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**Existence of a ground state solution for a class of  $p$ -laplace equations**

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## EXISTENCE OF A GROUND STATE SOLUTION FOR A CLASS OF $p$ -LAPLACE EQUATIONS

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**ABSTRACT.** According to a class of constrained minimization problems, the Schwartz symmetrization process and the compactness lemma of Strauss, we prove that there is a nontrivial ground state solution for a class of  $p$ -Laplace equations without the Ambrosetti-Rabinowitz condition.

**Keywords:** Ground state solution,  $p$ -Laplace equation, minimization problem, the Schwartz symmetrization process.

**MSC(2010):** Primary: 35J20; Secondary: 35J60.

### 1. Introduction

In [1, 2, 5, 6, 9], the authors studied the existence of a ground state solution for the following problem

$$(1.1) \quad \begin{cases} -\Delta u + W(x)u = g(x, u) + f \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

subject to the condition that  $W > 0$ . In the case  $W < 0$ , various difficulties arise in the study of (1.1). On this subject, the existence of solutions has been studied by Ghimenti, Micheletti, Benrhouna and Ounaies in [3, 4, 8, 11] under some special conditions.

It is well known that problems involving the  $p$ -Laplacian operator appear in many areas of applied mathematics and physics. For example, they may be found in the study of non-Newtonian fluids, non-linear

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elasticity and reaction-diffusions. In [7] and [12], the authors discussed the existence of a ground state solution and the asymptotic behavior of ground states for the following equation

$$(1.2) \quad -\Delta_p u + P(|x|)u^{p-1} = Q(|x|)u^{q-1},$$

under the condition that  $P(|x|) > 0$ . In [10], Liu studied the existence of ground states for a class of more general  $p$ -Laplacian equations.

To the best of author's knowledge, not much is known about the existence of a ground state solution to (1.2) and their general versions in  $\mathbb{R}^N$  under the condition  $P(|x|) < 0$ .

In this paper, we study the existence of a ground state solution for the following problem

$$(1.3) \quad \begin{cases} -\Delta_p u - |u|^{p-2}u + |u|^{q-2}u = f(u) \\ u > 0 \\ u \in W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \end{cases}$$

where  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ ,  $N \geq 3$ ,  $1 \leq q < p < N$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function satisfying the following standard condition

$$(1.4) \quad f(s) \leq C(s^{p^*-1} + s^{p-1}),$$

for all  $s > 0$  and some constants  $C > 0$ .

Let  $F(s) = \int_0^s f(t)dt$  and

$$(1.5) \quad G(s) = \frac{1}{p}|s|^p + F(s) - \frac{1}{q}|s|^q.$$

To guarantee the existence of a solution for problem (1.3), we suppose that there exists  $\xi > 0$  such that  $G(\xi) > 0$  which is a necessary condition for existence of a solution of problem (1.3) (see [5]).

It is worth pointing out that if there exist constants  $\lambda > 0$  and  $m \in (p, p^*)$  such that  $f(s) \geq \lambda s^{m-1}$  holds for every  $s > 0$ , then  $\lambda s^{m-1} \leq f(s) \leq C(s^{p^*-1} + s^{p-1})$  and  $G(s) = \frac{1}{p}|s|^p + F(s) - \frac{1}{q}|s|^q > 0$  can be satisfied by large enough  $s > 0$ . Therefore, the hypotheses  $f(s) \leq C(s^{p^*-1} + s^{p-1})$  for all  $s > 0$  and  $G(\xi) > 0$  for some  $\xi > 0$  are reasonable. The main result of this paper is

**Theorem 1.1.** *Suppose that there exists a constant  $C > 0$  such that  $f(s) \leq C(s^{p^*-1} + s^{p-1})$  for all  $s > 0$ . If there exists  $\xi > 0$  such that  $G(\xi) > 0$ , then (1.3) possesses a nontrivial ground state solution.*

Similar to [1], our result is obtained without the Ambrosetti-Rabinowitz condition and the condition that  $\frac{f(s)}{s}$  is increasing in  $(0, \infty)$ .

## 2. Notations and preliminaries

Since we seek positive solutions, without loss of generality, we may assume that  $f(s) = 0$  for  $s \leq 0$ . In order to discuss the existence of a ground state solution for (1.3), we consider the following minimization problem

$$(2.1) \quad A = \inf \left\{ \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p : u \in W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u) = 1 \right\},$$

where  $G(s)$  is defined in (1.5) and  $F(s) = \int_0^s f(t)dt$  with  $f$  satisfying condition (1.4).

Similar to [4] and [11], we let  $E = W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ . It is obvious that  $E$  is a Banach space under the following norm

$$\|u\| = \|\nabla u\|_p + \|u\|_q,$$

where  $\|\cdot\|_r$  denotes the standard norm in  $L^r(\mathbb{R}^N)$ .

We recall that the Schwartz symmetrized function  $f^*$  of  $f \in L^1(\mathbb{R}^N)$  is a radial, nonincreasing function of  $r = |x|$  such that

$$(2.2) \quad \int_{\mathbb{R}^N} H(f)dx = \int_{\mathbb{R}^N} H(f^*)dx$$

for every continuous function  $H$  with  $H(f)$  is integrable (for more details, please see [5]). Since (1.3) is an autonomous problem, by (2.2) we conclude that under the Schwartz symmetrization process we can minimize problem (2.1) on the space  $E_{rad}$ , the subspace of  $E$  formed by radially symmetric functions. Furthermore, according to the same method as in [5], we can easily prove that the set  $\{u \in W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) : \int_{\mathbb{R}^N} G(u) = 1\}$  is not empty.

## 3. Some lemmas

To prove Theorem 1.1, we need to establish some useful lemmas.

**Lemma 3.1.** *There exists a constant  $d > 0$  such that for any  $u \in E$  we have*

$$\frac{1}{q} \|u\|_q^q \geq \left(C + \frac{2}{p}\right) \|u\|_p^p - d \|u\|_{p^*}^{p^*},$$

where  $p^* = \frac{pN}{N-p} > p > q$ .

*Proof.* Consider the following function

$$h(s) = \frac{(C + \frac{2}{p})|s|^p - \frac{1}{q}|s|^q}{|s|^{p^*}}, \quad s \neq 0.$$

We observe that if  $0 < |s| < (\frac{1}{q(C+\frac{2}{p})})^{\frac{1}{p-q}}$ , then  $h(s) < 0$ . On the other hand, since  $p^* = \frac{pN}{N-p} > p > q$ , we have  $\lim_{|s| \rightarrow +\infty} h(s) = 0$ . Therefore we conclude that there exists  $d > 0$  such that

$$(3.1) \quad (C + \frac{2}{p})|s|^p - \frac{1}{q}|s|^q \leq d|s|^{p^*}.$$

Putting  $s = |u|$  in (3.1) and then integrating, the lemma is proved.  $\square$

**Lemma 3.2.** *Any minimizing sequence  $\{u_n\}$  for (2.1) is bounded in  $E_{rad}$ .*

*Proof.* If  $\{u_n\}$  is a minimizing sequence for (2.1), then we have

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p = A \text{ and } \int_{\mathbb{R}^N} G(u_n) = 1.$$

By (1.4), we obtain

$$(3.3) \quad F(s) = \int_0^s f(t)dt \leq C(s^{p^*} + s^p).$$

According to (1.5), (3.2) and (3.3), we get

$$(3.4) \quad 1 \leq \frac{1}{p} \|u_n\|_p^p + C \|u_n\|_p^p + C \|u_n\|_{p^*}^{p^*} - \frac{1}{q} \|u_n\|_q^q.$$

By Lemma 3.1 and (3.4), we get

$$(3.5) \quad 1 + \frac{1}{p} \|u_n\|_p^p \leq (C + d) \|u_n\|_{p^*}^{p^*}.$$

Since  $\lim_{n \rightarrow \infty} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p = A$ , then  $\int_{\mathbb{R}^N} |\nabla u_n|^p$  is bounded. By the Gagliardo-Nirenberg inequality we conclude that  $\|u_n\|_{p^*}^{p^*}$  is also bounded. Thus, it follows from (3.5) that  $\|u_n\|_p^p$  is bounded. By (3.4),  $\|u_n\|_q^q$  is bounded, and consequently, we conclude that  $\{u_n\}$  is bounded in  $E_{rad}$ .  $\square$

**Lemma 3.3.** *The number  $A$  given by (2.1) is positive, that is,  $A > 0$ .*

*Proof.* From the definition of  $A$ , it is clear that  $A \geq 0$ . Assume by contradiction that  $A = 0$ . Similar to [1], we let  $\{u_n\}$  be a minimizing sequence in  $E_{rad}$  to  $A = 0$ , then we have

$$\lim_{n \rightarrow \infty} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p = 0 \text{ and } \int_{\mathbb{R}^N} G(u_n) = 1.$$

Therefore, by the Gagliardo-Nirenberg inequality we conclude that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{p^*} = 0.$$

On the other hand, by (3.5) we have  $\|u_n\|_{p^*}^{p^*} \geq \frac{1}{C+d}$ . Therefore, we get a contradiction which means that  $A > 0$ .  $\square$

**Lemma 3.4.** ([5]) *If  $u \in L^p(\mathbb{R}^N)$ , and  $1 \leq p < +\infty$  is a radial nonincreasing function, then*

$$|u(x)| \leq |x|^{-\frac{N}{p}} \left( \frac{N}{|S^{N-1}|} \right)^{\frac{1}{p}} \|u\|_p, \quad x \neq 0,$$

where  $|S^{N-1}|$  is the volume of the unit sphere in  $\mathbb{R}^N$ .

**Lemma 3.5.** *The number  $A$  given by (2.1) is attained by some functions in the following set*

$$W = \{u \in W^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) : \int_{\mathbb{R}^N} G(u) = 1\}.$$

*Proof.* Let  $\{u_n\} \subset E_{rad}$  be a minimizing sequence for (2.1). By Lemma 3.2, we conclude that there is a subsequence of  $\{u_n\}$ , we also denoted  $\{u_n\}$  such that  $\{u_n\}$  converges weakly in  $E$  almost everywhere in  $\mathbb{R}^N$  to a function  $u \in E$ . Since every  $u_n$  is radial, nonnegative and nonincreasing with  $r = |x|$ , then  $u$  is radial, nonnegative and nonincreasing with  $r = |x|$ . Note that  $u_n \in L^q(\mathbb{R}^N)$ , and by Lemma 3.4 we have

$$(3.6) \quad |u_n(x)| \leq |x|^{-\frac{N}{q}} \left( \frac{N}{|S^{N-1}|} \right)^{\frac{1}{q}} \|u_n\|_q.$$

Since  $\|u_n\|_q^q$  is bounded, by (3.6) we conclude that there exists a constant  $b > 0$  such that  $|u_n(x)| \leq b|x|^{-\frac{N}{q}}$ . Therefore, we have

$$(3.7) \quad |u_n(x)|^p \leq b^p |x|^{-\frac{pN}{q}} \text{ and } |u_n(x)|^{p^*} \leq b^{p^*} |x|^{-\frac{p^*N}{q}}.$$

Since  $p > q$  and  $p^* > q$ , we have  $|x|^{-\frac{pN}{q}} \in L^1(\mathbb{R}^N)$  and  $|x|^{-\frac{p^*N}{q}} \in L^1(\mathbb{R}^N)$ . Thus, by (3.7) we get

$$(3.8) \quad F(u_n) \leq C(|u_n|^{p^*} + |u_n|^p) \leq C(b^p |x|^{-\frac{pN}{q}} + b^{p^*} |x|^{-\frac{p^*N}{q}}) \in L^1(\mathbb{R}^N).$$

Since  $\{u_n\}$  converges almost everywhere in  $\mathbb{R}^N$  to  $u$  and  $F$  is continuous, then we have  $F(u_n) \rightarrow F(u)$  almost everywhere. Therefore, by (3.8) and Lebesgue's dominated convergence theorem we obtain

$$(3.9) \quad F(u_n) \rightarrow F(u) \text{ in } L^1(\mathbb{R}^N).$$

On the other hand, since  $\|u_n\|_q^q$  and  $\|u_n\|_{p^*}^{p^*}$  are bounded,

$$(3.10) \quad \sup_n \int_{\mathbb{R}^N} (|u_n|^q + |u_n|^{p^*}) < +\infty.$$

By (3.6), we have  $u_n(x) \rightarrow 0$  as  $|x| \rightarrow +\infty$ , uniformly with respect to  $n$ . It follows from  $p^* > p > q \geq 1$  that

$$(3.11) \quad \lim_{|s| \rightarrow 0} \frac{|s|^p}{|s|^q + |s|^{p^*}} = \lim_{|s| \rightarrow 0} \frac{|s|^{p-q}}{1 + |s|^{p^*-q}} = 0,$$

and

$$(3.12) \quad \lim_{|s| \rightarrow +\infty} \frac{|s|^p}{|s|^q + |s|^{p^*}} = 0.$$

Since  $|u_n|^p$  converges to  $|u|^p$  almost everywhere in  $\mathbb{R}^N$ , by (3.10), (3.11), (3.12) and the compactness lemma of Strauss we conclude that

$$(3.13) \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} |u_n|^p = \int_{\mathbb{R}^N} |u|^p.$$

By (1.5), (3.9), (3.13) and Fatou's lemma, we have

$$(3.14) \quad 1 \leq \frac{1}{p} \int_{\mathbb{R}^N} |u|^p + \int_{\mathbb{R}^N} F(u) - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q.$$

The inequality (3.14) means that  $\int_{\mathbb{R}^N} G(u) \geq 1$ . If  $u$  is not in  $W$ , one should have

$$(3.15) \quad \int_{\mathbb{R}^N} G(u) > 1.$$

Similar to [1], we define a function  $h : [0, 1] \rightarrow \mathbb{R}$  as  $h(t) = \int_{\mathbb{R}^N} G(tu)$ . It is obvious that  $h$  is continuous. Since  $G(tu) = \frac{1}{p}|tu|^p + F(tu) - \frac{1}{q}|tu|^q$ ,  $F(tu) \leq C(|tu|^{p^*} + |tu|^p)$  and  $p^* > p > q \geq 1$ , we conclude that  $h(t) < 1$  for  $t$  close to 0. By (3.15), we have  $h(1) > 1$ . Therefore, there exists  $t_0 \in (0, 1)$  such that  $h(t_0) = 1$ , which means that  $t_0u \in W$ . On the other hand, since the minimizing sequence  $\{u_n\}$  for (2.1) converges weakly to  $u$ , then

$$(3.16) \quad \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \leq \liminf_{n \rightarrow +\infty} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p = A.$$

Since  $t_0 \in (0, 1)$  and  $t_0u \in W$ , by (3.16) we have

$$A \leq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla t_0u|^p = \frac{t_0^p}{p} \int_{\mathbb{R}^N} |\nabla u|^p < A.$$

This is a contradiction. Therefore,  $u \in W$  and  $\frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p = A$ . □

Let  $T(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p$  and  $V(u) = \int_{\mathbb{R}^N} G(u)$ . It is well known that  $T$  and  $V$  are  $C^1$  functionals on  $E$ .

**Lemma 3.6.** *Suppose that  $J(w) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla w|^p - \int_{\mathbb{R}^N} H(w)$  is a  $C^1$  function on a suitable Banach space. If  $u$  is a critical point of  $J$ , then*

$$(3.17) \quad (N - p) \int_{\mathbb{R}^N} |\nabla u|^p = pN \int_{\mathbb{R}^N} H(u).$$

*Proof.* Let  $\sigma > 0$  and

$$u_\sigma = u\left(\frac{x}{\sigma}\right) = u\left(\frac{x_1}{\sigma}, \frac{x_2}{\sigma}, \dots, \frac{x_N}{\sigma}\right) = u(y_1, y_2, \dots, y_N).$$

Direct calculation shows that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_\sigma|^p dx &= \frac{1}{\sigma^p} \int_{\mathbb{R}^N} \left\{ \left(\frac{\partial u}{\partial y_1}\right)^2 + \left(\frac{\partial u}{\partial y_2}\right)^2 + \dots + \left(\frac{\partial u}{\partial y_N}\right)^2 \right\}^{\frac{p}{2}} dx \\ &= \frac{1}{\sigma^p} \int_{\mathbb{R}^N} \sigma^N \left\{ \left(\frac{\partial u}{\partial y_1}\right)^2 + \left(\frac{\partial u}{\partial y_2}\right)^2 + \dots + \left(\frac{\partial u}{\partial y_N}\right)^2 \right\}^{\frac{p}{2}} dy \\ &= \sigma^{N-p} \int_{\mathbb{R}^N} |\nabla u|^p. \end{aligned}$$

Similarly, we have  $\int_{\mathbb{R}^N} H(u_\sigma) = \sigma^N \int_{\mathbb{R}^N} H(u)$ . Thus, we obtain

$$J(u_\sigma) = \frac{\sigma^{N-p}}{p} \int_{\mathbb{R}^N} |\nabla u|^p - \sigma^N \int_{\mathbb{R}^N} H(u).$$

Since  $u$  is a critical point of  $J$ , then  $\frac{d}{d\sigma}|_{\sigma=1} J(u_\sigma) = 0$ , which means that (3.17) holds. □

**Lemma 3.7.** *If  $u$  is a solution of (1.3), then  $S(u) = \frac{1}{N}T(u) > 0$ , where  $S(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p - \int_{\mathbb{R}^N} G(u)$ ,  $T(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p$ .*

*Proof.* By Lemma 3.6, we have

$$S(u) = \frac{1}{p} \left(1 - \frac{N-p}{N}\right) T(u) = \frac{1}{N} T(u) > 0.$$

□



#### 4. The proof of Theorem 1.1

*Proof.* Suppose that  $u_n, u, V$  and  $T$  are functions defined in Section 3. Since  $V$  and  $T$  are  $C^1$  functionals on  $E$ , there exists a Lagrange multiplier  $\theta$  such that  $T'(u) = \theta V'(u)$ . If  $\theta = 0$  or  $V'(u) = 0$ , then  $A = 0$  which contradicts Lemma 3.3. Therefore,  $\theta \neq 0$  and  $V'(u) \neq 0$ . Choose a function  $w \in C_0^\infty(\mathbb{R}^N)$  such that  $\langle V'(u), w \rangle > 0$ . It is obvious that  $V(u + \varepsilon w) = V(u) + \varepsilon \langle V'(u), w \rangle + o(\varepsilon)$  and

$$T(u + \varepsilon w) = T(u) + \varepsilon \theta \langle V'(u), w \rangle + o(\varepsilon) \text{ for } \varepsilon \rightarrow 0.$$

If  $\theta < 0$ , then one can find  $\varepsilon > 0$  small enough so that  $v = u + \varepsilon w$  satisfies  $V(v) > V(u) = 1$  and  $T(v) < T(u) = A$ . Therefore, there exists  $\sigma \in (0, 1)$  such that  $v_\sigma = v(\frac{x}{\sigma})$  satisfies  $V(v_\sigma) = 1$  and  $T(v_\sigma) < A$ , which is impossible. Hence  $\theta > 0$ . Thus  $u$  satisfies, at least in the distribution sense, the equation

$$-\Delta_p u = \theta(|u|^{p-2}u - |u|^{q-2}u + f(u)) \text{ in } \mathbb{R}^N.$$

Set  $u_\sigma = u(\frac{x}{\sigma})$ . Direct calculation shows that  $\nabla u_\sigma = \frac{1}{\sigma} \nabla u$  and  $|\nabla u_\sigma|^{p-2} = \frac{1}{\sigma^{p-2}} |\nabla u|^{p-2}$ . Therefore, we have

$$\Delta_p u_\sigma = |\nabla u_\sigma|^{p-2} \Delta u_\sigma + (p-2) |\nabla u_\sigma|^{p-3} \nabla u_\sigma \cdot \nabla |\nabla u_\sigma| = \frac{1}{\sigma^p} \Delta_p u.$$

Thus, we conclude that  $u(\frac{x}{\sqrt{\theta}}) = u_{\sqrt{\theta}}$  is a solution of problem (1.3). Using Lemma 3.6 and Lemma 3.7, similar to the method in the proof of Theorem 3 in [5], we have

$$0 < S(u_{\sqrt{\theta}}) \leq S(v),$$

where  $v$  is any solution of problem (1.3). Therefore,  $u_{\sqrt{\theta}}$  is a ground state solution of problem (1.3). □

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