Also, motivated by the work of Caffarelli and Friedman [5] we define

$$\mathcal{S}_m(u) = \{x_0; u(x) = H_m(x-x_0)+O(|x-x_0|^{m+\delta}), 0 \not\equiv H_m \in \Sigma_m, \delta > 0\},$$

where Σ_m denotes the space of all homogeneous harmonic polynomials of degree m.

Remark 3.5. *Note that in the definition of* $\mathcal{S}_m(u)$*, we can replace* H_m *by a harmonic polynomial (not necessary homogeneous)* P_m *of degree* m*. To see this it suffices to expand* P_m *into its Taylor series around* x_0*. Indeed, if* $P_m(x) = \sum_{i=k}^m Q_i(x-x_0)$*, where every* Q_i *is a homogeneous harmonic polynomial of degree* i *and* $k \geq 1$*, then* $x_0 \in \mathcal{S}_j(u)$*, where* j *is the least integer* ≥ 1 *such that* $Q_j \not\equiv 0$*.*

Now, we present one of our main results concerning the representation of free boundary points. We prove it for R^3, the proof however is valid for any dimension.

Theorem 3.6. *Let* $\Gamma(u)$ *be the free boundary, then*

$$\Gamma(u) = \bigcup_{m=1}^{[\beta]} \mathcal{S}_m(u) \cup \bigcap_{2<b<\beta} \mathcal{O}_b(u). \tag{3.10}$$

Moreover, if β *is a positive non-integer, then*

$$\Gamma(u) = \bigcup_{m=1}^{[\beta]} \mathcal{S}_m(u) \cup \mathcal{O}_\beta(u). \tag{3.11}$$

Furthermore, if K *is a compact subset of* D*, then the constant* C *in the definition of* $\mathcal{O}_b(u)$*,* $2 < b \leq \beta$ *is uniform in* x_0*,* $x_0 \in K \cap \Gamma(u)$ *and depends only on* q*,* u *and* K*.*

Proof. Suppose $x_0 \in \Gamma(u) \setminus \bigcup_{m=1}^{[\beta]} \mathcal{S}_m(u)$. Since $u(x_0) = 0$ and u is in C^2, thus there exists $C_0 > 0$ such that $|u(x)| \leq C_0|x-x_0|$ in $B_{r_0}(x_0)$

for some $r_0 > 0$. We can take $r_0 = 1$ due to the scaling property of solutions. Then it follows that

$$|\Delta u(x)| = \lambda^+(u^+)^{q-1} + \lambda^-(u^-)^{q-1} \le (\lambda^+ + \lambda^-)|u|^{q-1} \le \lambda C_0|x-x_0|^{(q-1)},$$

in $B_1(x_0)$, where $\lambda := \lambda^+ + \lambda^-$. Thus, by Lemma 3.4

$$u(x) = P_1(x) + R_1(x), \ in\ B_1(x_0),$$

where P_1 is a harmonic polynomial of degree $[q-1]+2 = 2$, and for $x \in B_1(x_0)$

$$|R_1(x)| \le C\lambda C_0 \frac{q-1}{\langle q-1\rangle}|x-x_0|^{q-1+2} = C\lambda C_0|x-x_0|^{q+1},$$

where C is a constant depending only on q, and on upper bounds on $u(x)$ and $|\nabla u(x)|$ for $x \in \partial B_1(x_0)$. Since P_1 is of degree $2 \le [\beta]$, by our assumption and Remark 3.5, we must have $P_1 \equiv 0$. It follows that $u(x) = R_1(x)$ in $B_1(x_0)$, hence

$$|\Delta u(x)| \le \lambda|u(x)|^{q-1} = \lambda|R_1(x)|^{q-1} \le \lambda(C\lambda C_0)^{q-1}|x-x_0|^{(q-1)(q+1)},$$

in $B_1(x_0)$. Replacing $(q-1)(q+1)$ (in the case it is an integer) with $(q-1)(q+1)-\varepsilon_1$ to get a non-integer, where $0 \le \varepsilon_1 < q-1$, and applying Lemma 3.4 once again we conclude that,

$$u(x) = P_2(x) + R_2(x), \ in\ B_1(x_0),$$

where P_2 is a harmonic polynomial of degree $2+[(q-1)(q+1)-\varepsilon_1]$ and

$$|R_2(x)| \le C\lambda(C\lambda C_0)^{q-1}\frac{(q-1)(q+1)-\varepsilon_1}{\langle (q-1)(q+1)-\varepsilon_1\rangle}|x-x_0|^{2+(q-1)(q+1)-\varepsilon_1}.$$

Since $1 < q < 2$ we get $q+1 < \beta = \frac{2}{2-q}$, thus

$$2 + [(q-1)(q+1)-\varepsilon_1] \le 2 + [(q-1)\beta] = [\beta].$$

Hence, by our assumption $P_2 \equiv 0$ and $u(x) = R_2(x)$ in $B_1(x_0)$. Repeating the above argument we are able to find sequences ε_j and β_j such that

$$|u(x)| \le C_j|x-x_0|^{\beta_j}, \ in\ B_1(x_0), \tag{3.12}$$

where $\beta_0 = 1, \varepsilon_0 = 0$ and for $j \ge 0$

$$\beta_{j+1} = 2 + (q-1)\beta_j - \varepsilon_j, \ 0 \le \varepsilon_j < (q-1)^j, \tag{3.13}$$

$$C_{j+1} = C\lambda C_j^{(q-1)}\frac{\beta_j-2}{\langle \beta_j - 2\rangle}. \tag{3.14}$$

By induction it is easy to see that $\beta_j \le \beta$. Indeed, we have $\beta_1 = 2 < \beta$, and if $\beta_j \le \beta$ then

$$\beta_{j+1} = 2 + (q-1)\beta_j - \varepsilon_j \le 2 + (q-1)\beta = \beta.$$

Taking $b_j = \beta - \beta_j$ in (3.13) we compute

$$\begin{aligned} b_{j+1} &= (q-1)b_j + \varepsilon_j = (q-1)^2 b_{j-1} + (q-1)\varepsilon_{j-1} + \varepsilon_j = ... = (q-1)^j b_1 \\ &+ \sum_{i=1}^{j} (q-1)^i \varepsilon_{j-i} \\ &\le (q-1)^j b_1 + j(q-1)^j \to 0,\ as\ n \to \infty, \end{aligned}$$

and consequently, $\beta_j \to \beta$ as $j \to \infty$. Now, if $b < \beta$ we can choose a $j_b \in \mathbb{N}$ so that $b < \beta_{j_b}$, hence

$$|u(x)| \le C_{j_b} |x - x_0|^{\beta_{j_b}} \le C_{j_b} |x - x_0|^b,\ in\ B_1(x_0).$$

This proves the first assertion. Now, suppose β is a positive non-integer, since $\beta_j \to \beta$ thus $\langle \beta_j \rangle \to \langle \beta \rangle > 0$, hence for a $j_0 \in \mathbb{N}$, $\langle \beta_j \rangle > \frac{\langle \beta \rangle}{2}$, for any $j \ge j_0$. Taking j_0 large enough we have $\beta_j \le 2 + \beta$ and from (3.14) with $\alpha := C\lambda \frac{2\beta}{\langle \beta \rangle}$ we get

(3.15)

$$C_{j+1} \le \alpha C_j^{q-1} \le \alpha^{1+(q-1)} C_j^{(q-1)^2} \le ... \le \alpha^{1+(q-1)+...(q-1)^{j-j_0}} C_{j_0}^{(q-1)^{j-j_0}}.$$

Since $0 < q - 1 < 1$, we have $1 + (q-1) + ...(q-1)^{j-j_0} < \frac{1}{2-q}$ and $(q-1)^{j-j_0} \to 0$ as $j \to \infty$. Thus, (3.15) implies that $\{C_j\}$ is a bounded sequence. Taking $j \to \infty$ in (3.12) we obtain

$$|u(x)| \le C|x - x_0|^\beta,\ in\ B_1(x_0),$$

which is the desired results.

For the last assertion, suppose that K is a compact subset of D. Since $u \in C_{loc}^{2,q-1}(D)$, for $x_0 \in K \cap \Gamma(u)$ we can choose a C_0 uniform in x_0 and depends only on u and K such that $|u(x)| \le C_0|x - x_0|$ in $B_{r_0}(x_0)$ for some $r_0 < \frac{1}{2} dist(K, \partial D)$. Starting the above proof with this C_0, the rest of the proof shows that C is independent of x_0. □

Theorem 3.7. *Let $n = 2$ and x_0 be a non-isolated point of $\mathcal{S}(u)$ then*

$$x_0 \in \bigcap_{b<\beta} \mathcal{O}_b(u), \tag{3.16}$$

and in the case $\beta \notin \mathbb{Z}$

$$x_0 \in \mathcal{O}_\beta(u). \tag{3.17}$$

Proof. For simplicity take $x_0 = 0$. We show that $0 \notin \bigcup_{m=1}^{[\beta]} \mathcal{S}_m(u)$ and then the conclusion follows from Theorem 3.6. Suppose this is not the case, so there is an $m \geq 1$ such that $0 \in \mathcal{S}_m(u)$, hence

$$u(x) = H_m(x) + O(|x|^{m+\delta}), \ \textit{in } B_1, \tag{3.18}$$

for some $\delta > 0$, where $0 \not\equiv H_m \in \Sigma_m$. Since $0 \in \mathcal{S}(u)$ and u is a C^2 function, then from (3.18) we get $H_m(0) = |\nabla H_m(0)| = 0$ follows that $m \geq 2$. Now, take a sequence of points $x_j \neq 0$ in $\mathcal{S}(u)$, where $x_j \to 0$. From (3.18) and the homogeneity of degree m of H_m we get

$$H_m(\frac{x_j}{2|x_j|}) = \frac{H_m(x_j)}{2^m|x_j|^m} = \frac{u(x_j)}{2^m|x_j|^m} + \frac{O(|x_j|^{m+\delta})}{2^m|x_j|^m} = \frac{O(|x_j|^{m+\delta})}{2^m|x_j|^m} \to 0.$$

Taking $y_j = \frac{x_j}{2|x_j|}$, there is a subsequence y_{j_i} and y with $|y| = \frac{1}{2}$ such that $y_{j_i} \to y$ and $H_m(y) = 0$. Since ∇H_m is also homogeneous (of degree $m-1$), similar to the above argument (starting with x_{j_i} instead of x_j) we can show that $\nabla H_m(y) = 0$, and from the harmonicity of H_m, $\lambda y \in \mathcal{S}(H_m)$ for every $\lambda \in R^+$. But this contradicts the fact that the singular set of every harmonic function in R^2 is isolated (for example, see [12], Lemma 2.4.1), so $H_m \equiv 0$ in B_1, which is a contradiction. $\square$

To get a similar result in the case $n \geq 3$, we need a density assumption on the singular set $\mathcal{S}(u)$ near x_0.

Definition 3.8. *For the set of points $x_1, ..., x_k$ in $S \subset R^n$, $k \leq n$, let $P^{x_0}_{x_1,...,x_k}$ be the k-dimensional parallelogram (with one"vertex" x_0) and vectors $\overrightarrow{x_0x_1}, ..., \overrightarrow{x_0x_k}$ as the edges. Indeed we have*

$$P^{x_0}_{x_1,...,x_k} = \{\sum_{i=1}^{k} t_i \overrightarrow{x_0x_i};\ 0 \leq t_i \leq 1\}.$$

Note that

$$\mathrm{Vol}_{R^k} P^{x_0}_{x_1,...,x_k} > 0 \Leftrightarrow \overrightarrow{x_0x_1}, ..., \overrightarrow{x_0x_k} \textit{ are linearly independent},$$

and

$$\mathrm{Vol}_{R^k} P^0_{r_1x_1,...,r_kx_k} = r_1...r_k \mathrm{Vol}_{R^k} P^0_{x_1,...,x_k}.$$

Now, let $S \subset R^n$ and $x_0 \in S$. We define

$$\delta_r(S, x_0, k) := \sup\{\frac{\mathrm{Vol}_{R^k} P^{x_0}_{x_1,...,x_k}}{r^k},\ x_1, ..., x_k \in S \cap B_r(x_0)\},$$

and $\delta_r(S, k) := \delta_r(S, 0, k)$.

Remark 3.9. *L. Caffarelli in* [4] *used the concept of minimal diameter, that measures the thinness of the zero set of solution at a given point. For a set $S \subset R^n$, define*

$$\delta'_r(S, x_0) := \frac{MD(S \cap B_r(x_0))}{r},$$

where $MD(A)$, called minimal diameter of A, is the infimum of distances between pairs of parallel planes such that A is contained in the strip determined by the plane. Many authors used the condition

$$\limsup_{r \to 0} \delta'_r(\Gamma(u), x_0) > 0,$$

where $x_0 \in \partial\Gamma(u) \cap D$ to analyze the free boundary. It is easy to see that there is a constant C so that $\delta'_r(S, x_0) \leq C\delta_r(S, x_0, k)$ for every $k \leq n$.

The following example shows that we can have $\delta'_r(S, x_0) = 0$ while $\delta_r(S, x_0, n-1) > C > 0$, for every $r > 0$.

Example 3.10. *Take $S = \{(0, y, |y|),\ y \in R\} \subset R^3$. Then S is contained in the strip determined by the planes $x = \pm\varepsilon$ for every $\varepsilon > 0$, so $\delta'_r(S, 0) = 0$ for every $r > 0$. But it is easy to see that $\delta_r(S, 0, 2) = \frac{\sqrt{3}}{2}$, for every $r > 0$.*

Theorem 3.11. *Let $n \geq 3$ and x_0 be a non-isolated point of $\mathcal{S}(u)$. Moreover,*

$$\limsup_{r \to 0} \delta_r(\mathcal{S}(u), x_0, n-1) > 0. \tag{3.19}$$

Then

$$x_0 \in \bigcap_{b < \beta} \mathcal{O}_b(u), \tag{3.20}$$

and in the case $\beta \notin \mathbb{Z}$

$$x_0 \in \mathcal{O}_\beta(u). \tag{3.21}$$

Proof. Without loss of generality take $x_0 = 0$, then similar to the proof of Theorem 3.7 it suffices to show $0 \notin \bigcup_{m=1}^{[\beta]} \mathcal{S}_m$. If this is not the case, then u satisfies (3.18) in the proof of Theorem 3.7. Take $\delta := \limsup_{r \to 0} \delta_r(\mathcal{S}(u), x_0, n-1)$, then from (3.19) we can find a sequence $\{r_j\}$ of real numbers and sequences $\{x_{1,j}\}$,..., $\{x_{n-1,j}\}$ in $\mathcal{S}(u)$ with $|x_{i,j}| \leq r_j \to 0$, $i = 1, ..., n-1$ such that

$$\mathrm{Vol}_{R^{n-1}} P^0_{x_{1,j},\ldots,x_{n-1,j}} \geq (\delta - \frac{1}{j}) r_j^{n-1}. \tag{3.22}$$

Now, suppose that $H_m \not\equiv 0$ in B_1. Similar to the proof of Theorem 3.7 we find $x_1, ..., x_{n-1} \in \mathcal{S}(H_m)$ with (possibly passing to subsequences) $\frac{x_{i,j}}{|x_{i,j}|} \to x_i$, $i = 1, ..., n-1$. But then from (3.22) we get

$$\begin{aligned}\mathrm{Vol}_{R^{n-1}} P^0_{x_1,...,x_{n-1}} &= \lim_{n\to\infty} \mathrm{Vol}_{R^{n-1}} P^0_{\frac{x_{1,j}}{|x_{1,j}|},...,\frac{x_{n-1,j}}{|x_{n-1,j}|}} \\ &\geq \overline{\lim}(\delta - \frac{1}{j}) \frac{r_j^{n-1}}{|x_{1,j}|...|x_{n-1,j}|} \geq \delta,\end{aligned}$$

that shows $x_1, ..., x_{n-1}$ are linearly independent. Therefore, $\dim \mathcal{S}(H_m) \geq n-1$. This contradicts the fact that the dimension of the singular set of a harmonic homogeneous polynomial is $\leq n-2$ [[13], page 5], also see [12]. Therefore, $H_m \equiv 0$ in B_1, a contradiction. □

4. **Structure of the singular set**

Let u be a C^2 solution of (1.2). By the implicit function theorem $\Gamma(u) \setminus \mathcal{S}(u) = \mathcal{S}_1(u)$ is an $(n-1)$-dimensional hypersurface at least locally. The following example shows that $dim\mathcal{O}_\beta(u)$ can take every number of $0, 1, ..., n$.

Example 4.1. *For $j = 1, 2, ..., n$ and $\beta = \frac{2}{2-q}$, $1 < q < 2$ the following functions*

$$u_j : R^n \to R, u_j(x_1, ..., x_n) = \gamma_j |(x_1, ..., x_j, 0, ..., 0)|^\beta,$$

are solutions of the equation

$$\Delta u(x) = |u(x)|^{q-1},$$

in B_1 for a suitable γ_j. Note that we have $dim\mathcal{O}_\beta(u_j) = n - j$, $j = 1, 2, ..., n$. Also the following function constructed in [17]

$$u(x) = (\frac{\sqrt{2}}{\beta})^\beta |x - s|^\beta, \ for\ s \leq x,$$

$$u(x) = 0, \ for\ x \leq s,$$

is a solution of

$$\Delta u(x) = (q-1)|u(x)|^{q-1}, \ in\ D = (-1, 1),$$

$$u_0(-1) = 0, u_0(1) = h > 0,$$

for small h and $s > -1$. Then $dim\mathcal{O}_\beta(u) = dim[-1, s] = 1$.

In the sequel we study the structure of $\mathcal{S}_m(u)$ for $m \geq 2$ and give some partial results by assuming some conditions on the range of q or the higher order regularity of solution u. To do this we use an approach similar to the one in [11], also see [12].
Let $1 < q < \frac{4}{3}$, then $2 < \beta = \frac{2}{2-q} < 3$ and thus from Theorem 3.6 we have

$$\mathcal{S}(u) = \mathcal{S}_2(u) \cup \mathcal{O}_\beta(u), \tag{4.1}$$

where by the definition of $\mathcal{S}_m(u)$

$$\mathcal{S}_2(u) = \{x_0; u(x) = H_2(x - x_0) + O(|x - x_0|^{2+\delta}), 0 \not\equiv H_2 \in \Sigma_2, \delta > 0\}.$$

Indeed, in this case we have

$$\mathcal{S}_2(u) = \{x_0; u(x) = H_2(x - x_0) + O(|x - x_0|^{2q}), 0 \not\equiv H_2 \in \Sigma_2\}. \tag{4.2}$$

To see this, let $x_0 \in \mathcal{S}_2(u)$, then there exists C such that $|u(x)| \leq C|x-x_0|^2$ in $B_{r_0}(x_0)$ for some $r_0 > 0$. From (1.2) it follows that $|\Delta u(x)| \leq C' |x-x_0|^{2(q-1)}$ in $B_{r_0}(x_0)$. Thus by Lemma 3.4

$$u(x) = P(x - x_0) + R(x), \ in \ B_{r_0}(x_0),$$

where P is a harmonic polynomial of degree $[2(q - 1)] + 2 = 2$ and $R(x) = O(|x - x_0|^{2q})$. Now, since $x_0 \in \mathcal{S}_2(u)$ and $2q > 2$ we must have $P(x) = x^T D^2 P(x_0) x \in \Sigma_2$ thus (4.2) holds. Also, note that since u is in C^2 and from Lemma 3.4 it is easy to show that

$$\mathcal{S}_2(u) = \{x_0; u(x_0) = |\nabla u(x_0)| = 0, D^2 u(x_0) \neq 0\}. \tag{4.3}$$

The next theorem shows that $\mathcal{S}_2(u)$ is locally countably $(n-2)$-rectifiable. The proof is essentially the same as that of Theorem 2.1 in [11] where the author studied the structure of the singular sets of solutions of homogeneous elliptic differential equations of the second order. The following lemma ([11], Lemma 2.3) is crucial for the proof.

Lemma 4.2. *Suppose $A \subset R^n$ has the following property: for any $x \in A$ there exists a j-dimensional linear subspace l_x such that for any sequence $\{x_k \subset A\}$ with $x_k \to x$, we have*

$$Angle\langle \overline{x x_k}, l_x \rangle \to 0.$$

Then A is on a countable union of j-dimensional Lipschitz graphs.

Theorem 4.3. *Let $1 < q < \frac{4}{3}$ and u be a solution of (1.2) in B_1. Then $\mathcal{S}_2(u)$ given in (4.1) is countably $(n-2)$-rectifiable.*

Proof. We show that

$$\mathcal{S}_2(u) = \bigcup_{j=0}^{n-2} \mathcal{S}_2^j(u),$$

where $\mathcal{S}_2^j(u)$ is on a countable union of j-dimensional C^1 manifolds, $j = 0, 1, ..., n-2$. To do this take $y \in \mathcal{S}_2(u)$ and let $H_{2,y}$ be the leading polynomial of u at y as in the definition of $\mathcal{S}_2(u)$. Since $H_{2,y}$ is a nonzero homogeneous harmonic polynomial of degree 2, thus

$$\mathcal{S}_2(H_{2,y}) = \{x; H_{2,y}(x) = |\nabla H_{2,y}(x)| = 0, D^2 H_{2,y}(x) \neq 0\}$$

is a linear subspace with $dim\mathcal{S}_2(H_{2,y}) \leq n-2$ [[11], page 10], also see [12]. Indeed, assuming $H_{2,y}(x) = \sum_{|\nu|=2} a_\nu x^\nu$, then if $z \in \mathcal{S}_2(H_{2,y})$ from $H_{2,y}(z) = |\nabla H_{2,y}(z)| = 0$ we get $H_{2,y}(x) = \sum_{|\nu|=2} a_\nu (x-z)^\nu$. Therefore,

$$H_{2,y}(x) = H_{2,y}(x+z),\ x \in R^n$$

which gives $H_{2,y}(x) = H_{2,y}(x+\lambda z)$ for $x \in R^n$ and $\lambda \in R$ (note that $H_{2,y}$ is a homogeneous harmonic polynomial). Therefore, $H_{2,y}(\lambda z) = |\nabla H_{2,y}(\lambda z)| = 0$ gives $\lambda z \in \mathcal{S}_2(H_{2,y})$. Now, it is easy to see that $\mathcal{S}_2(H_{2,y})$ is a linear subspace. To prove that $dim\mathcal{S}_2(H_{2,y}) \leq n-2$, take $d := dim\mathcal{S}_2(H_{2,y})$ then from the above fact that $H_{2,y}(x) = H_{2,y}(x+z)$, for $x \in R^n$ and $z \in \mathcal{S}_2(H_{2,y})$, $H_{2,y}$ must be a function of $n-d$ variables. But if $d = n-1$ then $H_{2,y}$ must be a second order harmonic polynomial of one variable which is impossible, so $d = n-2$. Now, define

$$\mathcal{S}_2^j(u) = \{y \in \mathcal{S}_2(u); dim\mathcal{S}_2(H_{2,y}) = j\},\ j = 0, 1, ..., n-2.$$

Let $y, y_k \in \mathcal{S}_2^j(u)$, with $y_k \to y$. Then we have

$$\begin{aligned} H_{2,y_k}(x-y_k) &= (x-y_k)^T D^2 u(y_k)(x-y_k) \to (x-y)^T D^2 u(y)(x-y) \\ &= H_{2,y}(x-y), \end{aligned} \tag{4.4}$$

uniformly in $C^2(B_1)$. Also, if we take $z_k = \frac{\overline{yy_k}}{|yy_k|} \to z$, then similar to the proof of Theorem 3.7 we can prove that $z \in \mathcal{S}_2(H_{2,y})$. Now, let $l_y := \mathcal{S}_2(H_{2,y})$, which is a j-dimensional linear subspace, then the latter fact together with (4.4) and the equality $Angle\langle w, l_y\rangle = Angle\langle \frac{w}{|w|}, l_y\rangle$, for $0 \neq w \in R^n$, show that

$$Angle\langle \overline{yy_k}, l_y\rangle \to 0.$$

Now, applying Lemma 4.2 completes the proof. $\square$

Remark 4.4. *If one can prove that a solution u of (1.2) is in $C^{[\beta]}_{loc}(D)$, then similar to the proof of Theorem 4.2 can prove that $\mathcal{S}_m(u)$ is locally countably $(n-2)$-rectifiable for $2 \le m \le [\beta]$. Indeed, for such u using Lemma 3.3 we have*

$$\begin{aligned}\mathcal{S}_m(u) &= \{x_0; u(x) = H_m(x-x_0)+O(|x-x_0|^{m+\delta}), 0 \not\equiv H_m \in \Sigma_m, \delta > 0\} \\ &= \{x_0 \in \mathcal{S}(u); D^\alpha u(x_0) = 0 \ for\ any\ |\alpha| < m, D^m u(x_0) \neq 0\}.\end{aligned}$$

Thus, using the fact that $dim\mathcal{S}_m(H_m) \le n-2$ for any non-zero homogeneous harmonic polynomial H_m of degree m and by a completely similar argument as above we can show that for any $2 \le m \le [\beta]$ there exists the following decomposition

$$\mathcal{S}_m(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^j_m(u),$$

where $\mathcal{S}^j_m(u)$ is on a countable union of j-dimensional C^1 manifolds for $j = 0, 1, ..., n-2$.

Acknowledgements

The author would like to thank an anonymous referee for his/her helpful comments and suggestions. Also, the author deeply thanks Professor Henrik Shahgholian for suggesting this problem and for many stimulating discussions, comments, suggestions and the continuous guidance and support. Moreover, this paper was prepared while the author was visiting the KTH Royal Institute of Technology, Stockholm, in the period of his sabbatical leave. It would be a pleasure to thank Iran University of Science and Technology (IUST) for its financial support and KTH for its hospitality. Also the author would like to thank Göran Gustafsson Foundation for partial support.

References

[1] H. W. Alt, L. Caffarelli and A. Friedman, Variational problems with two phases and their free boundaries, *Trans. Amer. Math. Soc.* **282** (1984), no. 2, 431–461.

[2] H. W. Alt and D. Phillips, A free boundary problem for semilinear elliptic equations, *J. Reine Angew. Math.* **368** (1986) 63–107.

[3] L. P. Bonorino, Regularity of the free boundary for some elliptic and parabolic problems, II, *Comm. Partial Differential Equations* **26** (2001), no. 3-4, 355–380.

[4] L. Caffarelli, The regularity of free boundaries in higher dimension, *Acta Math.* **139** (1977), no. 3-4, 155–184.

[5] L. Caffarelli and A. Friedman, The free boundary in the Thomas-Fermi atomic model, *J. Differential Equations* **32** (1979), no. 3, 335–356.

[6] L. A. Caffarelli, D. Jerison and C. E. Kenig, Some new monotonicity theorems with applications to free boundary problems, *Ann. Math. (2)* **155** (2002), no. 2, 369–404.

[7] L. Caffarelli and J. Salazar, Solutions of fully nonlinear elliptic equations with patches of zero gradient: Existence, regularity and convexity of level curves, *Trans. Amer. Math. Soc.* **354** (2002), no. 8, 3095–3115.

[8] A. Friedman and D. Philips, The free boundary problem of a semilinear elliptic equation, *Trans. Amer. Math. Soc.* **282** (1984) 153–182.

[9] M. Giaquinta and E. Giusti, Differentiability of minima of nondifferentiable functionals, *Invent. Math.* **72** (1983), no. 2, 285–298.

[10] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[11] Q. Han, Singular sets of solutions to elliptic equations, *Indiana Univ. Math. J.* **43** (1994), no. 3, 983–1002.

[12] Q. Han and F. H. Lin, Nodal sets of solutions of elliptic and parabolic equations, *Comm. Pure Appl. Math.* **44** (1991), no. 3, 287–308.

[13] R. Hardt, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili, Critical sets of solutions to elliptic equations, *J. Differential Geom.* **51** (1999), no. 2, 359–373.

[14] P. Hartman and A. Wintner, On the local behavior of solutions of non-parabolic partial differential equations, *Amer. J. Math.* **75** (1953) 449–476.

[15] R. Leitão, O. Queiroz and E. Teixeira, Regularity for degenerate two-phase free boundary problems, arXiv preprint arXiv:1202.5264, 2012-arxiv.org

[16] E. Lindgren and A. Petrosyan, Regularity of the free boundary in a two-phase semilinear problem in two dimensions, *Indiana Univ. Math. J.* **57** (2008), no. 7, 3397–3417.

[17] D. Phillips, A minimization problem and the regularity of solutions in the presence of a free boundary, *Indiana Univ. Math. J.* **32** (1983), no. 1, 1–17.

[18] D. Phillips, Hausdorff measure estimates of a free boundary for a minimum problem, *Comm. Partial Differential Equations* **8** (1983), no. 13, 1409–1454

[19] H. Shahgholian, $C^{1,1}$ regularity in semilinear elliptic problems, *Comm. Pure Appl. Math.* **56** (2003), no. 2, 278–281.

[20] H. Shahgholian, N. Uraltseva and G. S. Weiss, Global solutions of an obstacle-problem-like equation with two phases, *Monatsh. Math.* **142** (2004), no. 1-2, 27–34.

[21] H. Shahgholian and G. S. Weiss, The two-phase membrane problem–an intersection-comparison approach to the regularity at branch points, *Adv. Math.* **205** (2006), no. 2, 487–503.

[22] N. Uraltseva, Two-phase obstacle problem, *J. Math. Sci.* **106** (2001), no. 3, 3073–3077.

[23] G. S. Weiss, Partial regularity for weak solutions of an elliptic free boundary problem, *Comm. Partial Differential Equations* **23** (1998), no. 3-4, 439–455.

(Asadollah Aghajani) School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran

E-mail address: `aghajani@iust.ac.ir`