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NON EXISTENCE OF TOTALLY CONTACT UMBILICAL SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE SASAKIAN MANIFOLDS

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Abstract. We prove that there do not exist totally contact umbilical proper slant lightlike submanifolds of indefinite Sasakian manifolds other than totally contact geodesic proper slant lightlike submanifolds. We also prove that there do not exist totally contact umbilical proper slant lightlike submanifolds of indefinite Sasakian space forms.
Keywords: Slant lightlike submanifolds, totally contact umbilical lightlike submanifolds, totally contact geodesic lightlike submanifolds, indefinite Sasakian manifolds.
MSC(2010): Primary: 53C15; Secondary: 53C40, 53C50.

1. Introduction

In the theory of submanifolds of semi-Riemannian manifolds, it is interesting to study the theory of lightlike (degenerate) submanifolds due to the fact that in case of lightlike submanifolds, the intersection of normal vector bundle and the tangent bundle is non-trivial. Thus the study of lightlike submanifolds becomes more interesting and remarkably different from the study of non-degenerate submanifolds. Moreover, limited information available on the general theory of lightlike submanifolds makes this theory a topic of main discussion in the present scenario since it was initiated by Duggal and Bejancu in [7]. Chen [5, 6], introduced the notion of slant submanifolds as a generalization of holomorphic and

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totally real submanifolds for complex geometry and the subject was further extended by Lotta [10] for contact geometry. Cabrerizo et al. [3, 4] studied slant, semi-slant and bi-slant submanifolds in contact geometry. They all studied the geometry of slant submanifolds with positive definite metric therefore this geometry may not be applicable to the other branches of the mathematics and physics, where the metric is not necessarily definite. Thus the notion of slant lightlike submanifolds of indefinite Hermitian manifolds was introduced by Sahin [14]. Since there are significant applications of contact geometry in differential equations, optics, and phase spaces of a dynamical system (see Arnold [1], Maclane [11], Nazaikinskii et al. [12]) the notion of slant lightlike submanifolds of indefinite Sasakian manifolds was introduced by Sahin and Yildirim in [15] in which necessary and sufficient conditions for their existence is obtained.

In the present paper, we study the theory of slant lightlike submanifolds of indefinite Sasakian manifolds and prove that there do not exist totally contact umbilical proper slant lightlike submanifolds of indefinite Sasakian manifolds other than totally contact geodesic proper slant lightlike submanifolds (Theorem 4.5). We also prove that there do not exist totally contact umbilical proper slant lightlike submanifolds of indefinite Sasakian space forms (Theorem 4.6).

2. **Lightlike submanifolds**

An odd-dimensional semi-Riemannian manifold $\bar{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $(\phi, V, \eta, \bar{g})$, where ϕ is a $(1,1)$ tensor field, V is a vector field called the characteristic vector field, η is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$, satisfying

$$\phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1, \tag{2.1}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon\eta(X)\eta(Y) \quad \bar{g}(X, V) = \epsilon\eta(X), \tag{2.2}$$

for $X, Y \in \Gamma(T\bar{M})$, where $\epsilon = \pm 1$ and $T\bar{M}$ denotes the Lie algebra of vector fields on $\bar{M}$.

An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite Sasakian manifold if (see [9]),

$$(\bar{\nabla}_X \phi)Y = -\bar{g}(X, Y)V + \epsilon\eta(Y)X \text{ and } \bar{\nabla}_X V = \phi X, \tag{2.3}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$.

A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called an r-lightlike submanifold [7] if it admits a degenerate metric g induced from $\bar{g}$, whose radical distribution $RadTM = TM \cap TM^{\perp}$ is of rank r, where $0 < r < min\{m, n\}$. Let the screen distribution $S(TM)$ be a semi-Riemannian complementary distribution of $RadTM$ in TM, that is,

$$TM = RadTM \perp S(TM), \tag{2.4}$$

and $S(TM^{\perp})$ be the screen transversal vector bundle, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. For any local basis $\{\xi_i\}$ of $RadTM$, there exists a null vector bundle $ltr(TM)$ of $RadTM$ in $(S(TM))^{\perp}$ such that $\{N_i\}$ is a basis of $ltr(TM)$ satisfying

$$\bar{g}(N_i, N_j) = 0 \quad \text{and} \quad \bar{g}(N_i, \xi_j) = \delta_{ij}, \tag{2.5}$$

for any $i, j \in \{1, 2, ..., r\}$. Let $tr(TM)$ be the complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$, then

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.6}$$

and

$$\begin{aligned} T\bar{M}\,|_M &= TM \oplus tr(TM) \\ &= (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \end{aligned} \tag{2.7}$$

Let $\bar{\nabla}$ and ∇ denote the linear connections on $\bar{M}$ and M, respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M. The second fundamental form h is a symmetric bilinear form on $\Gamma(TM)$ and the shape operator A_U is a linear endomorphism of $\Gamma(TM)$.

Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then using (2.6), the Gauss and Weingarten formulae become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.9}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.10}$$

where we put $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l_X U = L(\nabla^{\perp}_X U)$ and $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, they are known as the lightlike second fundamental form and the screen second fundamental form on M. In particular, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.12}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2.6)-(2.7) and (2.9)-(2.12), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$. Let $\bar{P}$ be a projection of TM on $S(TM)$. Then using the decomposition $TM = RadTM \perp S(TM)$, we can write

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X \bar{P}Y,\ A^*_\xi X\}$ and $\{h^*(X, \bar{P}Y),\ \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. Here ∇^* and ∇^{*t} are linear connections on $S(TM)$ and $RadTM$, respectively. By using (2.9), (2.11) and (2.14), we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y). \tag{2.15}$$

3. **Slant lightlike submanifolds**

A lightlike submanifold has two distributions, namely the radical distribution and the screen distribution. The radical distribution is totally lightlike and it is not possible to define an angle between two vector fields of the radical distribution, where the screen distribution is non-degenerate. There are some definitions for the angle between two vector fields in Lorentzian setup [13], which is not appropriate for our goal. Therefore to introduce the notion of slant lightlike submanifolds one needs a Riemannian distribution. For such a distribution, Sahin and Yildirim [15] proved the following lemmas.

Lemma 3.1. *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then $\phi ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi ltr(TM) \cap \phi RadTM = \{0\}$.*

Lemma 3.2. *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2r$. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then any complementary distribution to $\phi ltr(TM) \oplus \phi RadTM$ in screen distribution S(TM) is Riemannian.*

Definition 3.3. *([15]) Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2r$. Then M is said to be a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:*

(A) *$RadTM$ is a distribution on M such that $\phi RadTM \cap RadTM = \{0\}$.*

(B) *For each nonzero vector field X tangent to $\bar{D} = D \perp \{V\}$ at $x \in U \subset M$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $\bar{D}_x$ is constant, that is, it is independent of the choice of $x \in U \subset M$ and $X \in \bar{D}_x$, where $\bar{D}$ is the complementary distribution to $\phi ltr(TM) \oplus \phi RadTM$ in screen distribution $S(TM)$.*

The constant angle $\theta(X)$ is called the slant angle of the distribution $\bar{D}$. A slant lightlike submanifold M is said to be proper if $\bar{D} \neq \{0\}$ and $\theta \notin \{0, \frac{\pi}{2}\}$.

Since a submanifold M is invariant (respectively, anti-invariant) if $\phi T_pM \subset T_pM$, (respectively, $\phi T_pM \subset T_pM^{\perp}$), for any $p \in M$. Using the above definition, it is clear that M is invariant (respectively, anti-invariant) if $\theta(X) = 0$, (respectively, $\theta(X) = \frac{\pi}{2}$).

For a slant lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$, the tangent bundle TM is decomposed as

$$TM = RadTM \perp (\phi RadTM \oplus \phi ltr(TM)) \perp \bar{D}, \tag{3.1}$$

where $\bar{D} = D \perp \{V\}$. Therefore, for any $X \in \Gamma(TM)$, we write

$$\phi X = TX + FX, \tag{3.2}$$

where TX is the tangential component of ϕX and FX is the transversal component of ϕX. Similarly, for any $U \in \Gamma(tr(TM))$, we write

$$\phi U = BU + CU, \tag{3.3}$$

where BU is the tangential component of ϕU and CU is the transversal component of ϕU. Using the decomposition in (3.1), we denote by P_1, P_2, Q_1, Q_2 and $\bar{Q}_2$ the projections on the distributions $RadTM$,

$\phi RadTM$, $\phi ltr(TM)$, D and $\bar{D} = D \perp V$, respectively. Then for any $X \in \Gamma(TM)$, we can write

$$X = P_1X + P_2X + Q_1X + \bar{Q}_2X, \tag{3.4}$$

where $\bar{Q}_2X = Q_2X + \eta(X)V$. Applying ϕ to (3.4), we obtain

$$\phi X = \phi P_1X + \phi P_2X + FQ_1X + TQ_2X + FQ_2X. \tag{3.5}$$

Then using (3.2) and (3.3), we get

$$\phi P_1X = TP_1X \in \Gamma(\phi RadTM), \quad \phi P_2X = TP_2X \in \Gamma(RadTM), \tag{3.6}$$

$$FP_1X = FP_2X = 0, \quad TQ_2X \in \Gamma(D), \quad FQ_1X \in \Gamma(ltr(TM)). \tag{3.7}$$

Lemma 3.4. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $FQ_2X \in \Gamma(S(TM^{\perp}))$, for any $X \in \Gamma(TM)$.*

Proof. Using (2.5) and (2.6), it is clear that $FQ_2X \in \Gamma(S(TM^{\perp}))$ if $g(FQ_2X, \xi) = 0$, for any $\xi \in \Gamma(RadTM)$. Therefore $g(FQ_2X, \xi) = g(\phi Q_2X - TQ_2X, \xi) = g(\phi Q_2X, \xi) = -g(Q_2X, \phi\xi) = 0$, implies the result.

Thus from Lemma (3.4), it follows that $F(D_p)$ is a subspace of $S(TM^{\perp})$. Hence, there exists an *invariant* subspace μ_p of $T_p\bar{M}$ such that

$$S(T_pM^{\perp}) = F(D_p) \perp \mu_p, \tag{3.8}$$

and consequently,

$$T_p\bar{M} = S(T_pM) \perp \{Rad(T_pM) \oplus ltr(T_pM)\} \perp \{F(D_p) \perp \mu_p\}. \tag{3.9}$$

Now, differentiating (3.5) and using (2.9)-(2.12), (3.2) and (3.3), for any $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla_X T)Y &= A_{FQ_1Y}X + A_{FQ_2Y}X + Bh(X,Y)\\ &\quad -g(X,Y)V + \epsilon\eta(Y)X,\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}D^s(X, FQ_1Y) + D^l(X, FQ_2Y) &= F\nabla_X Y - h(X, TY) + Ch^s(X,Y)\\ &\quad -\nabla^s_X FQ_2Y - \nabla^l_X FQ_1Y.\end{aligned} \tag{3.11}$$

□

Using the Sasakian property of $\bar{\nabla}$ with (2.8), we have the following lemmas.

Lemma 3.5. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then*

$$(\nabla_X T)Y = A_{FY}X + Bh(X,Y) - g(X,Y)V + \epsilon\eta(Y)X, \tag{3.12}$$

$$(\nabla^t_X F)Y = Ch(X,Y) - h(X,TY), \tag{3.13}$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (\nabla^t_X F)Y = \nabla^t_X FY - F\nabla_X Y. \tag{3.14}$$

Lemma 3.6. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then*

$$(\nabla_X B)U = A_{CU}X - TA_U X - g(X,U)V,$$
$$(\nabla^t_X C)U = -FA_U X - h(X, BU),$$

where $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$ and

$$(\nabla_X B)U = \nabla_X BU - B\nabla^t_X U, \quad (\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U. \tag{3.15}$$

Theorem 3.7. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then:*

(A) *The distribution $\bar{D}$ is integrable, if and only if, $h(X,TY) = h(Y,TX)$, $D^l(X, FQ_2Y) = D^l(Y, FQ_2X)$ and $\nabla^s_X FQ_2Y = \nabla^s_Y FQ_2X$, for any $X, Y \in \Gamma(\bar{D})$.*

(B) *The distribution $\phi ltr(TM)$ is integrable, if and only if, $A_{FQ_1Y}X = A_{FQ_1X}Y$, for any $X, Y \in \Gamma(\phi ltr(TM))$.*

Proof. Let $X, Y \in \Gamma(\bar{D})$ then using (3.11), we have $F\nabla_X Y = D^l(X, FQ_2Y) + h(X,TY) + \nabla^s_X FQ_2Y - Ch^s(X,Y)$. Here replacing X by Y and then subtracting the resulting equation from this equation, we get (A). Next, let $X, Y \in \Gamma(\phi ltr(TM))$ then using (3.12) and (3.14), we have

$$-T\nabla_X Y = A_{FQ_1Y}X + Bh(X,Y),$$

Then, similarly as above, we have $T[X,Y] = A_{FQ_1X}Y - A_{FQ_1Y}X$, that completes the proof of (B). □

In [15], Sahin and Yildirim proved the following theorem.

Theorem 3.8. *Let M be a lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is a slant lightlike submanifold, if and only if,*

(i) *$\phi(RadTM)$ is a distribution on M such that $\phi RadTM \cap RadTM = \{0\}$.*

(ii) $\bar{D} = \{X \in \Gamma(\bar{D}) : T^2X = -\lambda(X - \eta(X)V)\}$ *is a distribution such that it is complementary to* $\phi ltr(TM) \oplus \phi RadTM$*, where* $\lambda = -cos^2\theta$.

4. Totally contact umbilical slant lightlike submanifolds

Definition 4.1. *([16]). If the second fundamental form h of a submanifold, tangent to characteristic vector field V, of a Sasakian manifold $\bar{M}$ is of the form*

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y,V) + \eta(Y)h(X,V), \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called a totally contact umbilical and totally contact geodesic if $\alpha = 0$.

The above definition also holds for a lightlike submanifold M. For a totally contact umbilical lightlike submanifold M, we have

$$h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_L + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V), \tag{4.2}$$

$$h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha_S + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \tag{4.3}$$

where $\alpha_L \in \Gamma(ltr(TM))$ and $\alpha_S \in \Gamma(S(TM^\perp))$.

Lemma 4.2. *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then*

$$g(T\bar{Q}_2X, T\bar{Q}_2Y) = cos^2\theta[g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)], \tag{4.4}$$

and

$$g(F\bar{Q}_2X, F\bar{Q}_2Y) = sin^2\theta[g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)], \tag{4.5}$$

for any $X, Y \in \Gamma(TM)$.

Proof. From (2.2) and (3.2), we obtain

$$g(T\bar{Q}_2X, T\bar{Q}_2Y) = -g(\bar{Q}_2X, T^2\bar{Q}_2Y),$$

for any $X, Y \in \Gamma(TM)$. Then using the Theorem (3.8), assertions follows. □

Lemma 4.3. *Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $g(\nabla_X X, \phi\xi) = 0$, for any $X \in \Gamma(D)$ and $\xi \in \Gamma(RadTM)$.*

Proof. Let $X \in \Gamma(D)$, implying $X = Q_2X$. Then using (2.3), (2.9) and (2.12), for a totally contact umbilical slant lightlike submanifold, we have

$$\begin{aligned} g(\nabla_X X, \phi\xi) &= \bar{g}(\bar{\nabla}_X X, \phi\xi) = -\bar{g}(\bar{\nabla}_X TQ_2X, \xi) - \bar{g}(\bar{\nabla}_X FQ_2X, \xi) \\ &= -g(h^l(X, TQ_2X), \xi) - \bar{g}(D^l(X, FQ_2X), \xi) \\ &= -\bar{g}(D^l(X, FQ_2X), \xi), \end{aligned} \tag{4.6}$$

since for $X \in \Gamma(D)$, using (2.2), (3.2) and (4.2), we have $h^l(X, TQ_2X) = \{g(X, TQ_2X)\}\alpha_L = 0$. Moreover, $\eta(Q_2X) = 0$, $\eta(\xi) = 0$ and by replacing W by FQ_2X, Y by ξ in (2.13) and then using that M is a totally contact umbilical slant lightlike submanifold, we obtain

$$\begin{aligned} \bar{g}(D^l(X, FQ_2X), \xi) &= -\bar{g}(h^s(X, \xi), FQ_2X) \\ &= -g(X, \xi)g(\alpha_S, FQ_2X) \\ &= 0. \end{aligned} \tag{4.7}$$

Hence from (4.6) and (4.7), the result follows. □

Theorem 4.4. *Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then at least one of the following statements is true:*

(i) *M is an anti-invariant submanifold.*
(ii) *$D = \{0\}$.*
(iii) *If M is a proper slant lightlike submanifold, then $\alpha_S \in \Gamma(\mu)$.*

Proof: Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$, then for any $X = Q_2X \in \Gamma(D)$ with (4.1), we have

$$h(TQ_2X, TQ_2X) = g(TQ_2X, TQ_2X)\alpha,$$

while from (2.8), (4.4) and above equation, we get

$$\bar{\nabla}_{TQ_2X}TQ_2X - \nabla_{TQ_2X}TQ_2X = cos^2\theta[g(Q_2X, Q_2X)]\alpha.$$

Using (3.2) and the fact that $\bar{M}$ is a Sasakian manifold, we obtain

$$\phi\bar{\nabla}_{TQ_2X}Q_2X - g(TQ_2X, TQ_2X)V - \bar{\nabla}_{TQ_2X}FQ_2X - \nabla_{TQ_2X}TQ_2X$$
$$= cos^2\theta[g(Q_2X, Q_2X)]\alpha.$$

Then using (2.9)-(2.12) and (4.4), we get

$$\phi\nabla_{TQ_2X}Q_2X + \phi h^l(TQ_2X, X) + \phi h^s(TQ_2X, X) + A_{FQ_2X}TQ_2X$$
$$-\nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X) - \nabla_{TQ_2X}TQ_2X$$
$$= cos^2\theta[g(Q_2X, Q_2X)](\alpha + V).$$

Thus, using (3.2), (3.3), (4.2) and (4.3), we have

$$T\nabla_{TQ_2X}Q_2X + F\nabla_{TQ_2X}Q_2X + g(TQ_2X, X)\phi\alpha^l + g(TQ_2X, X)B\alpha^s$$
$$+g(TQ_2X), X)C\alpha^s + A_{FQ_2X}TQ_2X - \nabla^s_{TQ_2X}FQ_2X$$
$$-D^l(TQ_2X, FQ_2X) - \nabla_{TQ_2X}TQ_2X = cos^2\theta[g(Q_2X, Q_2X)](\alpha + V),$$

equating the transversal components, we get

$$F\nabla_{TQ_2X}Q_2X + g(TQ_2X, X)C\alpha^s - \nabla^s_{TQ_2X}FQ_2X$$
$$-D^l(TQ_2X, FQ_2X) = cos^2\theta[g(Q_2X, Q_2X)]\alpha. \tag{4.8}$$

On the other hand, (4.5) holds for any $X = Y \in \Gamma(D)$ and taking the covariant derivative with respect to TQ_2X, we obtain

$$g(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) = sin^2\theta g(\nabla_{TQ_2X}Q_2X, Q_2X). \tag{4.9}$$

Now, taking the inner product of (4.8) with FQ_2X, we obtain

$$g(F\nabla_{TQ_2X}Q_2X, FQ_2X) - g(\nabla^s_{TQ_2X}FQ_2X, FQ_2X)$$
$$= cos^2\theta[g(Q_2X, Q_2X)]g(\alpha_S, FQ_2X).$$

Then using (4.5) and (4.9), we get

$$cos^2\theta[g(Q_2X, Q_2X)]g(\alpha_S, FQ_2X) = 0. \tag{4.10}$$

Thus, from (4.10), it follows that either $\theta = \frac{\pi}{2}$ or $Q_2X = 0$ or $\alpha_S \in \Gamma(\mu)$. This completes the proof.

Theorem 4.5. *Every totally contact umbilical proper slant lightlike submanifold of an indefinite Sasakian manifold is totally contact geodesic.*

Proof. Since M is a totally contact umbilical slant lightlike submanifold for any $X = Q_2X \in \Gamma(D)$, using (4.1), we have $h(TQ_2X, TQ_2X)$ $= g(TQ_2X, TQ_2X)\alpha$, and using (4.4), we get

$$h(TQ_2X, TQ_2X) = cos^2\theta[g(Q_2X, Q_2X) - \eta(Q_2X)\eta(Q_2X)]\alpha$$
$$= cos^2\theta[g(Q_2X, Q_2X)]\alpha. \tag{4.11}$$

Using (2.1) and (3.11) for any $X \in \Gamma(D)$, we obtain

$$\begin{aligned} h(TQ_2X, TQ_2X) &= F\nabla_{TQ_2X}X + Ch(TQ_2X, X) - \nabla^s_{TQ_2X}FQ_2X \\ &\quad -D^l(TQ_2X, FQ_2X). \end{aligned} \tag{4.12}$$

Since M is a totally contact umbilical slant lightlike submanifold $Ch(TQ_2X, X) = g(TQ_2X, X)C\alpha = 0$, and using (4.11) and (4.12), we get

$$\begin{aligned} cos^2\theta[g(Q_2X, Q_2X)]\alpha &= F\nabla_{TQ_2X}X - \nabla^s_{TQ_2X}FQ_2X \\ &\quad -D^l(TQ_2X, FQ_2X). \end{aligned} \tag{4.13}$$

Taking the scalar product of both sides of (4.13) with respect to FQ_2X, we obtain

$$\begin{aligned} cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) &= \bar{g}(F\nabla_{TQ_2X}X, FQ_2X) \\ &\quad -\bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X), \end{aligned} \tag{4.14}$$

then using (4.5), we get

$$\begin{aligned} cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) &= sin^2\theta[g(\nabla_{TQ_2X}X, Q_2X)] \\ &\quad -\bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X). \end{aligned} \tag{4.15}$$

Now, for any $X = Q_2X \in \Gamma(D)$, (4.5) implies that

$$g(FQ_2X, FQ_2X) = sin^2\theta[g(Q_2X, Q_2X)].$$

Taking covariant derivative of this equation with respect to $\bar{\nabla}_{TQ_2X}$, we get

$$\bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) = sin^2\theta[g(\nabla_{TQ_2X}Q_2X, Q_2X)]. \tag{4.16}$$

Using (4.16) in (4.15), we obtain

$$cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) = 0. \tag{4.17}$$

Since M is a proper slant lightlike submanifold and g is a Riemannian metric on D we have $\bar{g}(\alpha_s, FQ_2X) = 0$. Thus, using Lemma (3.4) and the equation (3.8), we obtain

$$\alpha_S \in \Gamma(\mu). \tag{4.18}$$

Now, using the Sasakian property of $\bar{M}$, we have $\bar{\nabla}_X\phi Y = \phi\bar{\nabla}_XY - g(X, Y)V$, for any $X, Y \in \Gamma(D)$, and using (4.1) we obtain

$$\begin{aligned} \nabla_XTQ_2Y &+ g(X, TQ_2Y)\alpha - A_{FQ_2Y}X + \nabla^s_XFQ_2Y + D^l(X, FQ_2Y) \\ &= T\nabla_XY + F\nabla_XY + g(X, Y)\phi\alpha - g(X, Y)V. \end{aligned} \tag{4.19}$$

Taking the scalar product of both sides of (4.19) with respect to $\phi\alpha_S$ and using the fact that μ is an invariant subbundle of $T\bar{M}$, we obtain

$$\bar{g}(\nabla^s_X FQ_2X, \phi\alpha_S) = g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S). \tag{4.20}$$

Again using the Sasakian character of $\bar{M}$, we have $\bar{\nabla}_X\phi\alpha_S = \phi\bar{\nabla}_X\alpha_S$, this further implies that

$$\begin{aligned} -A_{\phi\alpha_S}X + \nabla^s_X\phi\alpha_S + D^l(X, \phi\alpha_S) &= -TA_{\alpha_S}X - FA_{\alpha_S}X + B\nabla^s_X\alpha_S \\ &\quad + C\nabla^s_X\alpha_S + \phi D^l(X, \alpha_S), \end{aligned} \tag{4.21}$$

taking the scalar product of both sides of above equation with respect to FQ_2Y and using invariant character of μ, that is, $C\nabla^s_X\alpha_S \in \Gamma(\mu)$ with (2.1) and (4.5), we get

$$\begin{aligned} \bar{g}(\nabla^s_X\phi\alpha_S, FQ_2Y) &= -g(FA_{\alpha_S}X, FQ_2Y) \\ &= -sin^2\theta[g(A_{\alpha_S}X, Q_2Y)]. \end{aligned} \tag{4.22}$$

Since $\bar{\nabla}$ is a metric connection we have $(\bar{\nabla}_Xg)(FQ_2Y, \phi\alpha_S) = 0$, this further implies that $\bar{g}(\nabla^s_X FQ_2Y, \phi\alpha_S) = \bar{g}(\nabla^s_X\phi\alpha_S, FQ_2Y)$, and using (4.22), we obtain

$$\bar{g}(\nabla^s_X FQ_2Y, \phi\alpha_S) = -sin^2\theta[g(A_{\alpha_S}X, Q_2Y)]. \tag{4.23}$$

From (4.20) and (4.23), we have

$$g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S) = -sin^2\theta g[(A_{\alpha_S}X, Q_2Y)], \tag{4.24}$$

and using (2.13) in (4.24), we obtain

$$\begin{aligned} g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S) &= -sin^2\theta[\bar{g}(h^s(Q_2X, Q_2Y), \alpha_S)] \\ &= -sin^2\theta[g(Q_2X, Q_2Y)]g(\alpha_S, \alpha_S), \end{aligned} \tag{4.25}$$

which implies that

$$(1 + sin^2\theta)[g(Q_2X, Q_2Y)]g(\alpha_S, \alpha_S) = 0.$$

Since M is a proper slant lightlike submanifold and g is a Riemannian metric on D we have

$$\alpha_S = 0. \tag{4.26}$$

Next, for any $X \in \Gamma(D)$, using the Sasakian character of $\bar{M}$, we have $\bar{\nabla}_X\phi X = \phi\bar{\nabla}_X X$, implying that $\nabla_X TQ_2X + h(X, TQ_2X) - A_{FQ_2X}X + \nabla^s_X FQ_2X + D^l(X, FQ_2X) = T\nabla_X X + F\nabla_X X + Bh(X, X) + Ch(X, X)$. Since M is a totally contact umbilical slant lightlike submanifold using

$h(X, TQ_2X) = 0$ and then comparing the tangential components, we obtain

$$\nabla_X TQ_2X - A_{FQ_2X}X = T\nabla_X X + Bh(X, X). \tag{4.27}$$

Taking the scalar product of both sides of (4.27) with respect to $\phi\xi \in \Gamma(\phi RadTM)$ and using the Lemma (4.3), we get

$$g(A_{FQ_2X}X, \phi\xi) + \bar{g}(h^l(Q_2X, Q_2X), \xi) = 0. \tag{4.28}$$

Now, using (2.11), we have

$$\bar{g}(h^s(X, \phi\xi), FQ_2X) + \bar{g}(\phi\xi, D^l(X, FQ_2X)) = g(A_{FQ_2X}X, \phi\xi).$$

Since M is a totally contact umbilical slant lightlike submanifold using (4.3) and (4.26) in the above equation, we have

$$g(A_{FQ_2X}X, \phi\xi) = 0. \tag{4.29}$$

Using (4.29) in (4.28), we obtain $\bar{g}(h^l(Q_2X, Q_2X), \xi) = 0$, and using (4.2) we have

$$g(Q_2X, Q_2X)\bar{g}(\alpha_L, \xi) = 0.$$

Since g is a Riemannian metric on D $\bar{g}(\alpha_L, \xi) = 0$, and using (2.5), we obtain

$$\alpha_L = 0. \tag{4.30}$$

Thus, from (4.26) and (4.30), the proof is complete. □

Next, denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively. Using (2.9)-(2.12), we have

$$\begin{aligned}
\bar{R}(X, Y)Z &= R(X, Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\
&\quad - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) \\
&\quad + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) \\
&\quad + (\nabla_X h^s)(Y, Z) - (\nabla_Y h^s)(X, Z) \\
&\quad + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)),
\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}
(\nabla_X h^s)(Y, Z) &= \nabla^s_X h^s(Y, Z) - h^s(\nabla_X Y, Z) \\
&\quad - h^s(Y, \nabla_X Z),
\end{aligned} \tag{4.32}$$

and

$$(\nabla_X h^l)(Y,Z) = \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z). \tag{4.33}$$

An indefinite Sasakian space form is a connected indefinite Sasakian manifold of constant holomorphic sectional curvature c and denoted by $\bar{M}(c)$. Then the curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by (see [9])

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{c+3\epsilon}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \frac{c-\epsilon}{4}\{\eta(X)\eta(Z)Y \\ &\quad -\eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)V - \bar{g}(Y,Z)\eta(X)V \\ &\quad +\bar{g}(\phi Y,Z)\phi X - \bar{g}(\phi X,Z)\phi Y - 2\bar{g}(\phi X,Y)\phi Z\}.\end{aligned} \tag{4.34}$$

for X, Y, Z vector fields on $\bar{M}$.

Theorem 4.6. *There do not exist totally contact umbilical proper slant lightlike submanifolds of an indefinite Sasakian space form $\bar{M}(c)$ such that $c \neq \epsilon$.*

Proof. Let M be a totally contact umbilical proper lightlike submanifold of $\bar{M}(c)$ such that $c \neq \epsilon$. Then, using (4.34), for any $X \in \Gamma(D)$, $Z \in \Gamma(\phi ltr(TM))$ and $\xi \in \Gamma(RadTM)$, we obtain

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -\frac{c-\epsilon}{2} g(\phi X,\phi X) g(\phi Z,\xi),$$

and using (2.2), we get

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -\frac{c-\epsilon}{2} g(Q_2X,Q_2X) g(\phi Z,\xi). \tag{4.35}$$

On the other hand, using (4.1) and (4.31), we get

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = \bar{g}((\nabla_X h^l)(\phi X,Z),\xi) - \bar{g}((\nabla_{\phi X} h^l)(X,Z),\xi). \tag{4.36}$$

Also, using (4.2) and (4.33), we have

$$(\nabla_X h^l)(\phi X,Z) = -g(\nabla_X \phi X,Z)\alpha_L - g(TQ_2X,\nabla_X Z)\alpha_L. \tag{4.37}$$

Similarly,

$$(\nabla_{\phi X} h^l)(X,Z) = -g(\nabla_{\phi X} X,Z)\alpha_L - g(X,\nabla_{\phi X} Z)\alpha_L. \tag{4.38}$$

Using (4.37) and (4.38) in (4.36), we obtain

$$\begin{aligned}\bar{g}(\bar{R}(X,\phi X)Z,\xi) &= -g(\nabla_X \phi X,Z)\bar{g}(\alpha_L,\xi) - g(\phi X,\nabla_X Z)\bar{g}(\alpha_L,\xi) \\ &\quad +g(\nabla_{\phi X} X,Z)\bar{g}(\alpha_L,\xi) + g(X,\nabla_{\phi X} Z)\bar{g}(\alpha_L,\xi).\end{aligned} \tag{4.39}$$

Now, using (2.3), we have

$$g(\phi X, \nabla_X Z) = -\bar{g}(\bar{\nabla}_X \phi X, Z) = -g(\nabla_X \phi X, Z), \tag{4.40}$$

and

$$g(X, \nabla_{\phi X} Z) = -\bar{g}(\bar{\nabla}_{\phi X} X, Z) = -g(\nabla_{\phi X} X, Z). \tag{4.41}$$

Using (4.40) and (4.41) in (4.39), we obtain

$$\bar{g}(\bar{R}(X, \phi X)Z, \xi) = 0. \tag{4.42}$$

Thus, using (4.42) in (4.35), we have

$$\frac{c-\epsilon}{2} g(Q_2 X, Q_2 X) g(\phi Z, \xi) = 0. \tag{4.43}$$

Since g is a Riemannian metric on D and $g(\phi Z, \xi) \neq 0$, (4.43) implies that $c = \epsilon$. This contradiction completes the proof. □

In [7], a minimal lightlike submanifold M was defined when M is a hypersurface of a 4-dimensional Minkowski space. Then in [2], a general notion of minimal lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ was introduced as follows:

Definition 4.7. *A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if*

(i) *$h^s = 0$ on $RadTM$ and*
(ii) *trace $h = 0$, where trace is written with respect to g restricted to $S(TM)$.*

We use the quasi orthonormal basis of M given by

$$\{\xi_1, \ldots.., \xi_r, \phi\xi_1, \ldots.., \phi\xi_r, V, e_1, \ldots.., e_q, \phi N_1, \ldots.., \phi N_r\},$$

such that $\{\xi_1, \ldots.., \xi_r\}$, $\{\phi\xi_1, \ldots.., \phi\xi_r\}$, $\{e_1, \ldots.., e_q\}$ and $\{\phi N_1, \ldots.., \phi N_r\}$ form a basis of $RadTM$, $\phi(RadTM)$, D and $\phi(ltr(TM))$, respectively.

Definition 4.8. *([8]) A lightlike submanifold is called irrotational if and only if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for all $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.*

Theorem 4.9. *Let M be an irrotational slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is minimal if and only if*

$$trace\ A_{W_k}|_{S(TM)} = 0, \quad and \quad trace\ A^*_{\xi_i}|_{S(TM)} = 0,$$

where $\{W_k\}_{k=1}^{l}$ is a basis of $S(TM^\perp)$ and $\{\xi_i\}_{i=1}^{r}$ is a basis of $RadTM$.

Proof. Since $\bar{\nabla}_V V = 0$ using (2.9), we get $h^l(V,V) = 0$ and $h^s(V,V) = 0$. Moreover, M is irrotational implying that $h^s(X,\xi) = 0$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. Thus h^s vanishes on $RadTM$. Hence M is minimal if and only if trace $h = 0$ on $S(TM)$, that is, M is minimal if and only if

$$\sum_{i=1}^{r} h(\phi\xi_i, \phi\xi_i) + \sum_{i=1}^{r} h(\phi N_i, \phi N_i) + \sum_{j=1}^{q} h(e_j, e_j) = 0.$$

Using (2.13) and (2.15) we obtain

$$\begin{aligned} \sum_{i=1}^{r} h(\phi\xi_i, \phi\xi_i) &= \sum_{i=1}^{r}\{\frac{1}{r}\sum_{a=1}^{r} g(A^*_{\xi_a}\phi\xi_i, \phi\xi_i)N_a \\ &\quad +\frac{1}{l}\sum_{k=1}^{l} g(A_{W_k}\phi\xi_i, \phi\xi_i)W_k\}. \end{aligned} \tag{4.44}$$

Similarly, we have

$$\begin{aligned} \sum_{i=1}^{r} h(\phi N_i, \phi N_i) &= \sum_{i=1}^{r}\{\frac{1}{r}\sum_{a=1}^{r} g(A^*_{\xi_a}\phi N_i, \phi N_i)N_a \\ &\quad +\frac{1}{l}\sum_{k=1}^{l} g(A_{W_k}e_j, e_j)W_k\}, \end{aligned} \tag{4.45}$$

and

$$\begin{aligned} \sum_{j=1}^{q} h(e_j, e_j) &= \sum_{j=1}^{q}\{\frac{1}{r}\sum_{i=1}^{r} g(A^*_{\xi_i}e_j, e_j)N_i \\ &\quad +\frac{1}{l}\sum_{k=1}^{l} g(A_{W_k}e_j, e_j)W_k\}. \end{aligned} \tag{4.46}$$

Thus our assertion follows from (4.44)-(4.46). □

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