Title:
Maximal elements of $\mathcal{F}_{C,\theta}$-majorized mappings and applications to generalized games

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MAXIMAL ELEMENTS OF $\mathcal{F}_{C,\theta}$-MAJORIZED MAPPINGS AND APPLICATIONS TO GENERALIZED GAMES

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ABSTRACT. In the paper, some new existence theorems of maximal elements for $\mathcal{F}_{C,\theta}$-mappings and $\mathcal{F}_{C,\theta}$-majorized mappings are established. As applications, some new existence theorems of equilibrium points for one-person games, qualitative games and generalized games are obtained. Our results unify and generalize most known results in recent literature.

Keywords: Maximal elements, generalized games, $\mathcal{F}_{C,\theta}$-majorized mappings, $FC$-space.

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1. Introduction

The existence of equilibrium is a main problem of investigating various kind of economic models in mathematical economics. In 1976, Borglin and Keiding [3] proved a new existence theorem for a compact generalized games with $KF$-majorized preference correspondence. Following the ideas, many authors have obtained the equilibrium existence theorem for generalized games, for example, Briec and Horvath [4], Tan and Yuan [27], Yuan and Tarafdar [28], Ansari and Yao [1], Kim et al. [21], Lin and Liu [22], Ding and Xia [9], Du and Deng [13], Deng and Du [7], Deng and Xia [8], Yang and Deng [29], Ding and Yao [10], and Hou [19] etc. In the setting, convexity assumptions play a crucial role. In 2005, Ding [11] introduce the concept of a $FC$-space, which is a topological space without any convexity structure and linear structure. Moreover,
FC-space include topological vector spaces, \(H\)-space \([17, 18]\), \(G\)-convex space \([24]\) and \(L\)-convex space \([2]\) as special cases. FC-space will be the framework of this paper.

In the paper, we introduce the notions of \(\mathcal{F}_{C,\theta}\)-mapping and \(\mathcal{F}_{C,\theta}\)-majorized mappings with transfer open lower sections in FC-space. We first establish some new existence theorems of maximal elements for \(\mathcal{F}_{C,\theta}\)-mappings and \(\mathcal{F}_{C,\theta}\)-majorized mappings in FC-space. Next, New notions for qualitative games and generalized games are introduced. As application of the existence theorems of maximal elements, we establish some existence theorems of equilibrium points for one-person games, qualitative games and generalized games. Our notions and results unify and generalize the corresponding notions and results introduced by many authors, for example, Borglin and Keiding \([3]\), Tan and Yuan \([27]\), Yuan and Tarafdar \([28]\), Ding and Xia \([9]\), Yang and Deng \([29]\), Chowdhury et al. \([5]\), Shen \([25]\), Lin \([23]\), Homidan et al. \([16]\) and etc.

Let \(X\) be a nonempty subset of topological space \(E\). We shall denote by \(2^X\) the family of all subsets of \(X\), by \(\langle X \rangle\) the family of all nonempty finite subsets of \(X\), by \(\text{int}_E(X)\) the interior of \(X\) in \(E\), and by \(\text{cl}_E(X)\) the closure of \(X\) in \(E\). Let \(\Delta_n\) denote the standard \(n\)-simplex, that is,

\[\Delta_n = \{ u \in \mathbb{R}^{n+1} : u = \sum_{i=1}^{n+1} \lambda_i(u)e_i, \lambda_i(u) \geq 0, \sum_{i=1}^{n+1} \lambda_i(u) = 1 \},\]

where \(e_i\) (\(i = 1, ..., n+1\)) is the \(i\)-th unit vector in \(\mathbb{R}^{n+1}\). In the paper, we suppose every topological space is Hausdorff.

If \(X\) and \(Y\) are topological spaces and \(T, S : X \to 2^Y\) are two mappings, for any \(D \subseteq X\) and \(y \in Y\), let \(S(D) = \cup_{x \in D} S(x)\) and \(S^{-1}(y) = \{ x \in X : y \in S(x) \}\). The notation \(\text{dom} S\) denotes the domain of \(S\), i.e., \(\text{dom} S = \{ x \in X : S(x) \neq \emptyset \}\), and \(T \cap S : X \to 2^Y\) is a mapping defined by \((T \cap S)(x) = T(x) \cap S(x)\) for each \(x \in X\). The graph of \(T\) is the set \(\text{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}\) and the mapping \(\overline{T} : X \to 2^Y\) is defined by \(\overline{T}(x) = \{ y \in Y : (x, y) \in \text{cl}_{X \times Y}(\text{Gr}(T))\}\). The mapping \(\text{cl}_T : X \to 2^Y\) is defined by \((\text{cl}_T)(x) = \text{cl}_Y(\overline{T}(x))\) for each \(x \in X\).

Let \(X\) be a nonempty set and \(Y\) be a topological space. The mapping \(F : X \to 2^Y\) is said to be transfer open valued on \(X\) if

\[\cup_{x \in X} F(x) = \cup_{x \in X} \text{int}_Y(F(x)).\]
The mapping $F : X \to 2^Y$ is said to be transfer closed valued on $X$ if
\[ \bigcap_{x \in X} F(x) = \bigcap_{x \in X} \text{cl}_Y(F(x)). \]
It is easy to prove that if $F$ is transfer open valued, then the mapping $G : X \to 2^Y$ with $G(x) = Y \setminus F(x)$ for each $x \in X$ is transfer closed valued.

**Definition 1.1.** [11] $(E; \varphi_N)$ is said to be a finitely continuous space (in short, FC-space), if $E$ is a topological space and for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle E \rangle$, there exists a continuous mapping $\varphi_N : \triangle_N \to 2^E$. If $X$ is a subset of $E$, $X$ is said to be an FC-subspace of $E$ if for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle E \rangle$ and for any $\{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\} \subset N \cap X$, $\varphi_N(\triangle_k) \subset X$, where $\triangle_k = \text{co}(\{e_{i_0}, \cdots, e_{i_k}\})$.

**Definition 1.2.** [29] Let $(E; \varphi_N)$ be an FC-space and $A$ be a subset of $E$, define the FC-hull of $A$ as follows:
\[ FC-(A) = \cap \{B \subset E : A \subset B \text{ and } B \text{ is an FC-subspace of } E\}. \]

**Definition 1.3.** Let $(X; \varphi_N)$ be an FC-space and $Y$ be a topological space. Let $\theta : X \to Y$ be a single valued mapping and $A : X \to 2^Y$ be a set-valued mapping. Then
(i) $A$ is said to be a $\mathcal{F}_{C,\theta}$-mapping if
(a) $\varphi_N(\triangle_n) \cap \bigcap_{x \in N} A^{-}(\theta(x)) = \emptyset$ for each $N \in \langle X \rangle$; and
(b) $A^{-} : Y \to 2^X$ is transfer open valued in $Y$.
(ii) $(B_x; N_x)$ is said to be a $\mathcal{F}_{C,\theta}$-majorant of $A$ at $x \in X$ if $B_x : X \to 2^Y$ is a set-valued mapping and $N_x$ is an open neighborhood of $x$ in $X$ such that
(a) $A(z) \subset B_x(z)$ for each $z \in N_x$; and
(b) $B_x$ is a $\mathcal{F}_{C,\theta}$-mapping.
(iii) $A$ is said to be a $\mathcal{F}_{C,\theta}$-majorized mapping if for each $x \in \text{dom } A$, there exists a $\mathcal{F}_{C,\theta}$-majorant $(B_x; N_x)$ of $A$ at $x$.

**Remark 1.4.** Definition 1.3 modifies Definition 1.2 of Shen [25] from $H$-space to FC-space, moreover, Definition 1.2 generalize the according notions in Borglin and Keiding [3], Tan and Yuan [27], Ding and Xia [9], Chowdhury et al. [5] and etc.

The following two propositions show that the intersection or union of finite transfer open valued mappings is still transfer open valued.

**Proposition 1.5.** Let $X$ be a nonempty subset of topological space $E$ and $Y$ be a topological space. For each $i \in I = \{1, 2, \cdots, n\}$, the mapping...
$S_i : X \rightarrow 2^Y$ is transfer open valued on $X$, then the mapping $\cap_{i=1}^n S_i$ is transfer open valued on $X$.

Proof. It is clear that $\cup_{x \in X} \text{int}_Y(\cap_{i=1}^n S_i(x)) \subset \cup_{x \in X} (\cap_{i=1}^n S_i(x))$, thus we only need to prove that $\cup_{x \in X} (\cap_{i=1}^n S_i(x)) \subset \cup_{x \in X} \text{int}_Y(\cap_{i=1}^n S_i(x))$. If $z \notin \cup_{x \in X} \text{int}_Y(\cap_{i=1}^n S_i(x))$, for each $x \in X$, $z \notin \text{int}_Y(\cap_{i=1}^n S_i(x))$, $z \in \text{cl}_Y(Y \backslash (\cap_{i=1}^n S_i(x))) = \text{cl}_Y(\cup_{i=1}^n Y \backslash S_i(x))$, then for each open neighborhood $N_z$ of $z$, there exists a $i_0 \in I$, such that $N_z \cap (Y \backslash S_{i_0}(x)) \neq \emptyset$. That is $z \in \text{cl}_Y(Y \backslash S_{i_0}(x))$, i.e. $z \notin \text{int}_Y(S_{i_0}(x))$. Since $S_{i_0}$ is transfer open valued, then $z \notin \cup_{x \in X} \text{int}_Y(S_{i_0}(x)) = \cup_{x \in X} (S_{i_0}(x))$. But $\cup_{x \in X} (\cap_{i=1}^n S_i(x)) \subset \cup_{x \in X} S_{i_0}(x)$, thus $z \notin \cup_{x \in X} (\cap_{i=1}^n S_i(x))$. That is $\cup_{x \in X} (\cap_{i=1}^n S_i(x)) \subset \cup_{x \in X} \text{int}_Y(\cap_{i=1}^n S_i(x))$. \hfill \square

Proposition 1.6. Let $X$ be a nonempty subset of the topological space $E$ and $Y$ be a topological space. If for each $i \in I = \{1, 2, \cdots, n\}$, the mapping $S_i : X \rightarrow 2^Y$ is transfer open valued on $X$, then the mapping $\cup_{i=1}^n S_i$ is transfer open valued on $X$.

Proof. It is clear that $\cup_{x \in X} \text{int}_Y(\cup_{i=1}^n S_i(x)) \subset \cup_{x \in X} (\cup_{i=1}^n S_i(x))$, thus we only need to prove that $\cup_{x \in X} (\cup_{i=1}^n S_i(x)) \subset \cup_{x \in X} \text{int}_Y(\cup_{i=1}^n S_i(x))$. If $z \notin \cup_{x \in X} \text{int}_Y(\cup_{i=1}^n S_i(x))$, for each $x \in X$, $z \notin \text{int}_Y(\cup_{i=1}^n S_i(x))$, $z \in \text{cl}_Y(Y \backslash (\cup_{i=1}^n S_i(x))) = \text{cl}_Y(\cup_{i=1}^n Y \backslash S_i(x))$, for each $i \in I$, that is $z \in \text{cl}_Y(Y \backslash S_i(x))$, $z \notin \text{int}_Y(S_i(x))$. Since $S_i$ is transfer open valued on $X$, $z \notin \cup_{x \in X} \text{int}_Y(S_i(x)) = \cup_{x \in X} S_i(x)$, that is $z \notin \cup_{x \in X} (\cup_{i=1}^n S_i(x))$. Hence, $\cup_{x \in X} (\cup_{i=1}^n S_i(x)) \subset \cup_{x \in X} \text{int}_Y(\cup_{i=1}^n S_i(x))$. \hfill \square

The following result is a special case of Lemma 2.3 in Yang [29].

Lemma 1.7. Let $X$ be a nonempty FC-subspace of a FC-space $(E; \varphi_N)$, $K$ be a nonempty compact subset of $X$ and $T : X \rightarrow 2^E$ be such that

(i) $T$ is transfer closed valued in $X$;

(ii) for each $N \in \langle X \rangle$, $\varphi_N(\Delta_n) \subset \cup_{x \in X} T(x)$;

(iii) for each $N \in \langle X \rangle$, there exists a compact FC-subspace $L_N$ of $E$ containing $N$ such that for each $y \in L_N \backslash K$, there is an $x \in L_N \cap X$ satisfying $y \notin \text{cl}_X(T(x))$. Then $K \cap \bigcap_{x \in X} T(x) \neq \emptyset$.

2. Existence theorem of maximal elements

Let $X$ be a topological space and $T : X \rightarrow 2^X$ be a mapping. A point $\hat{x} \in X$ is called a maximal element of $T$ if $T(\hat{x}) = \emptyset$.

In this section, we shall establish some new existence theorems of maximal elements for $\mathcal{F}_{C;\theta}$-mapping and $\mathcal{F}_{C;\theta}$-majorized mapping defined on noncompact FC-space.
Theorem 2.1. Let $X$ be a nonempty FC-subspace of an FC-space $(E; \varphi_N)$, $K$ be a nonempty compact subset of $X$ and $Y$ be a topological space. Suppose $A : X \to 2^Y$ be a $\mathcal{F}_{C,\theta}$-mapping such that

(i) $\theta : X \to Y$ is a single valued mapping with $\theta(X) = Y$; and

(ii) for each $N \in \langle X \rangle$, there exists a compact FC-subspace $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$ satisfying $y \in \text{int}_X(A^-(\theta(x)))$. Then there exists a point $\hat{x} \in K$ such that $A(\hat{x}) = \emptyset$.

Proof. Since $A$ is a $\mathcal{F}_{C,\theta}$-mapping then

(a) $\varphi_N(\Delta_n) \cap \bigcap_{x \in N} A^-(\theta(x)) = \emptyset$ for each $N \in \langle X \rangle$; and

(b) $A^- : Y \to 2^X$ is transfer open valued in $Y$.

Define a mapping $B : X \to 2^X$ by $B(x) = X \setminus A^-(\theta(x))$, for each $x \in X$. Then we claim that $B$ is transfer closed valued in $X$. Indeed, we only need to prove that $A^- \circ \theta : X \to 2^X$ is transfer open valued in $X$. Put $x_0 \in X$, $z_0 \in A^-\circ\theta(x_0) \subset \bigcup_{y \in Y} A^-\circ\theta(y)$. Since $A^-\circ\theta$ is transfer open valued in $X$, i.e., $\bigcup_{y \in Y} A^-\circ\theta(y) = \bigcup_{y \in Y} \text{int}_X(A^-\circ\theta(y))$, there exists a point $y' \in Y$ such that $z_0 \in \text{int}_X(A^-\circ\theta(y'))$. By (i), there exists a point $x' \in X$ so that $\theta(x') = y'$. Thus, $z_0 \in \text{int}_X(A^-\circ\theta(x')) \subset \bigcup_{x \in X} \text{int}_X(A^-\circ\theta(x))$, and $\bigcup_{x \in X} A^-\circ\theta(x) = \bigcup_{x \in X} \text{int}_X(A^-\circ\theta(x))$, therefore, $A^- \circ \theta$ is transfer open valued in $X$.

By (a), $\varphi_N(\Delta_n) \subset \bigcup_{x \in N} (X \setminus A^-\circ\theta(x)) = \bigcup_{x \in N} B(x)$ for each $N \in \langle X \rangle$.

By (ii), for each $N \in \langle X \rangle$, there exists a compact FC-subspace $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$ satisfying $y \in \text{int}_X(A^-\circ\theta(x)))$. This follows $y \notin X \setminus \text{int}_X(A^-\circ\theta(x))) = \text{cl}_X(B(x))$. Therefore, $B$ satisfies all the hypotheses of Lemma 1.7. By Lemma 1.7, $K \cap \bigcap_{x \in X} B(x) \neq \emptyset$. Then there exists a point $\hat{x} \in K$ such that $\hat{x} \notin A^-\circ\theta(x)$ for each $x \in X$. That is $\theta(x) \notin A(\hat{x})$ for each $x \in X$, thus $A(\hat{x}) = \emptyset$ by (i). This completes the proof. $\square$

For a topological space $(X, \tau)$, the compactly generated extension of the topology $\tau$ is the new topology consisting of all compactly closed respectively, open] subsets. In this way, we have the following modified form of Theorem 2.1 which is equivalent to Theorem 3.1 of Yang and Deng [29].

Theorem 2.2. Let $X$ be a nonempty FC-subspace of an FC-space $(E; \varphi_N)$ and $K$ be a nonempty compact subset of $X$. Suppose the mapping $A : X \to 2^X$ be such that

(i) $x \notin FC(A(x))$ for each $x \in X$;
(ii) $A^- : X \to 2^X$ is transfer compactly open valued in $X$;
(iii) for each $N \in \langle X \rangle$, there exists a compact $FC$-subspace $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$ satisfying $y \in \text{int}_X(A^-(x))$. Then there exists a point $\hat{x} \in K$ such that $A(\hat{x}) = \emptyset$.

Proof. Replace the topology of $E$ by its compactly generated extension, then $(E; \varphi_N)$ with this new topology is another $FC$-space. It is easy to prove that $x \not\in FC-(A(x))$ for all $x \in X$ implies $\varphi_N(\Delta_n) \cap (\bigcap_{x \in X} A^-(x)) = \emptyset$ for each $N \in \langle X \rangle$. Let $\theta = I_X$ be the identity mapping on $X$, then $A$ becomes a $\mathcal{F}_{C,\theta}$-mapping. All the hypotheses of Theorem 2.1 are satisfied. By Theorem 2.1, there exists a point $\hat{x} \in K$ such that $A(\hat{x}) = \emptyset$. This completes the proof. \hfill $\square$

Remark 2.3. Theorem 2.1 improves Corollary 2.2 of Shen [25] from $CH$-space to $FC$-space. Moreover, Theorem 2.1 generalizes Theorem 3.1 of Ding and Xia [9], Theorem 3.1 of Chowdhury et al. [5] and Theorem 6 of Lin [23] with weaker assumptions.

Moreover, it is easy to prove that Theorem 2.2 is equivalent to the following fixed point theorem.

Theorem 2.4. Let $X$ be a nonempty $FC$-subspace of an $FC$-space $(E; \varphi_N)$ and $K$ be a nonempty compact subset of $X$. Suppose the mapping $A : X \to 2^X$ be such that
(i) $A(x) \neq \emptyset$ for each $x \in K$;
(ii) $A^- : X \to 2^X$ is transfer open valued in $X$;
(iii) for each $N \in \langle X \rangle$, there exists a compact $FC$-subspace $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$ satisfying $y \in \text{int}_X(A^-(x))$.
Then there exists a point $\hat{x} \in K$ such that $\hat{x} \in FC-(A(\hat{x}))$.

Remark 2.5. Theorem 2.4 generalizes Theorem 3.2 of Yuan and Tarafdar [28] and Theorem 3.2 of Ding and Xia [9] in several aspects.

Theorem 2.6. Let $X$ be a nonempty $FC$-subspace of an $FC$-space $(E; \varphi_N)$, $K$ be a nonempty compact subset of $X$ and $Y$ be a topological space. Suppose $A : X \to 2^Y$ is a $\mathcal{F}_{C,\theta}$-majorized mapping such that
(i) $\theta : X \to Y$ is a single valued mapping with $\theta(X) = Y$;
(ii) there exists a paracompact subset $G$ of $X$ such that $\{x \in X : A(x) \neq \emptyset\} \subseteq G$; and
(iii) for each $N \in \langle X \rangle$, there exists a compact $FC$-subspace $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$.
satisfying \( y \in \text{int}_X(A^-(\theta(x))) \).

Then there exists a point \( \hat{x} \in K \) such that \( A(\hat{x}) = \emptyset \).

Proof. Suppose that \( A(x) \neq \emptyset \) for each \( x \in X \). By (ii), \( X \) is paracompact. Since \( A \) is a \( \mathcal{F}_{C,\theta} \)-majorized mapping, for each \( x \in X \), there exists an open neighborhood \( N_x \) of \( x \) in \( X \) and a mapping \( B_x : X \rightarrow 2^Y \) such that

(a) \( A(z) \subset B_x(z) \) for each \( z \in N_x \);
(b) \( \varphi_N(\Delta_n) \cap \bigcap_{i \in N} B_x^-(\theta(z)) = \emptyset \) for each \( N \in \langle X \rangle \); and
(c) \( B_x^- : Y \rightarrow 2^X \) is transfer open valued in \( Y \).

By Theorem VIII.1.4 of Dugundji [14], the open covering \( \{ N_x : x \in X \} \) of \( X \) has an open precise neighborhood finite refinement \( N'_x \) with \( \text{cl}_X N'_x \subset N_x \). For each \( x \in X \), define \( B'_x : X \rightarrow 2^Y \) by

\[
B'_x(z) = \begin{cases} 
B_x(z) & \text{if } z \in \text{cl}_X N'_x, \\
Y & \text{if } z \notin \text{cl}_X N'_x.
\end{cases}
\]

and \( B : X \rightarrow 2^Y \) by \( B(z) = \cap_{x \in X} B'_x(z) \) for each \( z \in X \).

For each \( N \in \langle X \rangle \), \( t \in \cap_{z \in N} B^- \theta(\theta(z)), \) then for each \( z \in N, \theta(z) \in B(t). \) Since \( t \in X, \) then there exists an \( x_0 \in X \) such that \( t \in \text{cl}_X N'_{x_0}, \theta(z) \in B(t) \subset B_{x_0}(t). \) By (b), \( t \notin \varphi_N(\Delta_n) \). Thus we have \( \varphi_N(\Delta_n) \cap \bigcap_{i \in N} B^- \theta(\theta(z)) = \emptyset. \)

Now, we show \( B^- : Y \rightarrow 2^X \) is transfer open valued in \( Y \). For each \( x \in X, \ y \in Y, \) we have

\[
(B'_x)^-(y) = \{ z \in \text{cl}_X N'_x : y \in B'_x(z) \} \cup \{ z \in X \setminus \text{cl}_X N'_x : y \in B'_x(z) \}
\]

\[
= \{ z \in \text{cl}_X N'_x : y \in B_x(z) \} \cup (X \setminus \text{cl}_X N'_x)
\]

\[
= [B_x^-(y) \cap (\text{cl}_X N'_x)] \cup (X \setminus \text{cl}_X N'_x)
\]

\[
(2.1)
\]

(2.1)

For each \( y \in Y \) let \( t \in B^-(y) \) be arbitrarily fixed. Since \( \{ N'_x : x \in X \} \) is a neighborhood finite refinement, there exists an open neighborhood \( V_t \) of \( t \) in \( X \) such that \( \{ x \in X : V_t \cap N'_x \neq \emptyset \} = \{ x_1, x_2, \ldots, x_n \}. \) If \( x \notin \{ x_1, x_2, \ldots, x_n \}, \) then \( V_t \cap N'_x = V_t \cap \text{cl}_X N'_x = \emptyset. \) Thus \( B'_x(z) = Y \) for all \( z \in V_t, \) that is \( B(z) = \cap_{i=1}^n B'_{x_i}(z) \) for each \( z \in V_t. \) By formula (2.1), we have

\[
B^-(y) = \{ z \in X : y \in B(z) \} \supset \{ z \in V_t : y \in \bigcap_{i=1}^n B'_{x_i}(z) \}
\]

\[
= V_t \cap \bigcap_{i=1}^n (B'_{x_i})^-(y) = \bigcap_{i=1}^n \{ V_t \cap [B^-_{x_i}(y) \cup (X \setminus \text{cl}_X N'_{x_i})] \}.
\]
By Proposition 1.5 and Proposition 1.6, $B^{-} : Y \to 2^X$ is transfer open valued in $Y$. Thus $B$ is a $\mathcal{F}_{C;g}$-mapping.

By (a) and the definition of $B$, we have $A(z) \subseteq B(z)$ for each $z \in X$, thus by the assumption (iii), for each $N \in (X)$, there exists a compact $FC$-subspace $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$ satisfying $y \in \text{int}_X(A^{-}(\theta(x))) \subseteq \text{int}_X(B^{-}(\theta(x)))$. All conditions of Theorem 2.1 are satisfied. By Theorem 2.1, there exists a point $\hat{x} \in K$ such that $B(\hat{x}) = \emptyset$. Since $A(z) \subseteq B(z)$ for each $z \in K$, thus $A(\hat{x}) = \emptyset$, which is a contradiction. Hence, there exists a point $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$. By condition (iii), $\hat{x}$ must be in $K$. This completes the proof.

\begin{remark}
Theorem 2.6 improves Theorem 1 of Ding and Tan [12] from paracompact topological vector spaces to nonparacompact $FC$-space. Moreover, Theorem 2.4 generalizes Theorem 3.3 of Ding and Xia [9], Theorem 3.3 of Yang and Deng [29] and Theorem 2.3 of Shen [25].
\end{remark}

3. Existence of equilibria points

Let $I$ be a (finite or infinite) set of players. Let its strategy set $X$ be a nonempty $FC$-subspace of an $FC$-space $(E; \varphi_N)$, and $Y_i$ be a topological space for each $i \in I$ with $Y = \prod_{i \in I} Y_i$. Let $P_i : X \to 2^{Y_i}$ be the preference correspondence of $i$th player. The collection $\Lambda = (X; Y_i; P_i)_{i \in I}$ will be called a qualitative game. A point $\hat{x} \in X$ is said to be an equilibrium of the qualitative game, if $P_i(\hat{x}) = \emptyset$ for each $i \in I$.

A generalized game (=abstract economy) is a quintuple family $\Lambda = (X; Y_i; A_i; B_i; P_i; \theta_i)_{i \in I}$ where $X$ is a nonempty $FC$-subspace of an $FC$-space $(E; \varphi_N)$, $I$ is a (finite or infinite) set of players such that for each $i \in I$, $Y_i$ is a topological space with $Y = \prod_{i \in I} Y_i$. Let $A_i, B_i : X \to 2^{Y_i}$, $\theta_i : X \to Y_i$ be the constraint correspondences and let $P_i : X \to 2^{Y_i}$ be the preference correspondence. An equilibrium of the generalized game $\Lambda$ is a point $\hat{x} \in X$ such that for each $i \in I, \theta_i(\hat{x}) \in B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. If $X = Y = \prod_{i \in I} Y_i$ and $\theta_i = \pi_i : Y \to Y_i$ is the projection of $Y$ onto $Y_i$, then our definition of an equilibrium point coincides with the standard definition given by Yang and Deng [29], moreover, our definition of an equilibrium point generalizes the standard definition; e.g., Borglin and Keiding [3], Chowdhury et al. [5], Gale and Mas-Colell [15], Kim [20], Yannelis and Prabhakar [30], Cubiotti and Yao [6].

As an application of Theorem 2.6, we firstly prove the following existence theorem of equilibrium points for one person game.
Theorem 3.1. Let $X$ be a nonempty paracompact $FC$-subspace of an $FC$-space $(E; \varphi_N)$, $K$ be a nonempty compact subset of $X$ and $Y$ be a topological space. Suppose the mappings $A, B, P : X \rightarrow 2^Y$ are such that

(i) $\theta : X \rightarrow Y$ is a single valued and continuous mapping with $\theta(X) = Y$;

(ii) $\text{dom} A = X$, $A$ is a $\mathcal{F}_{C, \beta}$-mapping and $P$ is a $\mathcal{F}_{C, \beta}$-majorized mapping;

(iii) for each $N \in \langle X \rangle$, there exists a compact $FC$-subspace $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$ satisfying $y \in \text{int}_X((A \cap P)^-(\theta(x)))$.

Then there exists a point $\hat{x} \in K$ such that $\theta(\hat{x}) \in \overline{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Proof. Let $W = \{x \in X : \theta(x) \notin \overline{B}(x)\}$, then $W$ is open in $X$ since $\theta$ is continuous. Define $Q : X \rightarrow 2^Y$ by

$$Q(z) = \begin{cases} A(z) \cap P(z) & \text{if } z \notin W; \\ A(z) & \text{if } z \in W. \end{cases}$$

By (ii), $P$ is a $\mathcal{F}_{C, \beta}$-majorized mapping, for each $x \in \text{dom} P$, there exists an open neighborhood $M_x$ of $x$ in $X$ and $\psi_x : X \rightarrow 2^Y$ such that

(a) $\varphi_N(\Delta_n) \cap \bigcap z \in N \psi^-_x(\theta(z)) = \emptyset$ for each $N \in \langle X \rangle$ and $P(z) \subset \psi_x(z)$ for all $z \in M_x$;

(b) $\psi^-_x : Y \rightarrow 2^X$ is transfer open valued in $Y$.

Now, for each $x \in X$ with $Q(x) \neq \emptyset$, let

$$N_x = \begin{cases} M_x & \text{if } x \notin W; \\ W & \text{if } x \in W. \end{cases}$$

and define $\Psi_x : X \rightarrow 2^Y$ by

$$\Psi_x(z) = \begin{cases} A(z) \cap \psi_x(z) & \text{if } z \notin W; \\ A(z) & \text{if } z \in W. \end{cases}$$

Then for each $y \in Y$,

$$\Psi^-_x(y) = \{z \in X \setminus W : y \in \Psi_x(z)\} \cup \{z \in W : y \in \Psi_x(z)\}$$

$$= \{z \in X \setminus W : y \in A(z) \cap \psi_x(z)\} \cup \{z \in W : y \in A(z)\}$$

$$= [(X \setminus W) \cap A^-(y) \cap \psi^-_x(y)] \cup [W \cap A^-(y)]$$

$$= [W \cup \psi^-_x(y)] \cap A^-(y) = [W \cap A^-(y)] \cup [\psi^-_x(y) \cap A^-(y)],$$

thus

$$\Psi^-_x(\theta(z)) = [W \cap A^-(\theta(z))] \cup [\psi^-_x(\theta(z)) \cap A^-(\theta(z))].$$
Hence we have

(a') \(\varphi_N(\Delta_n) \cap \bigcap_{z \in N} \Psi_x(z) = \emptyset\) for each \(N \in \langle X \rangle\) by (ii) and (a);
(b') for each \(z \in N_x\), \(Q(z) \subseteq \Psi_x(z)\) by (a); and
(c') \(\Psi_x : Y \to 2^X\) is transfer open valued in \(Y\) by Proposition 1.5 and Proposition 1.6. Therefore, \((\Psi_x; N_x)\) is a \(\mathcal{F}_{C,\theta}\)-majorant of \(Q\) at \(x\).

From the definition of \(Q\), it follows that \((A \cap P)(z) \subseteq Q(z)\) for each \(z \in X\). By (iii), for each \(N \in \langle X \rangle\), there exists a compact \(FC\)-subspace \(L_N\) of \(E\) containing \(N\) such that for each \(y \in L_N \setminus K\), there is an \(x \in L_N \cap X\) satisfying \(y \in \text{int}_X((A \cap P)(\theta(x))) \subseteq \text{int}_X(Q(\theta(x)))\). All conditions of Theorem 2.6 are satisfied. By Theorem 2.6, there exists a point \(\hat{x} \in K\) such that \(Q(\hat{x}) = \emptyset\). By the definition of \(Q\), we have \(\theta(\hat{x}) \in B(\hat{x})\) and \(A(\hat{x}) \cap P(\hat{x}) = \emptyset\). This completes the proof. \(\square\)

Remark 3.2. Theorem 3.1 unifies Theorem 4.1 of Chowdhury et al. [5], Theorem 3.1 of Shen [25] and Theorem 2 in Ding and Tan [12] where \(X = Y\) and \(\theta = I_X\) is the identical mapping.

By using Theorem 2.6, we shall show an equilibrium existence theorem for a qualitative game.

Theorem 3.3. Let \(\Lambda = (X; Y_i; P_i)_{i \in I}\) be a qualitative game. Where \(X\) is a nonempty paracompact \(FC\)-subspace of an \(FC\)-space \((E; \varphi_N)\), \(K\) be a nonempty compact subset of \(X\) and \(Y_i(i \in I)\) is a topological space with \(Y = \prod_{i \in I} Y_i\). For each \(i \in I\), suppose the mapping \(P_i : X \to 2^{Y_i}\) be such that

(i) \(\theta_i : X \to Y_i\) is a single valued mapping with \(\theta_i(X) = Y_i\);
(ii) \(W_i = \text{dom} P_i\) is open and \(P_i : X \to 2^{Y_i}\) is a \(\mathcal{F}_{C,\theta_i}\)-majorized mapping; and
(iii) for each \(N \in \langle X \rangle\), there exists a compact \(FC\)-subspace \(L_N\) of \(E\) containing \(N\) such that for each \(y \in L_N \setminus K\), there is an \(x \in L_N \cap X\) satisfying \(y \in \text{int}_X(P_i^{-1}(\theta_i(x)))\).

Then \(\Lambda\) has an equilibrium point in \(\hat{x} \in K\).

Proof. For each \(x \in X\), let \(I(x) = \{i \in I : P_i(x) \neq \emptyset\}\). For each \(i \in I\), define \(P_i'(x) : X \to 2^Y\) by

\[P_i'(x) = \pi_i^{-1}(P_i(x)),\] for each \(x \in X\),

where \(\pi_i : Y \to Y_i\) is the projection of \(Y\) onto \(Y_i\). Furthermore, define \(P : X \to 2^Y\) by

\[P(x) = \begin{cases} \cap_{i \in I(x)} P_i'(x) & \text{if } I(x) \neq \emptyset; \\ \emptyset & \text{if } I(x) = \emptyset. \end{cases}\]
Clearly, for each \( x \in \text{dom } P \) if and only if \( I(x) \neq \emptyset \). Let \( x \in \text{dom } P \), for any fixed \( i \in I(x) \), by (ii), there exist an open neighborhood \( N_x \) of \( x \) in \( X \) and a mapping \( \psi_{i,x} : X \to 2^{Y_i} \) such that

(a) \( \varphi_N(\Delta_n) \cap \bigcap_{z \in N} \psi_{i,x}^{-1}(\theta_i(z)) = \emptyset \) for each \( N \in \langle X \rangle \) and \( P_i(z) \subset \psi_{i,x}(z) \) for each \( z \in N_x \);

(b) \( \psi_{i,x}^{-1} : Y_i \to 2^X \) is transfer open valued in \( Y_i \).

By (ii), we may assume that \( N_x \subset W_i \) so that \( P_i(z) \neq \emptyset \) for all \( z \in N_x \).

Define \( \Psi_x : X \to 2^Y \) by

\[
\Psi_x(z) = \pi_i^{-1}(\psi_{i,x}(z)) \quad \text{for each } z \in X.
\]

For each \( y \in Y \),

\[
\Psi_x^{-1}(y) = \{ z \in X : y \in \Psi_x(z) \} = \{ z \in X : y_i \in \psi_{i,x}(z) \} = \psi_{i,x}^{-1}(y_i).
\]

By condition (iii), for each \( z \in N_x \), we get

\[
\varphi_N(\Delta_n) \cap \bigcap_{z \in N} \Psi_x^{-1}(z) = \varphi_N(\Delta_n) \cap \bigcap_{z \in N} \psi_{i,x}^{-1}(z) = \emptyset,
\]

and for each \( z \in N_x \),

\[
P(z) = \bigcap_{i \in I(z)} P_i(z) \subset P_i(z) = \pi_i^{-1}(P_i(z)) \subset \pi_i^{-1}(\psi_{i,x}(z)) = \Psi_x(z).
\]

(\( b' \)) by (b) and formula (3.1), \( \Psi_x^{-1} : Y \to 2^X \) is transfer open valued in \( Y \).

Hence \( (N_x; \Psi_x) \) is a \( \mathcal{F}_{C,\theta} \)-majorant of \( P \) at \( x \), \( P \) is a \( \mathcal{F}_{C,\theta} \)-majorized mapping.

By the definitions of \( P \), we have \( P^{-1}(y) = P_i^{-1}(\pi_i(y)) \) for each \( y \in Y \). By condition (iii), for each \( N \in \langle X \rangle \), there exists a compact FC-subspace \( L_N \) of \( E \) containing \( N \) such that for each \( y \in L_N \setminus K \), there is an \( x \in L_N \cap X \) satisfying \( y \in \text{int}_X(P^{-1}(\theta(x))) \). All hypotheses of Theorem 2.6 are satisfied. By Theorem 2.6, there exists a point \( \hat{x} \in K \) such that \( P(\hat{x}) = \emptyset \), which implies \( I(\hat{x}) = \emptyset \). Therefore, \( P_i(\hat{x}) = \emptyset \) for all \( i \in I \). The proof is completed.

\[\square\]

**Remark 3.4.** If for each \( i \in I \), \( X_i = Y_i \), \( X = \prod_{i \in I} X_i \) and \( \theta_i = \pi_i \) is the projection from \( X \) onto \( X_i \), Theorem 3.3 unifies Theorem 3.2 of Shen [25] from \( CH \)-space to \( FC \)-space. Moreover, Theorem 3.2 improves
Corollary 3 of Borglin and Keiding [3], Theorem 3.1 of Tan and Yuan [27] and Theorem 4.2 of Chowdhury et al. [5].

**Theorem 3.5.** Let \( \Lambda = (X; Y_i; A_i; B_i; P_i; \theta_i)_{i \in I} \) be a generalized qualitative game. Where \( X \) is a nonempty paracompact FC-space of an FC-space \((E; \varphi_N)\), \( K \) be a nonempty compact subset of \( X \) and \( Y_i (i \in I) \) is a topological space with \( Y = \prod_{i \in I} Y_i \). For each \( i \in I \), suppose the mapping \( A_i, B_i, P_i : X \to 2^{Y_i} \) be such that

(i) \( W_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset \} \) is open in \( X \);

(ii) \( \theta_i : X \to Y_i \) is a single valued and continuous mapping with \( \theta_i(X) = Y_i \);

(iii) \( \text{dom } A_i = X \), \( A_i \) is a \( \mathcal{F}_{C; \theta_i} \)-mapping and \( P_i \) is a \( \mathcal{F}_{C; \theta_i} \)-majorized mapping; and

(iv) for each \( N \in (X) \), there exists a compact FC-space \( L_N \) of \( E \) containing \( N \) such that for each \( y \in L_N \setminus K \), there is an \( x \in L_N \cap X \) satisfying \( y \in \text{int}_X((A_i \cap P_i)(\theta_i(x))) \).

Then \( \Lambda \) has an equilibrium point in \( \hat{x} \in K \).

**Proof.** For each \( i \in I \), let \( U_i = \{x \in X : \theta_i(x) \notin B_i(x)\} \), then \( U_i \) is open in \( X \) by (ii). Define \( Q_i : X \to 2^{Y_i} \) by

\[
Q_i(x) = \begin{cases} 
A_i(x) \cap P_i(x) & \text{if } x \notin U_i; \\
A_i(x) & \text{if } x \in U_i.
\end{cases}
\]

Now, we will show that the qualitative game \( \Lambda' = (X; Y_i; Q_i)_{i \in I} \) satisfies all assumptions of Theorem 3.3. For each \( i \in I \), the set

\[
\{x \in X : Q_i(x) \neq \emptyset\} = \{x \in U_i : A_i(x) \neq \emptyset\} \\
\cup \{x \in X \setminus U_i : A_i(x) \cap P_i(x) \neq \emptyset\} \\
= U_i \cup [(X \setminus U_i) \cap W_i] = U_i \cup W_i
\]

is open in \( X \).

Since \( P_i \) is a \( \mathcal{F}_{C; \theta_i} \)-majorized mapping, for each \( x \in \text{dom } P_i \), there exist an open neighborhood \( M_x \) of \( x \) in \( X \) and a mapping \( \psi_{i, x} : X \to 2^{Y_i} \) such that

(a) \( \varphi_N(\Delta_n) \cap \bigcap_{z \in N} \psi_{i, z}(\theta_i(z)) = \emptyset \) for each \( N \in (X) \) and \( P_i(z) \subset \psi_{i, x}(z) \) for each \( z \in N_x \);

(b) \( \psi_{i, x} : Y_i \to 2^X \) is transfer open valued in \( Y_i \).

Now, for each \( x \in X \) with \( Q_i(x) \neq \emptyset \), let

\[
N_x = \begin{cases} 
M_x & \text{if } x \notin U_i; \\
U_i & \text{if } x \in U_i.
\end{cases}
\]
and define $\Psi_{i,x} : X \to 2^{Y_i}$ by

$$
\Psi_{i,x}(z) = \begin{cases} 
A_i(z) \cap \psi_{i,x}(z) & \text{if } z \not\in U_i; \\
A_i(z) & \text{if } z \in U_i.
\end{cases}
$$

For each $y \in Y_i$, we have

$$
\Psi^-_{i,x}(y) = \{ z \in X \setminus U_i : y \in \Psi_{i,x}(z) \} \cup \{ z \in U_i : y \in \Psi_{i,x}(z) \} \\
= \{ z \in X \setminus U_i : y \in A_i(z) \cap \psi_{i,x}(z) \} \cup \{ z \in U_i : y \in A_i(z) \} \\
= [(X \setminus U_i) \cap A_i^-(y) \cap \psi_{i,x}^-(y)] \cup [U_i \cap A_i^-(y)] \\
= [U_i \cup \psi_{i,x}^-(y)] \cap A_i^-(y) \\
= [U_i \cap A_i^-(y)] \cup [\psi_{i,x}^-(y) \cap A_i^-(y)].
$$

Thus, we have

(a') $\varphi_N(\triangle_n) \cap \bigcap_{z \in N} \Psi^-_{i,x}(\theta_i(z)) = \emptyset$ for each $N \in \langle X \rangle$ and $Q_i(z) \subset \Psi_{i,x}(z)$ for each $z \in N_x$ by (a) and (iii);

(b') By Proposition 1.5 and Proposition 1.6, $\Psi^-_{i,x} : Y_i \to 2^X$ is transfer open valued in $Y_i$. Hence $(N_x ; \Psi_{i,x})$ is a majorant of $Q_i$ at $x$, i.e., $Q_i$ is a $\mathcal{F}_c,\theta_i$-majorized mapping.

By condition (iv) and the definition of $Q_i$, for each $N \in \langle X \rangle$, there exists a compact $FC$-subspace $L_N$ of $E$ containing $N$ such that for each $y \in L_N \setminus K$, there is an $x \in L_N \cap X$ satisfying $y \in \text{int}_X((A_i \cap P_i)^-(\theta_i(x))) \subset \text{int}_X(Q_i^-\theta_i(x)))$. All assumptions of Theorem 3.3 are satisfied. By Theorem 3.3, there exists a point $\hat{x} \in K$ such that $Q_i(\hat{x}) = \emptyset$ for all $i \in I$. By the definition of $Q_i$, we must have that for each $i \in I$, $\theta_i(\hat{x}) \in B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. $\Box$

**Remark 3.6.** Theorem 3.5 generalized Theorem 3.3 of Shen [25], Theorem 4 of Ding and Tan [12] and Corollary 3.4 of Tan and Yuan [26] in weaker assumptions.

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